On the Limiting Performance of Turbo-Hadamard Codes

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***Abstract*—In this letter, we employ the extrinsic information transfer (EXIT) chart technique to assess the limiting performance of turbo-Hadamard codes and identify the optimized design parameters for such codes. It is shown that for a sufficiently long code length and a sufficiently large number of iterations, a carefully designed low rate turbo-Hadamard code can potentially achieve successful decoding at $E_b/N_0 \approx -1.3$ dB, which is about 0.29 dB from the ultimate Shannon limit.**

***Index Terms*—Extrinsic information transfer (EXIT) charts, low-rate Shannon limit, turbo-Hadamard codes.**

I. Introduction

SUBSTANTIAL effort has been devoted to devising codes capable of attaining the theoretical capacity of communication systems. For modest rates over an additive white Gaussian noise (AWGN) channel, near capacity performance has been reported using turbo codes [1] and low-density parity-check (LDPC) codes [2]. For very low-rate cases, the ultimate AWGN capacity is the so-called Shannon limit of $E_b/N_0 \approx -1.59$ dB. Various low-rate codes based on turbo code and repeat accumulate code structures have been devised to approach this limit [3]–[6]. Reference [7] presents simulation results demonstrating a bit error rate (BER) 10^{-5} at $E_b/N_0 \approx -1.2$ dB for the so-called turbo-Hadamard codes, about 0.39 dB away from the Shannon limit.

It has been shown recently that low-rate codes can be used to replace the traditional coding-spreading structure of the direct-sequence spread spectrum systems, which leads to significant performance improvements [8]. This can be a useful application of the codes discussed in this paper.

Theoretically, it is interesting to verify just how close we can approach to the Shannon limit. This requires performance assessment of infinitely long code lengths. The related simulation work is computationally demanding. The issue is particularly challenging for low rate codes, since for a fixed length of information bits, the length of a codeword is inversely proportional to rate. The extrinsic information transfer (EXIT) chart technique [9] is a useful approach in this context. This technique can predict the convergence behavior of an iterative decoder for an infinitely long concatenated code. Only a partial simulation of the component codes with modestly long lengths is involved. (The lengths used should be sufficient to remove the termination effect.) Thus the cost issue is greatly eased.

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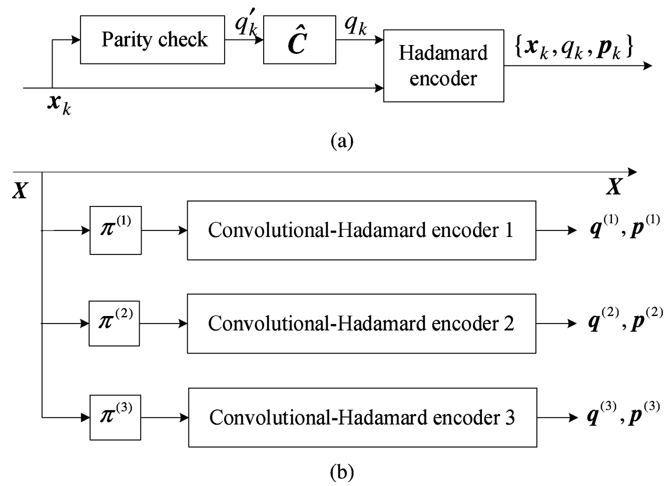

Fig. 1. (a) A convolutional-Hadamard encoder. (b) A turbo-Hadamard encoder with three component convolutional-Hadamard encoders.

In this letter, we employ the EXIT chart technique to assess the limiting performance of turbo-Hadamard codes and identify the optimized design parameters for such codes. It is shown that with infinitely long code length, a low rate turbo-Hadamard code can potentially achieve successful decoding at $E_b/N_0 \approx -1.3$ dB. For verification, we observe in simulations a BER of 10^{-5} at $E_b/N_0 \approx -1.25$ dB using a code length of 65 536. This is quite close to the prediction of the EXIT analysis.

