

Low-Rate Repeat-Zigzag-Hadamard Codes

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Abstract—In this paper, we propose a new class of low-rate error correction codes called repeat-zigzag-Hadamard (RZH) codes featuring simple encoder and decoder structures, and flexible coding rate. RZH codes are serially concatenated turbo-like codes where the outer code is a repetition code and the inner code is a punctured zigzag-Hadamard (ZH) code. By analyzing the code structure of RZH codes, we prove that both systematic and nonsystematic RZH codes are good codes, in the sense that for an RZH code ensemble, there exists a positive number γ_0 such that for any binary-input memoryless channel whose Bhattacharyya noise parameter is less than γ_0 , the average block error probability of maximum-likelihood (ML) decoding approaches zero. Two decoding algorithms—serial and parallel decoders for RZH codes—are proposed. We then employ the extrinsic information transfer (EXIT) chart technique to design irregular RZH codes. Results show that the optimized irregular RZH codes exhibit a performance that is very close to capacity in the low-rate regime.

Index Terms—Irregular code design, low-rate codes, serial concatenation, zigzag-Hadamard codes.

I. INTRODUCTION

THE ultimate Shannon capacity of an additive white Gaussian noise (AWGN) channel in terms of signal-to-noise ratio (SNR) per information bit is about -1.6 dB for codes with rates approaching zero [1]. Hadamard codes [2], [3] and superorthogonal convolutional codes [4] are traditional low-rate channel coding schemes that offer low coding gains and hence their performance is far away from the Shannon limit. Recently, along with the discovery of turbo codes [5], low-rate concatenated coding schemes have been proposed in [6], [7] which offer higher performance whereas they also incur higher complexity due to the complex trellis structure that specifies the codes.

The repeat-accumulate (RA) codes proposed in [8] have been proved to achieve the Shannon limit in AWGN channels when the coding rate approaches zero. Another class of advanced codes, the low-density parity-check (LDPC) codes [9], also have capacity-approaching capability for various code rates when the ensemble profiles are optimized. However, in the low-rate region, both RA codes and LDPC codes suffer from performance loss and extremely slow convergence speed with

iterative decoding. Constructed from Hadamard code arrays, the recently proposed low-rate turbo-Hadamard codes [10] offer the bit-error-rate (BER) of 10^{-5} at $E_b/N_0 = -1.2$ dB, which is only around 0.4 dB away from the Shannon limit. More recently, built on zigzag codes [11], parallel-concatenated zigzag-Hadamard (PCZH) codes are proposed in [12] which offer a similar performance as that of the turbo-Hadamard codes, with much simpler encoder and decoder structures.

In this paper, we consider a new class of low-rate codes called repeat-zigzag-Hadamard (RZH) codes which are essentially serially concatenated turbo-like codes where the outer code is a repetition code and the inner code is a punctured ZH code. By analyzing the code structure of RZH codes, we prove that RZH codes are good in the sense that for an RZH code ensemble, there exists a positive number γ_0 such that for any binary-input memoryless channel whose Bhattacharyya noise parameter [13] is less than γ_0 , the block error probability of the maximum-likelihood (ML) decoder approaches zero. Two low-complexity decoding algorithms—serial and parallel decoders for RZH codes—are then proposed. Moreover, irregular RZH (IRZH) codes are also proposed and the extrinsic information transfer (EXIT) chart technique [14], [15] are employed to optimize the code profiles. With the optimized ensemble profiles, the IRZH codes offer near-capacity performance in the low-rate regime. Similar to parallel concatenated ZH codes, RZH codes enjoy simpler encoder and decoder structures when compared to turbo-Hadamard codes. Another advantage of the proposed RZH code over the turbo-Hadamard codes and parallel concatenated ZH codes is that it exhibits a more flexible code structure, with a number of parameters (e.g., degrees of outer codes) that can be tuned according to a particular design criterion (e.g., the coding rate).

The remainder of this paper is organized as follows. In Section II, we introduce the RZH codes and their unpunctured version. In Section III, we prove that RZH codes are good codes. Two low-complexity decoding algorithms are presented in Section IV. Section V presents the code design for IRZH codes where the EXIT chart method is used to optimize the ensemble profile. Numerical results and discussions are also provided in Section V. Section VI contains the conclusions.

II. REPEAT-ZIGZAG-HADAMARD CODES

A. Hadamard Codes and Zigzag-Hadamard Codes

A Hadamard codeword is obtained from a Hadamard matrix. Starting from $\mathbf{H}_1 = [+1]$, an $n \times n$ ($n = 2^r$, where r is called the order of the Hadamard code) Hadamard matrix \mathbf{H}_n over $\{+1, -1\}$ can be constructed recursively as

$$\mathbf{H}_n = \begin{bmatrix} +\mathbf{H}_{\frac{n}{2}} & +\mathbf{H}_{\frac{n}{2}} \\ +\mathbf{H}_{\frac{n}{2}} & -\mathbf{H}_{\frac{n}{2}} \end{bmatrix}. \quad (1)$$

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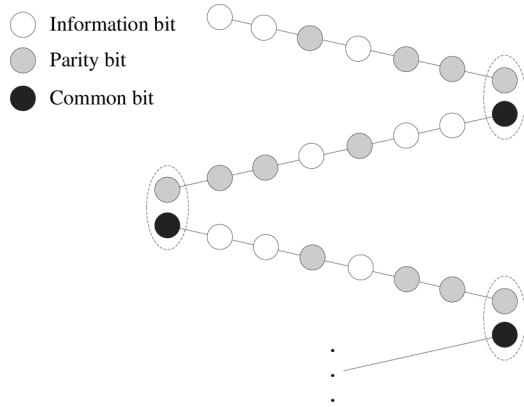


Fig. 1. Graphical representation of an unpunctured systematic ZH code with $r = 3$. The first bit of the first segment is omitted.

