

On the Impact of Antenna Correlation in Multi-User MIMO Systems with Rate Constraints

Hao Wang, Peng Wang, Li Ping, *Senior Member, IEEE*, and Xiaokang Lin

Abstract—This letter is concerned with multi-user multiple-input multiple-output (MIMO) systems with rate constraints. We show that antenna correlation at mobile units (MUs) is actually beneficial from the capacity point of view. This finding is useful in practice as minimizing the physical size of MUs is highly desirable, which may result in antenna correlation.

Index Terms—Multiple-input multiple-output (MIMO), antenna correlation, rate constraints.

I. INTRODUCTION

ANTENNA correlation in a multiple-input multiple-output (MIMO) system has two consequences [1]. First, it reduces the degree of freedom; the related disadvantage is most noticeable in the high rate (or power) regime. Second, it increases the potential of beamforming gain¹; the related advantage is most noticeable in the low rate (or power) regime. These two opposite effects result in a cross point between the capacity curves for systems with and without antenna correlation. Below this point, correlation is advantageous and *vice versa*. In a conventional single-user MIMO system, such a cross point occurs at a relatively low information rate, so antenna correlation is generally regarded as a negative factor [1][2].

In this letter, we investigate the impact of antenna correlation at mobile units (MUs) on the capacity of multi-user MIMO systems with rate constraints [3]. We show numerically that the cross point mentioned above occurs at a rate increasing with the number of MUs (denoted by K below). This implies that the range where antenna correlation is beneficial increases with K . We also quantify this advantage analytically in the limiting case of $K \rightarrow \infty$. Our results indicate that antenna correlation at MUs is potentially beneficial for multi-user MIMO systems with rate constraints.

Our discussions are based on capacity analysis (i.e., the theoretical limits of system performance), which distinguishes this letter from some existing work on related issues. For example, [4] and [5] are for specific transmission strategies in time-division multiple-access format. The advantage of

antenna correlation reported in [4][5] results purely from beamforming gain; while that reported in this letter results from, besides beamforming gain, the space diversity related to the locations of multiple MUs.

The notations used in this letter are as follows. For any matrix \mathbf{A} , \mathbf{A}^* is the conjugate transpose of \mathbf{A} . For a Hermitian matrix \mathbf{A} , $\text{tr}(\mathbf{A})$, $\mathbf{A}^{1/2}$ and $\lambda_i(\mathbf{A})$ denote, respectively, its trace, principal square root and i th eigenvalue after sorting in descending order. In particular, $\lambda_{\max}(\mathbf{A})$ is the maximum eigenvalue of \mathbf{A} . $\mathbb{E}\{\cdot\}$ denotes expectation.

II. SYSTEM MODEL

Our focus is on MIMO multiple access channels (MACs), but the results can be extended to MIMO broadcast channels using the duality principle [6]. Consider a K -user MIMO system over a quasi-static flat fading MAC with M antennas at the base station (BS) and N antennas at each MU. Denote by \mathbf{H}_k and \mathbf{x}_k the channel matrix and transmitted signal of MU k , respectively. The received signal \mathbf{y} is given by

$$\mathbf{y} = \sum_{k=1}^K \mathbf{H}_k \mathbf{x}_k + \mathbf{n} \quad (1)$$

where \mathbf{n} is a vector of complex additive white Gaussian noise samples with zero mean and unit variance. $\{\mathbf{H}_k\}$ are independent and identically distributed (i.i.d.) among different MUs and perfectly known at both transmitters and the receiver.

We only consider the fading correlation among antennas at each MU due to its limited physical size and assume independent fading among antennas at the BS. Following [1][4], we write \mathbf{H}_k as

$$\mathbf{H}_k = \sqrt{s_k} \mathbf{W}_k \mathbf{T}_k^{1/2} \quad (2)$$

where s_k is a scalar denoting the gain of path loss and lognormal fading², \mathbf{W}_k denotes Rayleigh fading whose entries are i.i.d. complex circular symmetric Gaussian variables with zero mean and unit variance, and \mathbf{T}_k is a semi-positive definite matrix that characterizes the antenna correlation effect at MU k and is normalized to $\text{tr}(\mathbf{T}_k) = N$.