II. Turbo-Hadamard Codes

Turbo-Hadamard codes [7] are constructed by concatenating several convolutional-Hadamard codes in parallel. A convolutional-Hadamard encoder is shown in Fig. 1(a). The information bit stream $\boldsymbol{X}$ is segmented into blocks: $\boldsymbol{X} = \{\boldsymbol{x}_k\}$. Each $\boldsymbol{x}_k$ contains r information bits. A parity check q'_k is generated from each block $\boldsymbol{x}_k$. The bit stream $\{q'_k\}$ is used to drive a rate-1, memory-length-s recursive systematic convolutional encoder $\hat{C}$, producing (q_k). Then each $(r+1)$-tuple $(\boldsymbol{x}_k, q_k)$ is used to encode a length-2^r biorthogonal Hadamard code to create codewords $\{\boldsymbol{x}_k, q_k, \boldsymbol{p}_k\}$. An example of a turbo-Hadamard encoder composed of three convolutional-Hadamard codes concatenated in parallel is given in Fig. 1(b), where $(\boldsymbol{q}^{(m)}, \boldsymbol{p}^{(m)}, m = 1, 2, 3)$ are the redundancy outputs in the mth component code and $\{\pi^{(m)}, m = 1, 2, 3\}$ are interleavers. The overall codeword is $\{\boldsymbol{X}, \boldsymbol{q}^{(1)}, \boldsymbol{p}^{(1)}, \ldots, \boldsymbol{q}^{(3)}, \boldsymbol{p}^{(3)}\}$.

The basic principle of the global decoder is outlined in Fig. 2. Denote by $\boldsymbol{Z}^{(m)}$ the channel observations of the mth component

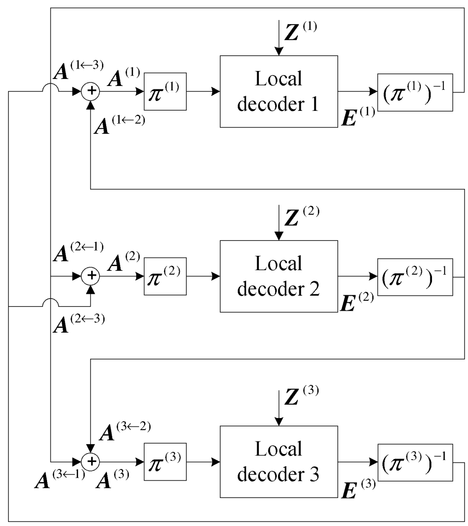

Fig. 2. Decoder of a parallel concatenated code with three constituent codes.

code. Denote by $\boldsymbol{A}^{(m)}$ the input *a priori* log-likelihood ratios (LLRs) to the mth component code for the information bits $\boldsymbol{X}$. Initially, we set $\boldsymbol{A}^{(m)}$, $\forall\, m$, to zero. At each iteration, the local decoder-m performs *a posteriori* probability (APP) decoding based on $\boldsymbol{Z}^{(m)}$ and $\boldsymbol{A}^{(m)}$. Its output is a vector of the so-called extrinsic LLR values for $\boldsymbol{X}$, denoted by $\boldsymbol{E}^{(m)}$. $\boldsymbol{A}^{(m)}$ is then updated as $\boldsymbol{A}^{(m)} = \sum_{m' \neq m} \boldsymbol{A}^{(m \leftarrow m')}$, where $\boldsymbol{A}^{(m \leftarrow m')}$ is obtained from $\boldsymbol{E}^{(m')}$ after interleaving. Full details of the turbo decoding principle can be found in [10].

III. EXIT Chart Analysis

Consider the EXIT chart introduced in [9]. In general, an M-dimensional plot is required to characterize a concatenated code with M component codes [11]. However, when all of the component codes are identical, the issue is greatly simplified, as discussed below.