A length- 2^r Hadamard codeword set is formed by the columns of the bi-orthogonal Hadamard matrix $\pm \mathbf{H}_n$, denoted by $\{\pm \mathbf{h}^j : j = 0, 1, \dots, 2^r - 1\}$, in a binary $\{0, 1\}$ form, i.e., $+1 \rightarrow 0$ and $-1 \rightarrow 1$. Each Hadamard codeword carries $(r+1)$ bits of information. In systematic encoding, the bit indexes $\{0, 1, 2, 4, \dots, 2^{r-1}\}$ are used as information bit positions of the length- 2^r Hadamard code. The other $(2^r - r - 1)$ bits are parity bits. Denote \mathcal{H} as a Hadamard encoder. Then the Hadamard encoding can be represented by

$$\mathbf{c} = \mathcal{H}(d_0, d_1, \dots, d_r) \quad (2)$$

where \mathbf{c} is the resulting binary codeword. Clearly, \mathbf{c} is a column of either $+\mathbf{H}_n$ or $-\mathbf{H}_n$ after the mapping of $+1 \rightarrow 0$ and $-1 \rightarrow 1$. Hereafter, for simplicity, we will drop the subscript n in \mathbf{H}_n .

As shown in Fig. 1, a zigzag-Hadamard code [12] is described by a simple zigzag graph with each segment being a length- 2^r Hadamard code. The data stream \mathbf{d} over $\{0, 1\}$ is first segmented into blocks $\{\mathbf{d}_j\}$, where $\mathbf{d}_j = [d_{j,1}, d_{j,2}, \dots, d_{j,r}]$, $j = 1, 2, \dots, J$. Then as shown in Fig. 1, for a systematic ZH encoder, with the last parity bit of the previous coded Hadamard segment being the first input bit to the Hadamard encoder for the current segment, the coded bits of the j th segment are obtained by the Hadamard encoder with

$$\mathbf{c}_j = \mathcal{H}(c_{j-1, 2^r-1}, \mathbf{d}_j) \quad (3)$$

where the codeword $\mathbf{c}_j = [c_{j,0}, c_{j,1}, \dots, c_{j,2^r-1}]^T$, with $c_{j,0} = c_{j-1, 2^r-1}$ being the *common bit* that connects the current segment to the previous segment and forms the zigzag structure, $c_{j, 2^{i-1}} = d_{j,i}$, $i = 1, 2, \dots, r$, being the systematic bits, and all other $(2^r - r - 1)$ bits being the parity bits. Denote $q_j = c_{j,0}$ and let \mathbf{p}_j represent the parity bits, i.e., $\mathbf{p}_j = \{c_{j,i}, i \neq 0, i \neq 2^{m-1}, m = 1, \dots, r\}$. The j th coded bits segment is then denoted by a triplet, i.e., $\mathbf{c}_j = \{\mathbf{d}_j, q_j, \mathbf{p}_j\}$, $j = 1, \dots, J$, and the overall ZH code sequence is denoted by $\mathbf{c} = \{\mathbf{d}, \mathbf{q}, \mathbf{p}\}$. Obviously, the above ZH code is a *systematic* code. Note that the common bit of the first segment can be freely assigned and is usually assumed to be 1.

ZH codes are convolutional-like codes, and we have the following results.

Definition 2.1: A binary convolutional-like code is said to be weight-recursive if any information sequence with Hamming distance one will produce a codeword with Hamming weight $\rightarrow \infty$ when its length $\rightarrow \infty$.

Theorem 2.1: A systematic ZH code with even Hadamard order r is weight-recursive, and a systematic ZH code with odd Hadamard order r is not weight-recursive.

The proof of the above theorem is given in [12]. In what follows, we only consider ZH codes with even Hadamard order.

B. RZH Codes

In this paper, we propose an alternative code structure to parallel concatenated zigzag-Hadamard codes and turbo-Hadamard codes, namely, the RZH codes. The structure of a systematic RZH code is shown in Fig. 2. An information bit block \mathbf{u} of length k is encoded by an outer repetition encoder E_1 of rate $1/d_v$ into a codeword \mathbf{b} of length $N = kd_v$, which is permuted by an interleaver Π of length N to form \mathbf{d} , and then encoded by an inner recursive systematic ZH encoder E_2^r . Since the outer code is a repetition code and the inner ZH code is a systematic code, we have d_v copies of the information bits in the resulting ZH codeword. To achieve a higher rate, we only transmit one copy of the information bits and puncture the other $d_v - 1$ copies. Also notice that the common bit of a ZH segment is a repetition of the last parity bit of the previous Hadamard segment and hence can also be punctured. After puncturing, the inner code is a punctured ZH code with which we obtain the systematic RZH code as shown in Fig. 2, with the parity bits \mathbf{p} being the output of the punctured ZH code. Let n be the length of the ultimate codeword $\mathbf{c} = \{\mathbf{u}, \mathbf{p}\}$, then the overall code is an (n, k) linear block code and the overall coding rate is given by

$$R_c = \frac{r}{d_v(2^r - r - 1) + r} \quad (4)$$

where r is the order of the Hadamard code for the inner encoder. If the information bits \mathbf{u} are also punctured, we obtain the non-systematic RZH codes, and the coding rate is given by

$$R_c = \frac{r}{d_v(2^r - r - 1)}. \quad (5)$$

In case the inner code has no puncturing, we have the unpunctured RZH codes whose structure is shown in Fig. 3. Although in general the unpunctured code cannot constitute a good structure since it contains an embedded repetition code, it is introduced here for the purpose of analysis. In fact, the performance of an unpunctured RZH code with a large Hadamard order can be very good when the coding rate is low, as will be seen in Section V. For the unpunctured code, the overall code rate is given by $R_c = R_1 R_2$, where $R_1 = 1/d_v$ is the rate of the outer repetition code and R_2 is the rate of the inner ZH code. Since normally the first common bit of the ZH code is omitted, the overall rate of the unpunctured RZH code is given by

$$R_c = \frac{k}{\left(\frac{k d_v}{r}\right) 2^r - 1} = \frac{r}{d_v 2^r - r/k}. \quad (6)$$

With $k \rightarrow \infty$, we have $R_c \rightarrow \frac{r}{d_v 2^r}$.