III. PERFORMANCE COMPARISON FOR A FINITE K

This letter is concerned with multi-user MIMO systems with rate constraints, in which each MU must transmit a certain amount of information within a fixed time period. The applications of such systems include delay-sensitive services such as speech and real-time video. In this case, minimizing the system sum power is an appropriate objective [3].

²We assume equal path loss and lognormal fading for all the antenna links seen by a particular MU.

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H. Wang and X. Lin are with the Department of Electronic Engineering, Tsinghua University, Beijing, P. R. China (e-mail: hao-wang04@mails.tsinghua.edu.cn, linxk@sz.tsinghua.edu.cn).

P. Wang and L. Ping are with the Department of Electronic Engineering, City University of Hong Kong, Hong Kong SAR, P. R. China (e-mail: {pengwang, eeliping}@cityu.edu.hk).

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¹Beamforming gain is a term commonly used for antenna arrays. It is used in this letter to refer to the advantage of the effect that the transmitted power is concentrated in some particular directions.

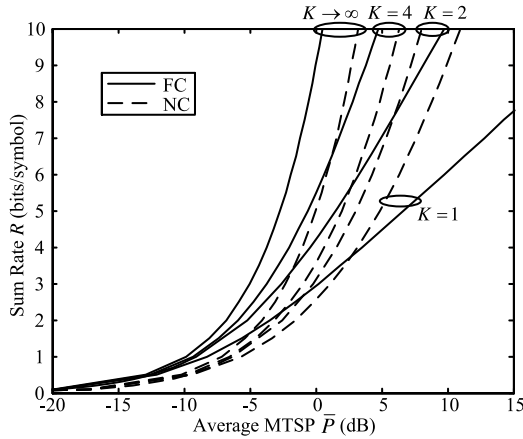


Fig. 1. Performance comparison between MIMO systems with FC and NC. $M = 8$, $N = 4$ and $\varepsilon = 0.01$. The numbers of MUs K are marked on the curves.

For simplicity, we assume that each MU has the same rate constraint R/K (i.e., a MU transmits at a fixed rate R/K), where R is referred to as the system sum rate. Denote by P the minimum transmitted sum power (MTSP) of the system in (1) for each channel realization. To obtain P , we need to jointly optimize the transmit covariance matrices of all MUs and the decoding order of the successive interference cancellation at the BS [3][7]. In general, there is no closed-form expression for P , but it can be evaluated numerically [7].

In this letter, we are interested in the average performance of the system in (1). Define \bar{P} as the average MTSP, i.e.,

$$\bar{P} = \mathbb{E}\{P\} \quad (3)$$

where the expectation is taken over the joint distribution of $\{\mathbf{H}_k\}$. Figure 1 shows the numerical results for the average performance of MIMO systems with $M = 8$ and $N = 4$ in two extreme scenarios, namely, full correlation (FC) and no correlation (NC). For the former, $\{\mathbf{T}_k\}$ are i.i.d. rank-1 matrices; while for the latter, $\{\mathbf{T}_k\}$ are identity matrices, i.e., $\mathbf{T}_k = \mathbf{I}, \forall k$. All MUs are independent and uniformly distributed in an edge-length-1 hexagon cell with fourth power path-loss law and the standard deviation of their normalized lognormal fading is 8. To avoid extremely large transmission power, a MU doesn't transmit under deep fading. The probability of such an event for each MU is set at $\varepsilon = 0.01$.

From Fig. 1, we have the following observations.

- When K is finite, there is a cross point for each pair of curves representing the FC and NC scenarios. Below this point, FC performs better and *vice versa*.
- The corresponding rate of the cross point increases with K , which implies that the advantage of FC becomes more noticeable when K increases. Specifically, the cross points are at $R = 3.8, 7.6$ and 16 bits/symbol, respectively, for $K = 1, 2$ and 4 . (The cross point of $R = 16$ bits/symbol for $K = 4$ is outside the range shown in this figure.) We will discuss the situation for $K \rightarrow \infty$ (the related curves are obtained by (4)) in Section IV.

The above observations indicate that antenna correlation at MUs is potentially beneficial for multi-user MIMO systems.

IV. ASYMPTOTIC ANALYSIS

Due to the lack of closed-form solutions to the problem in (3), we resorted to numerical results in the last section. However, in some asymptotic situations, concise closed-form solutions can be obtained as discussed below, which may provide more insights into the problem.