We denote by $I(\boldsymbol{X}; \boldsymbol{A}^{(m \leftarrow m')})$ the mutual information between $\boldsymbol{X}$ and $\boldsymbol{A}^{(m \leftarrow m')}$ and by $I(\boldsymbol{X}; \boldsymbol{E}^{(m)})$ the mutual information between $\boldsymbol{X}$ and $\boldsymbol{E}^{(m)}$. We assume that the codes are symmetrical, i.e., all of the local decoders in Fig. 2 are identical and all $\boldsymbol{Z}^{(m)}$, $m = 1, 2, \ldots, M$, are statistically equivalent. Then during a given iteration, $I(\boldsymbol{X}; \boldsymbol{A}^{(m \leftarrow m')})$, $\forall\, m, m'$, are equal and also $I(\boldsymbol{X}; \boldsymbol{E}^{(m)})$, $\forall\, m$, are equal. Thus we can denote

$$I(\boldsymbol{X}; \boldsymbol{A}^{(m \leftarrow m')}) = I_A \text{ and } (\boldsymbol{X}; \boldsymbol{E}^{(m)}) = I_E. \qquad (1)$$

We plot I_A versus I_E in a two-dimensional plane as indicated in Fig. 3. If the first intersection of the curve with the diagonal (0,0) to (1,1) is at point (1, 1), as in Fig. 3(a), it implies that at this point the mutual information $I_E = I_A = 1$ and the iterative process converges. Otherwise if the first intersection $I_E = I_A < 1$, as in Fig. 3(b), there will be no further improvement of the decoding outputs when the iterative process continues (as $\boldsymbol{A}^{(m \leftarrow m')}$ is formed by $\boldsymbol{E}^{(m')}$ with appropriate interleaving in the previous iteration). We can thus conclude that convergence to a very low BER is possible only if the first intersection of the I_E vs. I_A curve with the diagonal is at $(1, 1)$. As indicated in [11, p. 272], the above technique is equivalent to the projection of a multi-dimensional surface onto a two-dimensional curve based on the symmetric condition in (1).

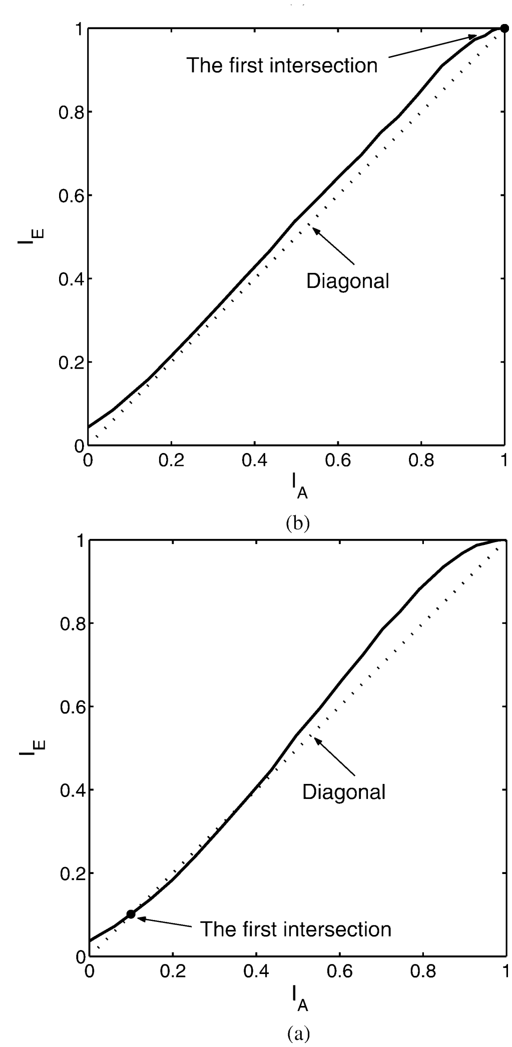

Fig. 3. Illustration of convergence condition. (a) Convergent. (b) Un-convergent.

IV. Limiting Performance of Turbo-Hadamard Codes

Following Section II and [7], a turbo-Hadamard code is defined by a number of parameters, namely, M the number of component codes, r the order of the Hadamard code (its length $= 2^r$), s the memory length of the convolutional code, and (G_d, G_n) the generator polynomials given in octal, where G_d represents the recursive denominator and G_n the numerator. Based on the EXIT technique, we conducted an exhaustive search of all combinations of the following parameter values in order to evaluate the impact of these parameters on the limiting performance of turbo-Hadamard codes:

- $M = 2, 3, 4,$ and 5
- $r = 6, 7, 8$ and 9
- $s = 1, 2, 3$ and 4 [using some well-known generator polynomials (G_d, G_n)].