III. RZH CODES ARE GOOD CODES

In this section, we prove that the proposed RZH codes are "good" in the following sense. For an RZH code ensemble, there

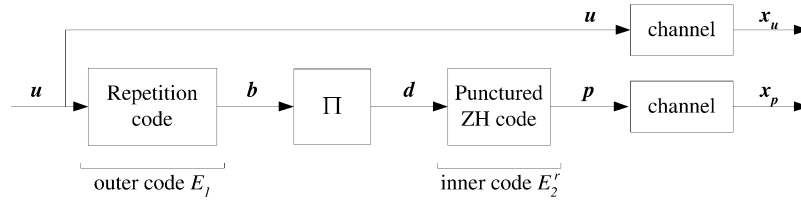


Fig. 2. Systematical RZH encoder.

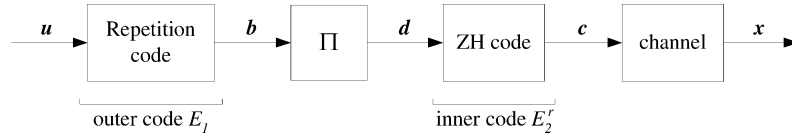


Fig. 3. Unpunctured RZH encoder.

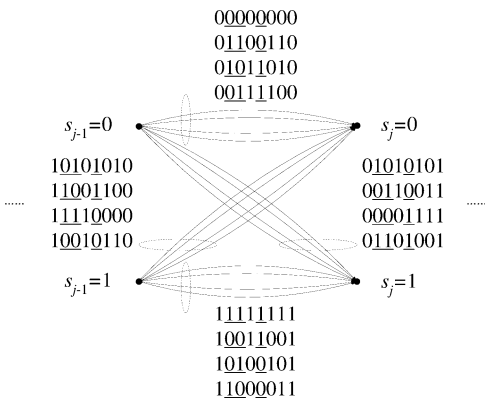


Fig. 4. An example of the j th section of the trellis based on a ZH code with $r = 3$.

exists a positive number γ_0 such that for any binary-input memoryless channel whose Bhattacharyya noise parameter [13] is less than γ_0 , the block error probability of the ML decoder approaches zero, at least as fast as $n^{-\beta}$, where n is the length of the RZH codeword and β is the “interleaver gain” exponent defined in [16].

A. Bound of Weight Enumerators of ZH Codes

In what follows, we first adapt the analysis on the weight structure of convolutional codes in [17] to obtain the weight structure of unpunctured ZH codes, which will then be used to prove the goodness of RZH codes.

Since ZH codes are convolutional-like codes, they can be represented by trellis. Fig. 4 depicts an example of the j th section of the two-state ZH trellis with $r = 3$. The codewords associated with the edges are also depicted in the figure with the underlined bits being the systematic bits. Since there are multiple parallel edges between s_{j-1} and s_j , the definition of a *detour* in [17] and consequently the results of the weight structure for recursive convolutional codes used in [13] cannot be applied to ZH codes. Let $w(s_{j-1} \xrightarrow{\ell_j} s_j)$ denote the output weight of the Hadamard codeword associated with the ℓ_j th edge from s_{j-1} to s_j .

Definition 3.2: The Detours of a systematic ZH code is defined in terms of the sequence of consecutive edges with

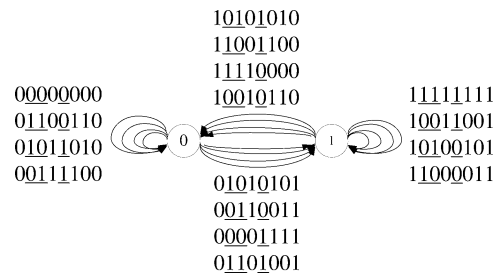


Fig. 5. The two-state diagram for a ZH code with $r = 3$.

nonzero weights, starting and ending in the zero state, and taking on nonzero state values in between. Mathematically, given a length- n ZH codeword, the total length of its trellis is given by $L_{\text{Trellis}} = n/2^r$. The starting point b_i and the ending point e_i of the i -th detour associated with this codeword can be defined as follows. Let $e_0 = 0$, then b_i and e_i are given by

$$b_i = \min\{j \in \{e_{i-1} + 1, \dots, L_{\text{Trellis}}\} : w(s_{j-1} \xrightarrow{\ell_j} s_j) \neq 0\} - 1,$$

$$e_i = \min\{j \in \{b_i + 1, \dots, L_{\text{Trellis}}\} : s_j = 0\}.$$

The length of the i -th detour is given by $(e_i - b_i)$, and the number of detours is denoted by t .

Note that due to the existence of multiple transitions from zero state to zero state, the above definition of detour is different from that of convolutional codes. It is clear that the minimum length of a detour is 1 which corresponds to an edge with nonzero weight from zero state to zero state. Also, since the weight of any edge of a detour is either 2^{r-1} or 2^r , the total weight h of any nonzero codeword is always divisible by 2^{r-1} .

Since the encoding of a ZH code is a Markov process, it can also be represented by a two-state diagram as shown in Fig. 5. To facilitate the analysis, we have the following definition.

Definition 3.3: A path through the systematic ZH code state diagram is called admissible if it does not contain an initial segment corresponding to cycles at the zero state with zero weight.

Lemma 3.1: For the recursive systematic ZH code with degree r , any admissible path of length no less than 1 has weight no less than 2^{r-1} .

Proof: Due to the structure of ZH codes, there are only three possible weights: 0, 2^{r-1} , and 2^r associated with each segment, and the only zero-weight codeword is associated with the cycle at the zero state. According to the definition of admissible path, the weight of the first segment is not zero. Hence the weight of any admissible path of length at least 1 has weight no less than 2^{r-1} . \square

Lemma 3.2: If a nonzero ZH codeword consists of several detours and has weight h , then the total length of all its detours is bounded by $h/2^{r-1}$ and the input weight w is bounded by $rh/2^{r-1}$.