A. Correlation Gain for Infinite M and K

The discussion in this subsection explains the rationale behind the advantage provided by antenna correlation at MUs observed in Fig. 1. It is based on the underlying principle discussed in [3]: the performance of the system in (1) is determined by the distribution of the maximum eigenvalue of $\mathbf{H}_k \mathbf{H}_k^*$ (i.e., $\lambda_{\max}(\mathbf{H}_k \mathbf{H}_k^*)$) when K is large. It can be shown that antenna correlation increases $\lambda_{\max}(\mathbf{H}_k \mathbf{H}_k^*)$ statistically. Consequently we can expect that antenna correlation is potentially beneficial to the system performance.

Specifically, since $\{\mathbf{H}_k\}$ are i.i.d. among different MUs, the cumulative distribution function (CDF) of $\lambda_{\max}(\mathbf{H}_k \mathbf{H}_k^*) = s_k \cdot \lambda_{\max}(\mathbf{W}_k \mathbf{T}_k \mathbf{W}_k^*)$, denoted by $F(\cdot)$, is the same for all MUs. The following closed-form solution to (3) in the limiting case of $K \rightarrow \infty$ is derived in [3].

$$\lim_{K \rightarrow \infty} \bar{P} = \int_{\varepsilon}^1 R \ln 2 \cdot 2^{R(t-\varepsilon)/M} / F^{-1}(t) dt \quad (4)$$

where $F^{-1}(\cdot)$ is the inverse of $F(\cdot)$. The following lemma is based on [2].

Lemma 1: For an arbitrary semi-positive matrix \mathbf{T}_k , the distribution of $\lambda_{\max}(\mathbf{W}_k \mathbf{T}_k \mathbf{W}_k^*)$ converges to a Gaussian one with mean $M \cdot \lambda_{\max}(\mathbf{T}_k)$ and variance $M \cdot (\lambda_{\max}(\mathbf{T}_k))^2$ as $M \rightarrow \infty$.

Consider the FC and NC scenarios in Fig. 1 and define

$$G = \bar{P}_{\text{NC}} / \bar{P}_{\text{FC}} \quad (5)$$

where \bar{P}_{FC} and \bar{P}_{NC} are obtained from \bar{P} in (3) by setting $\{\mathbf{T}_k\}$ to be i.i.d. rank-1 matrices (i.e., FC) and $\mathbf{T}_k = \mathbf{I}, \forall k$ (i.e., NC), respectively. We call G the correlation gain. It characterizes the impact of full antenna correlation at MUs.

Let $F_{\text{FC}}^{-1}(\cdot)$ and $F_{\text{NC}}^{-1}(\cdot)$ be $F^{-1}(\cdot)$ for FC and NC, respectively. From Lemma 1 and the fact that $\lambda_{\max}(\mathbf{T}_k) = N$ for FC (as \mathbf{T}_k is a rank-1 matrix with $\text{tr}(\mathbf{T}_k) = N$) and $\lambda_{\max}(\mathbf{T}_k) = 1$ for NC (as $\mathbf{T}_k = \mathbf{I}$), we have

$$F_{\text{FC}}^{-1}(t) = N \cdot F_{\text{NC}}^{-1}(t) \quad (6)$$

for any probability t when $M \rightarrow \infty$. Then based on (4), (5) and (6), the following theorem can be shown.

Theorem 1: For a MIMO system with $K \rightarrow \infty$ and $M \rightarrow \infty$, the correlation gain $G = N$.

This theorem indicates that FC outperforms NC in all rate range for infinite K and M .