We observed that increasing the number or complexity of the component codes did not improve the limiting performance. A typical example can be seen in Fig. 3, where the EXIT curves

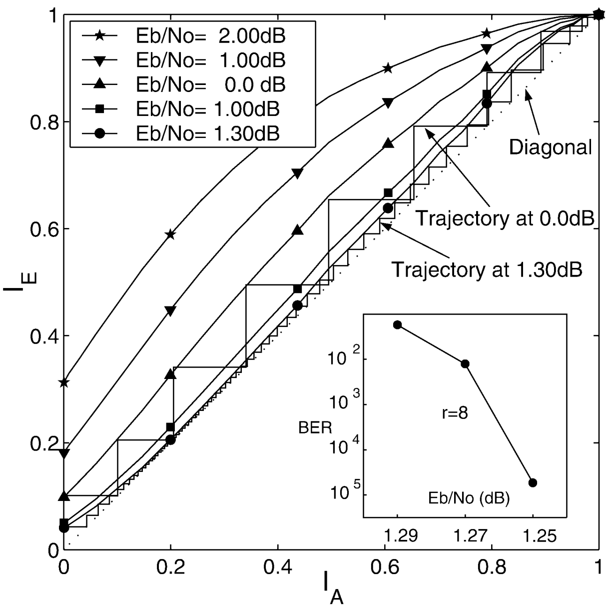

Fig. 4. EXIT charts for $M = 3$, $r = 8$ and code $(G_d, G_n) = (03, 01)_8$ at different E_b/N_0 values. Two decoding trajectories at -1.30 dB and 0.0 dB are also shown. The inset plot shows the simulated BER versus E_b/N_0 performance.

denoted by Fig. 3(a) and (b) are for codes with the same $M(= 3)$, $r(= 8)$ and $E_b/N_0 = -1.30$ dB but with different s (= 1 and 2, respectively). We can see that the curve in Fig. 3(b) with $s = 2$ intersects with the diagonal at $I_A < 1$. This violates the convergence condition discussed above. Similar behaviors are also observed when r and M are increased based on both EXIT chart analysis and simulation. Similar observations have been reported for other turbo-type codes [12].

We found that the parameter combination that provides the best performance is ($M = 3$, $r = 8$ and $s = 1$) with $(G_d, G_n) = (03, 01)_8$ and an overall rate of 1/94 (Note: The corresponding convolutional code is simply a binary accumulator with transfer function $1/(D+1)$). The EXIT curves for this choice of parameter values and different E_b/N_0 values are shown in Fig. 4. Each intersection of these curves with the diagonal line is at (1,1), indicating the possibility of convergence. The curve with $E_b/N_0 = -1.30$ dB nearly touches the diagonal line, indicating that any further reduction of SNR may potentially lead to a decoding failure.

The iterative decoding trajectories (interleaving length $= 65\,536$) at -1.30 dB and 0.0 dB are also shown in Fig. 4. These are simulation results with 62 and 10 decoding iterations, respectively, each vertical-horizontal step indicating one complete decoding iteration. The trajectories deviate very little from their corresponding EXIT curves, indicating the accurate convergence prediction capability of the EXIT technique. However, the trajectory with the zigzag path at $E_b/N_0 = -1.30$ dB shows that convergence may be very slow for low BER.

This analysis indicates that it is possible to achieve successful decoding at $E_b/N_0 = -1.30$ dB provided the block size is sufficiently long, but the simulation is very time consuming. We have carried out simulations to verify this prediction based on a block size of 65 536 using 100 iterations. The result is shown in the inset plot in Fig. 4. The BER of 10^{-5} is observed at $E_b/N_0 \approx -1.25$ dB.

V. Conclusions

We have assessed the limiting performance of turbo-Hadamard codes using the EXIT chart technique. Our study indicates that, among the parameters evaluated, the best achievable performance of the turbo-Hadamard code is at $E_b/N_0 = -1.30$ dB, about 0.29 dB from the theoretical low-rate Shannon limit. However, this appears to exclude the possibility to push further toward the Shannon limit based on the current structure of turbo-Hadamard codes. Thus we may require other solutions, perhaps involving the introduction of irregularity into the code design following the design philosophy of irregular LDPC codes [2].

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