Proof: From the definitions of admissible path and detour, it is clear that each segment of a detour is a length-1 admissible path. According to Lemma 3.1, any admissible path of length-1 has weight no less than 2^{r-1} , hence for a ZH codeword with weight h , the total number of segments of all its detours is no more than $h/2^{r-1}$. Since the input weight of any segment in a detour is no more than r , the input weight w is no more than $rh/2^{r-1}$. \square

Given an (n, k) systematic ZH code C , let $C_{w, \leq h}$ denote the set of codewords of C with weight at most h and input weight of w , and the cumulative input-output weight enumerator (CIOWE) $A_{w, \leq h}(C) = |C_{w, \leq h}|$ be the number of elements in the set $C_{w, \leq h}$.

Theorem 3.2: For an (n, k) recursive systematic ZH code C with a given r , the CIOWE is bounded by

$$A_{w, \leq h}(C) \leq \theta^w \binom{rh/2^{r-1}}{w} \sum_{t=1}^{\lfloor w/2 \rfloor} \binom{k/r}{t} \quad (7)$$

for $w \geq 2$, where $\theta > 1$ is a constant independent of k and r .

Proof: Here we split $C_{w, \leq h}$ into $R_{w, \leq h}$ and $T_{w, \leq h}$ where R denotes codewords with only regular detours whereas T denotes codewords with a truncated detour at the end. Note that each nonzero codeword consists of at least one detour and that an element of $R_{w, \leq h}$ consists of at most $\lfloor w/2 \rfloor$ detours since each regular detour requires an input weight of at least 2 whereas an element of $T_{w, \leq h}$ can have as many as $\lceil w/2 \rceil$ detours.

Let $C_{(w_1, \dots, w_t), \leq h}$ be the set of codewords with weight at most h consisting of t detours, with the i th detour having input weight w_i . Clearly

$$|R_{w, \leq h}| = \sum_{t=1}^{\lfloor w/2 \rfloor} \sum_{(w_1, \dots, w_t): \sum w_i = w; w_i \geq 2, 1 \leq i \leq t} |R_{(w_1, \dots, w_t), \leq h}| \quad (8)$$

$$|T_{w, \leq h}| = \sum_{t=1}^{\lceil w/2 \rceil} \sum_{(w_1, \dots, w_t): \sum w_i = w; w_i \geq 2, 1 \leq i < t; w_t \geq 1} |T_{(w_1, \dots, w_t), \leq h}|. \quad (9)$$

In order to bound $|R_{(w_1, \dots, w_t), \leq h}|$ and $|T_{(w_1, \dots, w_t), \leq h}|$, let $C_{((w_1, b_1), \dots, (w_t, b_t)), \leq h}$ be the subset of $C_{(w_1, \dots, w_t), \leq h}$ where the i th detour starts at position b_i . We have the codeword $\mathbf{c} = \{\mathbf{d}, \mathbf{q}, \mathbf{p}\} \in C_{((w_1, b_1), \dots, (w_t, b_t)), \leq h}$ where \mathbf{d} denotes the information bits, \mathbf{q} denotes the connection bits and \mathbf{p} denotes

all the parity bits. We now exhibit an injective map from $C_{((w_1, b_1), \dots, (w_t, b_t)), \leq h}$ into the set of binary $(rh/2^{r-1})$ -tuples of weight w . The map is composed of two simpler maps $\{\mathbf{d}, \mathbf{q}, \mathbf{p}\} \rightarrow \mathbf{d} \rightarrow \mathbf{d}'$. Clearly the first map $\{\mathbf{d}, \mathbf{q}, \mathbf{p}\} \rightarrow \mathbf{d}$ is injective. The second map $\mathbf{d} \rightarrow \mathbf{d}'$ simply involves deleting all components of \mathbf{d} which are not part of a detour. Note that since (w_i, b_i) , $1 \leq i \leq t$ is known it is possible to reconstruct \mathbf{d} from \mathbf{d}' . This shows that $\mathbf{d} \rightarrow \mathbf{d}'$ and therefore $\{\mathbf{d}, \mathbf{q}, \mathbf{p}\} \rightarrow \mathbf{d} \rightarrow \mathbf{d}'$ is injective. It is clear that the weight of \mathbf{d}' is still w . As shown in Lemma 3.2, the total number of edges of all t detours is bounded above by $h/2^{r-1}$. Also note that every edge has r binary input bits, therefore the length of \mathbf{d}' is bounded by $rh/2^{r-1}$. Hence we have $|R_{((w_1, b_1), \dots, (w_t, b_t)), \leq h}|, |T_{((w_1, b_1), \dots, (w_t, b_t)), \leq h}| \leq |C_{((w_1, b_1), \dots, (w_t, b_t)), \leq h}| \leq \binom{rh/2^{r-1}}{w}$. Since there are at most $\binom{k/r}{t}$ starting values for t regular detours, it follows that $|R_{(w_1, \dots, w_t), \leq h}| \leq \binom{rh/2^{r-1}}{w} \binom{k/r}{t}$. Moreover, since there are at most $\binom{k/r}{t-1} h/2^{r-1}$ starting values for $(t-1)$ regular and one truncated terminating detour, it follows that $|T_{(w_1, \dots, w_t), \leq h}| \leq \binom{rh/2^{r-1}}{w} \binom{k/r}{t-1} h/2^{r-1}$.