B. Correlation Gain for a Finite M and Infinite K

For the case of a finite M and infinite K , we can derive the closed-form expression for the correlation gain by applying the exact CDF of the maximum eigenvalue of the complex Wishart matrix derived in [8] to $F^{-1}(\cdot)$ in (4). However,

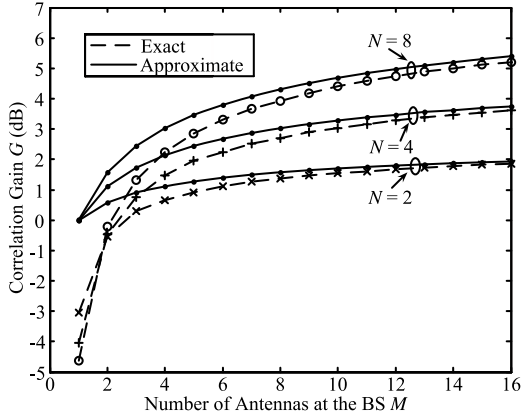


Fig. 2. The correlation gain for MIMO systems with different M and N . $\varepsilon = 0.01$ and $K \rightarrow \infty$. The numbers of antennas at each MU N are marked on the curves.

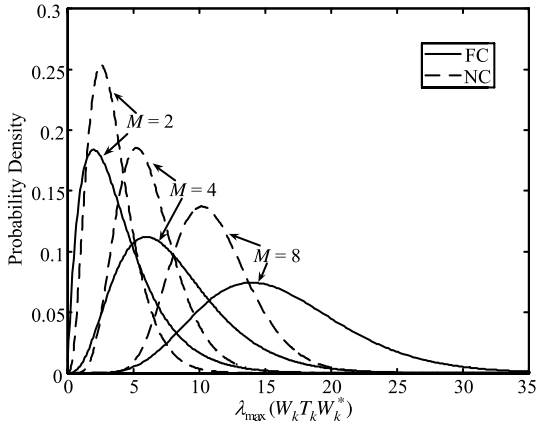


Fig. 3. PDFs of $\lambda_{\max}(\mathbf{W}_k \mathbf{T}_k \mathbf{W}_k^*)$ for FC and NC. $N = 2$ and the numbers of antennas at the BS M are marked on the curves.

the resultant expression is very complicated and not easily tractable. Instead, we consider an approximate method below.

From Lemma 1, $\lambda_{\max}(\mathbf{W}_k \mathbf{T}_k \mathbf{W}_k^*) / \mathbb{E}\{\lambda_{\max}(\mathbf{W}_k \mathbf{T}_k \mathbf{W}_k^*)\}$ converges to 1 with probability one as $M \rightarrow \infty$. This suggests to approximate $\lambda_{\max}(\mathbf{W}_k \mathbf{T}_k \mathbf{W}_k^*)$ by its mean. Since $\text{tr}(\mathbf{T}_k) = N$, we have

$$\sum_i \mathbb{E}\{\lambda_i(\mathbf{W}_k \mathbf{T}_k \mathbf{W}_k^*)\} = \mathbb{E}\{\text{tr}(\mathbf{W}_k \mathbf{T}_k \mathbf{W}_k^*)\} = MN \quad (7)$$

for an arbitrary semi-positive matrix \mathbf{T}_k . From (7), $\mathbb{E}\{\lambda_{\max}(\mathbf{W}_k \mathbf{T}_k \mathbf{W}_k^*)\} = MN$ for FC as the rank of $\mathbf{W}_k \mathbf{T}_k \mathbf{W}_k^*$ is one; while $\mathbb{E}\{\lambda_{\max}(\mathbf{W}_k \mathbf{T}_k \mathbf{W}_k^*)\} = \mathbb{E}\{\lambda_{\max}(\mathbf{W}_k \mathbf{W}_k^*)\}$ for NC as $\mathbf{T}_k = \mathbf{I}$. Then (6) can be rewritten as

$$F_{\text{FC}}^{-1}(t) / F_{\text{NC}}^{-1}(t) \approx MN / \mathbb{E}\{\lambda_{\max}(\mathbf{W}_k \mathbf{W}_k^*)\} \quad (8)$$

for any probability t . Based on (4) and (8), the following approximation of G in (5) at $K \rightarrow \infty$ can be shown.

$$\lim_{K \rightarrow \infty} G \approx MN / \mathbb{E}\{\lambda_{\max}(\mathbf{W}_k \mathbf{W}_k^*)\}. \quad (9)$$

Figure 2 demonstrates the accuracy of the approximation involved in (9) for different M and N . The parameters for channel fading are the same as those in Fig. 1. We set $R/M = 2$ bits/symbol for different M . The dashed curves are obtained using the exact solution to (5), while the solid ones using the approximation in (9). It is observed in Fig. 2

that when M is modestly large (e.g., from 8 to 16), there is good agreement between the exact and approximate results. For each fixed N , the two curves gradually approach N when $M \rightarrow \infty$, but their convergence speeds are low because $\mathbb{E}\{\lambda_{\max}(\mathbf{W}_k \mathbf{W}_k^*)\}$ converges to $M \cdot \lambda_{\max}(\mathbf{I}) = M$ slowly [2].