We can then bound $|R_{w, \leq h}|$ by

$$\begin{aligned} |R_{w, \leq h}| &= \sum_{t=1}^{\lfloor w/2 \rfloor} \sum_{(w_1, \dots, w_t): \sum w_i = w; w_i \geq 2, 1 \leq i \leq t} |R_{(w_1, \dots, w_t), \leq h}| \\ &\leq \sum_{t=1}^{\lfloor w/2 \rfloor} \sum_{(w_1, \dots, w_t): \sum w_i = w; w_i \geq 2, 1 \leq i \leq t} \binom{rh/2^{r-1}}{w} \binom{k/r}{t} \\ &\stackrel{(a)}{\leq} \sum_{t=1}^{\lfloor w/2 \rfloor} \binom{w-t-1}{t-1} \binom{rh/2^{r-1}}{w} \binom{k/r}{t} \\ &\stackrel{(b)}{\leq} \theta^w \binom{rh/2^{r-1}}{w} \sum_{t=1}^{\lfloor w/2 \rfloor} \binom{k/r}{t}. \end{aligned}$$

Note that (a) follows from the fact that there are exactly $\binom{w-1}{t-1}$ ordered partitions of a positive integer w into t positive components, and if all the t components are required to be no less than 2, it is equivalent to partition $w-t$ into t ordered positive parts, which has exactly $\binom{w-t-1}{t-1}$ solutions [17]. To see (b) note that there is a constant $\theta > 1$ such that $\binom{w-t-1}{t-1} \leq \theta^w$. Clearly, θ is independent of k and r .

Similarly, we can bound $|T_{w, \leq h}|$ by

$$\begin{aligned} |T_{w, \leq h}| &= \sum_{t=1}^{\lceil w/2 \rceil} \sum_{(w_1, \dots, w_t): \sum w_i = w; w_i \geq 2, 1 \leq i < t; w_t \geq 1} |T_{(w_1, \dots, w_t), \leq h}| \\ &\leq \sum_{t=1}^{\lceil w/2 \rceil} \sum_{(w_1, \dots, w_t): \sum w_i = w; w_i \geq 2, 1 \leq i < t; w_t \geq 1} \binom{rh/2^{r-1}}{w} \binom{k/r}{t-1} h/2^{r-1} \end{aligned}$$

$$\begin{aligned} &\leq \sum_{t=1}^{\lceil w/2 \rceil} \binom{w-t}{t-1} \binom{rh/2^{r-1}}{w} \binom{k/r}{t-1} h/2^{r-1} \\ &\leq \theta^w \binom{rh/2^{r-1}}{w} \sum_{t=1}^{\lfloor w/2 \rfloor} \binom{k/r}{t}. \end{aligned}$$

Since $|C_{w,\leq h}| = |R_{w,\leq h}| + |T_{w,\leq h}|$, we have (7). \square

B. Bound of Weight Enumerators of Unpunctured RZH Codes

With the upper bound (7) for the inner code, we can obtain an upper bound of the weight enumerator for the overall concatenated code.

For the (n, k) unpunctured RZH code ensemble \mathcal{C}_n , the average weight enumerator of weight h is given by

$$\bar{A}_h^{(n)} \triangleq \frac{1}{|\mathcal{C}_n|} \sum_{C \in \mathcal{C}_n} A_h(C), \quad \text{for } h = 0, 1, \dots, n \quad (10)$$

where $A_h(C)$ is the weight enumerator of weight h for C . Similarly, the average accumulative weight enumerator $\bar{A}_{\leq h}^{(n)}$, the average input-output weight enumerator (IOWE) $\bar{A}_{w,h}^{(n)}$, and the average CIOWE $\bar{A}_{w,\leq h}^{(n)}$ can be defined.

The average IOWE of the unpunctured RZH code ensemble can be obtained from the weight enumerator of the outer repetition code C_1 and the IOWE of the inner ZH code C_2

$$\bar{A}_h^{(n)} = \sum_{d=d_1}^N \frac{A_d^{[1]} A_{d,h}^{[2]}}{\binom{N}{d}}, \quad (11)$$

where d_1 is the minimum distance of the outer code and N is the interleaver size. Hence

$$\bar{A}_{\leq h}^{(n)} = \sum_{d=d_1}^N \frac{A_d^{[1]} A_{d,\leq h}^{[2]}}{\binom{N}{d}}. \quad (12)$$

Given $h, d \leq \frac{rh}{2^{r-1}}$. Let $\mu = \frac{r}{2^{r-1}}$, and L_1 be the trellis length for C_1 . Applying Theorem A.1 in [13] to the outer code and Theorem 3.2 to the inner code, we have

$$\bar{A}_{\leq h}^{(n)} \leq \sum_{d=d_1}^{\mu h} \theta^d \frac{\binom{L_1}{\lfloor d/d_1 \rfloor} \binom{\mu h}{d}}{\binom{N}{d}} \sum_{t=1}^{\lfloor d/2 \rfloor} \binom{N/r}{t}. \quad (13)$$

Suppose that the rate of C_1 is R_1 and the rate of C_2 is R_2 , then we have $N = R_2 n = \beta n$, and $L_1 = R_1 N = R_1 \beta n = \alpha n$, where $\beta = R_2$ and $\alpha = \beta R_1$. Also let $\gamma = \beta/r$, then

$$\bar{A}_{\leq h}^{(n)} \leq \sum_{d=d_1}^{\mu h} \theta^d \frac{\binom{\alpha n}{\lfloor d/d_1 \rfloor} \binom{\mu h}{d}}{\binom{\beta n}{d}} \sum_{t=1}^{\lfloor d/2 \rfloor} \binom{\gamma n}{t}. \quad (14)$$

For $\delta = h/n$ small enough, we have

$$\binom{\gamma n}{t} \leq \binom{\gamma n}{\lfloor d/2 \rfloor} \quad (15)$$

for any $1 \leq t \leq \lfloor d/2 \rfloor$. Hence, replacing the inner sum in (14) with $\lfloor d/2 \rfloor$ times the right-hand side of (15), we have

$$\begin{aligned} \bar{A}_{\leq h}^{(n)} &\leq \sum_{d=d_1}^{\mu h} \theta^d \lfloor d/2 \rfloor \\ &\quad \times \frac{\binom{\alpha n}{\lfloor d/d_1 \rfloor} \binom{\mu h}{d}}{\binom{\beta n}{d}} \binom{\gamma n}{\lfloor d/2 \rfloor} \\ &\stackrel{(a)}{<} \sum_{d=d_1}^{\mu h} (2\theta)^d \\ &\quad \times \frac{\binom{\alpha n}{\lfloor d/d_1 \rfloor} \binom{\mu h}{d}}{\binom{\beta n}{d}} \binom{\gamma n}{\lfloor d/2 \rfloor} \end{aligned} \quad (16)$$

where step (a) follows from the fact that $\lfloor d/2 \rfloor < 2^d$.

C. Unpunctured RZH Codes Are Good

The ‘‘noisiness’’ of a memoryless binary input channel can be characterized by its Bhattacharyya noise parameter γ . For BIAWGN with noise variance $\sigma^2 = \frac{N_0}{2}$, γ is given by [13]

$$\gamma \triangleq e^{-1/2\sigma^2}. \quad (17)$$

Assuming maximum-likelihood (ML) decoding, we have the following result showing that unpunctured RZH codes are good.

Theorem 3.3: For the unpunctured RZH ensemble, if the minimum distance of the outer repetition code satisfies $d_1 \geq 3$, then there exists a positive number γ_0 such that for any binary-input memoryless channel whose Bhattacharyya noise parameter $\gamma < \gamma_0$ the ensemble word error probability

$$\bar{P}_W^{(n)} < \mathcal{O}\left(n^{-(d_1 - \lfloor d_1/2 \rfloor - 1) + \epsilon}\right)$$

for arbitrary $\epsilon > 0$.

To prove the above result, let D_n be a fixed sequence of integers satisfying $\frac{D_n}{n^\epsilon} \rightarrow 0$ for all $\epsilon > 0$, and $\frac{\log n}{D_n} \rightarrow 0$ with $n \rightarrow \infty$. Also, the n th innominate sum $Z^{(n)}(D) \triangleq \sum_{h=1}^D \bar{A}_h^{(n)}$ where D is an integer with $1 \leq D \leq n$ was defined in [13]. Because of [13, Th. 5.1 and Corollary 5.2], it is sufficient to prove the following lemma.

Lemma 3.3: For the unpunctured RZH ensemble, if $d_1 \geq 3$

$$Z^{(n)}(D_n) < \mathcal{O}\left(n^{-(d_1 - \lfloor d_1/2 \rfloor - 1) + \epsilon}\right)$$

for arbitrary $\epsilon > 0$.

Proof: With the bound (16), we have

$$\begin{aligned} Z^{(n)}(D_n) &= \bar{A}_{\leq D_n}^{(n)} \\ &< \sum_{d=d_1}^{\mu D_n} (2\theta)^d \frac{\binom{\alpha n}{\lfloor d/d_1 \rfloor} \binom{\mu D_n}{d}}{\binom{\beta n}{d}} \binom{\gamma n}{\lfloor d/2 \rfloor} \end{aligned}$$

$$\stackrel{(a)}{\leq} \sum_{d=d_1}^{\mu D_n} \Theta^{d_n \lfloor d/d_1 \rfloor + \lfloor d/2 \rfloor - d} D_n^{2d}$$

$$\stackrel{(a)}{\leq} \mathcal{O} \left(n^{-(d_1 - \lfloor d_1/2 \rfloor - 1) + \epsilon} \right).$$

In step (a), we have used the following inequalities: $\binom{\gamma n}{\lfloor \nu D_n \rfloor} \leq (\gamma n)^{\lfloor \nu D_n \rfloor}$, $\binom{\alpha n}{\lfloor d/d_1 \rfloor} \leq (\alpha n)^{\lfloor d/d_1 \rfloor}$, $\binom{\mu D_n}{d} \leq (\mu D_n)^d$, and $\binom{\beta n}{d} \geq (\beta n)^d / d^d \geq (\beta n)^d / (\mu D_n)^d$. For step (b), the sum is upper bounded by μD_n times the largest term and it is easy to show that the $d = d_1$ term is the largest one as n becomes large. Since $D_n = o(n^\epsilon)$ for any positive ϵ , the conclusion follows. \square

D. RZH Codes Are Good

To see that RZH code ensembles are also good, we have

Theorem 3.4: For the RZH ensemble, if the minimum distance of the outer repetition code satisfies $d_1 \geq 3$, then there exists a positive number γ_0 such that for any binary-input memoryless channel whose Bhattacharyya noise parameter $\gamma < \gamma_0$ the ensemble word error probability

$$\bar{P}_W^{(n)} \leq \mathcal{O} \left(n^{-(d_1 - \lfloor d_1/2 \rfloor - 1) + \epsilon} \right)$$

for arbitrary $\epsilon > 0$.

Different from the unpunctured RZH code, the inner encoder of a systematic RZH code will puncture the repetition bits of information bits \mathbf{u} and connection bits; for a nonsystematic RZH code, the information bits \mathbf{u} are also punctured. To prove RZH codes are good, we first consider $r = 2$ in which case the resulting RZH code is an RA code and hence the equality holds. For $r \geq 4$, we first prove the following lemma for the inner punctured ZH encoder.

Lemma 3.4: In case that $r \geq 4$, if a nonzero punctured ZH codeword (either with or without \mathbf{u}) consists of several detours and has weight h , the total length of all its detours is bounded by $h/(2^{r-1} - r - 1)$ and the input weight w is bounded by $rh/(2^{r-1} - r - 1)$.

Proof: We use the fact that the output weight of any admissible path of a punctured ZH code (either with or without \mathbf{u}) is at least $(2^{r-1} - r - 1)$. Since in order to have $(2^{r-1} - r - 1) > 0$, we should have $r \geq 4$. Then similar to the proof of Lemma 3.2, we have the above result. \square

With Lemma 3.4, we can obtain the following result.