Note from Fig. 2 that the correlation gain (based on the exact calculation) is negative for $M = 1$ or 2, i.e., FC is inferior to NC in this case. Such an observation is explained as follows. FC increases the potential of beamforming gain and thus increases the mean of $\lambda_{\max}(\mathbf{W}_k \mathbf{T}_k \mathbf{W}_k^*)$, which is generally advantageous. However, FC also increases the variance of $\lambda_{\max}(\mathbf{W}_k \mathbf{T}_k \mathbf{W}_k^*)$, which is generally disadvantageous (due to the increased probability of very small $\lambda_{\max}(\mathbf{W}_k \mathbf{T}_k \mathbf{W}_k^*)$). These two opposite effects can be seen from the PDFs of $\lambda_{\max}(\mathbf{W}_k \mathbf{T}_k \mathbf{W}_k^*)$ for FC and NC plotted in Fig. 3. The results in Fig. 2 indicate that the advantage related to beamforming gain is dominant except for $M = 1$ or 2. Thus if the BS is equipped with more than two antennas, we should consider exploiting the gain provided by antenna correlation at MUs.

Based on (7), it can be shown that $MN / \mathbb{E}\{\lambda_{\max}(\mathbf{W}_k \mathbf{W}_k^*)\}$ in (9) is larger than one for $\forall M \neq 1$ and $\forall N \neq 1$, regardless of the system sum rate R . Then we conjecture that FC outperforms NC in all rate range even for a finite M (in the case of $M > 2$), although we cannot prove this rigorously so far.

V. CONCLUSIONS

This letter shows that antenna correlation at MUs is potentially beneficial for multi-user MIMO systems with rate constraints. The key is that the space diversity related to user locations compensates the loss of degree of freedom due to antenna correlation. This finding is useful in practice as minimizing the physical size of MUs is highly desirable, which may result in antenna correlation. Although the discussions of this letter are mainly related to full correlation, we have observed in our simulations that partial correlation can also provide a potential advantage. Quantifying this advantage analytically is an interesting future research topic.

REFERENCES

- [1] C.-N. Chuah, D. Tse, J. Kahn, and R. Valenzuela, "Capacity scaling in MIMO wireless systems under correlated fading," *IEEE Trans. Inf. Theory*, vol. 48, pp. 637-650, Mar. 2002.
- [2] C. Martin and B. Ottersten, "Asymptotic eigenvalue distributions and capacity for MIMO channels under correlated fading," *IEEE Trans. Wireless Commun.*, vol. 3, no. 4, pp. 1350-1358, July 2004.
- [3] P. Wang and L. Ping, "On multi-user gain in MIMO systems with rate constraints," in *Proc. IEEE GlobeCom*, Washington DC, USA, Nov. 2007.
- [4] R. Louie, M. McKay, and I. Collings, "Impact of correlation on the capacity of multiple access and broadcast channels with MIMO-MRC," *IEEE Trans. Wireless Commun.*, vol. 7, no. 6, pp. 2397-2407, June 2008.
- [5] P. Viswanath, D. Tse, and R. Laroia, "Opportunistic beamforming using dumb antennas," *IEEE Trans. Inf. Theory*, vol. 48, no. 6, pp. 1277-1294, June 2002.
- [6] S. Vishwanath, N. Jindal, and A. Goldsmith, "Duality, achievable rates, and sum-rate capacity of Gaussian MIMO broadcast channels," *IEEE Trans. Inf. Theory*, vol. 49, pp. 2658-2668, Oct. 2003.
- [7] J. Lee and N. Jindal, "Symmetric capacity of MIMO downlink channels," in *Proc. IEEE Int. Symp. Inform. Theory (ISIT)*, Seattle, July 2006.
- [8] M. McKay, "Random matrix theory analysis of multiple antenna communication systems," Ph.D. dissertation, University of Sydney, 2006.