Theorem 3.5: For an RZH code where the inner code is an (n, k) punctured recursive ZH code C with $r \geq 4$, the CIOWE of C can be bounded by

$$A_{w, \leq h}(C) \leq \theta^w \binom{rh/(2^{r-1} - r - 1)}{w} \sum_{t=1}^{\lfloor w/2 \rfloor} \binom{k/r}{t} \quad (18)$$

for $w \geq 2$, where $\theta > 1$ is a constant independent of k and r .

Proof: The proof is similar to that of Theorem 3.2. Again, we split $C_{w, \leq h}$ into $R_{w, \leq h}$ and $T_{w, \leq h}$ where R denotes codewords with only regular detours whereas T denotes codewords with a truncated detour at the end. We still have (8) and (9).

For a systematic RZH code, we have the codeword $\mathbf{c} = \{\mathbf{u}, \mathbf{p}\} \in C_{((w_1, b_1), \dots, (w_t, b_t)), \leq h}$ where \mathbf{u} denotes the information bits of the RZH code and is a punctured version of \mathbf{d} , as shown in Fig. 2; \mathbf{p} denotes the parity bits. We now exhibit an injective map from $C_{((w_1, b_1), \dots, (w_t, b_t)), \leq h}$ into the set of binary $(rh/(2^{r-1} - r - 1))$ -tuples of weight w . The map is composed of three simpler maps $\{\mathbf{u}, \mathbf{p}\} \rightarrow \{\mathbf{d}, \mathbf{q}, \mathbf{p}\} \rightarrow \mathbf{d} \rightarrow \mathbf{d}'$. Since the puncturing scheme is known and the puncturing simply involves removing the repetition bits, we can easily reconstruct \mathbf{d} and \mathbf{q} from \mathbf{u} and \mathbf{p} , hence the first map $\{\mathbf{u}, \mathbf{p}\} \rightarrow \{\mathbf{d}, \mathbf{q}, \mathbf{p}\}$ is injective. The second and the third maps are also injective. Then with Lemma 3.4, it is easy to prove (18) where the procedure is similar to that of the proof for Theorem 3.2.

In case that the RZH code is nonsystematic, we have the codeword $\mathbf{c} = \mathbf{p} \in C_{((w_1, b_1), \dots, (w_t, b_t)), \leq h}$ where \mathbf{p} denotes the parity bits. Again, we need to exhibit an injective map from $C_{((w_1, b_1), \dots, (w_t, b_t)), \leq h}$ into the set of binary $(rh/(2^{r-1} - r - 1))$ -tuples of weight w . The map is composed of three simpler maps $\mathbf{p} \rightarrow \{\mathbf{q}, \mathbf{p}\} \rightarrow \mathbf{d} \rightarrow \mathbf{d}'$. To see that the first map is injective, it is clear that from \mathbf{p} we can reconstruct \mathbf{q} . Hence eventually, what we need to do is to prove that we can reconstruct \mathbf{d} from \mathbf{p} and \mathbf{q} . Consider the j th segment in the trellis. Suppose that the map from $\{q_j, p_j\}$ to \mathbf{d}_j is not injective, then we have at least two distinct \mathbf{d}_j , namely, $\mathbf{d}_{j,1}$ and $\mathbf{d}_{j,2}$, which produce the same $\{q_j, p_j\}$. Consequently, the two unpunctured j th segment Hadamard codewords are $\{\mathbf{d}_{j,1}, q_j, p_j\}$ and $\{\mathbf{d}_{j,2}, q_j, p_j\}$ with a distance of at most r between them. However, for $r \geq 4$, the minimum distance of Hadamard code is 2^{r-1} which is larger than r . This means that we must have $\mathbf{d}_{j,1} = \mathbf{d}_{j,2}$, hence the map $\{q_j, p_j\} \rightarrow \mathbf{d}_j$ is injective and so is the map $\{\mathbf{q}, \mathbf{p}\} \rightarrow \mathbf{d}$. The third map is also injective. Then with Lemma 3.4, it is easy to prove (18) for nonsystematic codes. \square

With Theorem 3.5, following the proof procedure of Lemma 3.3, it is easy to obtain the following lemma which completes the proof of Theorem 3.4.

Lemma 3.5: For the RZH ensemble with $r \geq 4$, if $d_1 \geq 3$

$$Z^{(n)}(D_n) < \mathcal{O} \left(n^{-(d_1 - \lfloor d_1/2 \rfloor - 1) + \epsilon} \right)$$

for arbitrary $\epsilon > 0$.

IV. EFFICIENT DECODING ALGORITHMS

A. Belief Propagation Decoding of Un-Punctured RZH Codes

Since the unpunctured RZH codes are standard serial concatenated codes, here we first introduce the decoding scheme for unpunctured codes, and the decoding of RZH codes as shown in Fig. 2 can be easily extended. An unpunctured RZH code can be represented by its Tanner graph [18]. As shown in Fig. 6, there are three sets of nodes in the Tanner graph of an unpunctured RZH code: the *variable nodes of information bits*, corresponding to \mathbf{u} , the *variable nodes of common bits*, corresponding to \mathbf{q} , and the *Hadamard check nodes*, corresponding to the Hadamard code constraints. Note that in Fig. 6, the parity bits of Hadamard codewords are embedded in their Hadamard check nodes. Those information bits that are repeated d_v times are represented by information bit variable nodes with degree d_v , since they participate in d_v Hadamard code constraints,

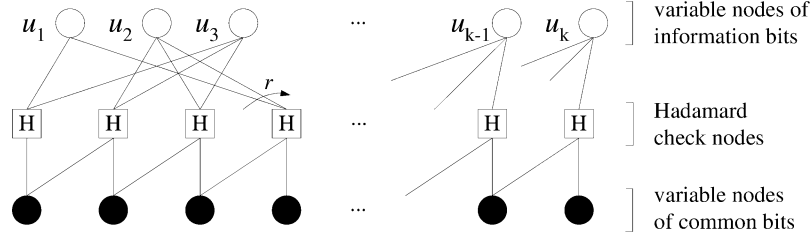
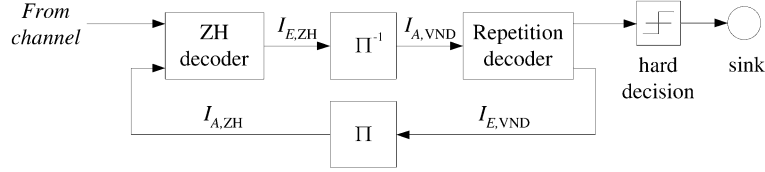

 Fig. 6. Tanner graph of an unpunctured (n, k) RZH code.


Fig. 7. Turbo-like decoding of unpunctured RZH codes.

and the common bit variable nodes can be viewed as degree-2 information bit variable nodes. Each Hadamard check node is connected to r information bit nodes and to two common bit nodes. The connections between Hadamard check nodes and information bit nodes are determined by the edge interleaver and are highly randomized, whereas the connections between Hadamard check nodes and common bit nodes are arranged in a highly structured zigzag pattern.

Based on the Tanner graph, iterative decoding algorithms can be derived. Here we consider two belief-propagation (BP) based decoding schemes for unpunctured RZH codes: the turbo-like serial decoding and the LDPC-like parallel decoding. In general, the serial decoding takes less iteration number than the parallel one, whereas the latter enjoys the advantage of easy hardware implementation with parallel processing. There is no fundamental difference in terms of bit error rate (BER) performance.

Turbo-like Decoding:

Since the unpunctured RZH codes are serially concatenated codes, the standard turbo-like decoding can be employed as shown in Fig. 7, which consists of an inner ZH decoder and an outer information bit node decoder (which is essentially a repetition decoder), with extrinsic LLRs of \mathbf{d} exchanged between the two decoders. The encoding of a ZH code is a Markov process and similar to the decoding of zigzag codes, the *a posteriori probability* (APP) decoding of a ZH code can be accomplished with the two-way algorithm [12] where the low-complexity fast-Hadamard-transform (FHT)-based Hadamard decoding is employed.

For RZH codes, only the extrinsic LLRs of \mathbf{d} are exchanged between the inner ZH decoder and the outer repetition decoder. Note that for the code word $\mathbf{c} = \{c_j\}$, we have $c_{j,2^i-1} = d_{j,i}$, for $i = 1, 2, \dots, r$ and $j = 1, 2, \dots, J$, hence we have

$$L_H^A(c_{j,2^i-1}) = L_{zh}^{\text{in}}(d_{j,i}),$$

for $i = 1, 2, \dots, r$, and $j = 1, 2, \dots, J$, (19)

where $\{L_H^A(c_{j,2^i-1})\}$ are the *a priori* LLRs for the Hadamard coded bits $\{c_{j,2^i-1}\}$, and $\{L_{zh}^{\text{in}}(d_{j,i})\}$ are the extrinsic LLRs of $\{d_{j,i}\}$ obtained from the outer repetition decoder in the previous iteration, initialized to zeros for the first iteration. After the two-way APP ZH decoding, we have the APP LLRs

$\{L_H^{\text{APP}}(c_{j,2^i-1})\}$. Then the extrinsic information of \mathbf{d} passing from the ZH decoder to the repetition decoder is given by

$$L_{zh}^{\text{out}}(d_{j,i}) = L_H^{\text{APP}}(c_{j,2^i-1}) - L_{zh}^{\text{in}}(d_{j,i}),$$

for $i = 1, 2, \dots, r$, and $j = 1, 2, \dots, J$, (20)

which will be fed to the repetition decoder after deinterleaving.

Consider only a particular information bit node of degree d_v which has d_v incoming messages from the edge interleaver. The repetition decoder decodes by computing, for $i = 1, 2, \dots, d_v$

$$L_{\text{rep}}^{\text{out}}(i) = \sum_{j \neq i} L_{\text{rep}}^{\text{in}}(j) \quad (21)$$

where $L_{\text{rep}}^{\text{in}}(j)$ is the j th *a priori* LLR value from the ZH decoder to the information bit node, and $L_{\text{rep}}^{\text{out}}(i)$ is the i th extrinsic LLR value coming out of the information bit node. The output extrinsic LLRs of the repetition decoder are then interleaved and fed to the ZH decoder as $\{L_{zh}^{\text{in}}(d_{j,i})\}$ after reindexing.

LDPC-like Decoding:

For LDPC-like decoding, the common bit nodes are treated as degree-2 information bit nodes. In this case, all Hadamard check nodes and all information bit nodes are activated alternately and in parallel. Every time the Hadamard check nodes are activated, the J parallel Hadamard decoders firstly compute the APP LLRs of the information bits as in [10] based on the channel observation $\{x_j\}$ and the *a priori* LLRs passed from the information nodes, with which the extrinsic LLRs of the information bits can be obtained by subtracting the *a priori* LLRs from the APP LLRs. The resulting extrinsic LLRs are then fed to the information bit nodes as the *a priori* information, with which the information bit nodes are activated in parallel and compute the extrinsic LLRs according to (21) for the next iteration.

B. Decoding of RZH Codes

The decoding of systematic RZH codes is similar to that of the unpunctured case. The only difference is the treatment of the channel observation for the information bits. As shown in Fig. 2, the information bits are transmitted only once, then similar to the decoding of systematic RA codes, it is more convenient for the channel observation of the information bits to participate in the repetition decoding. Fig. 8 depicts the structure

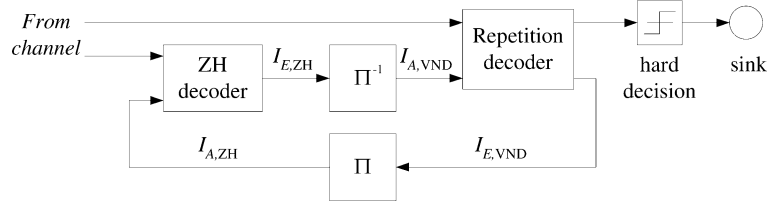


Fig. 8. Turbo-like decoding of systematic RZH codes.

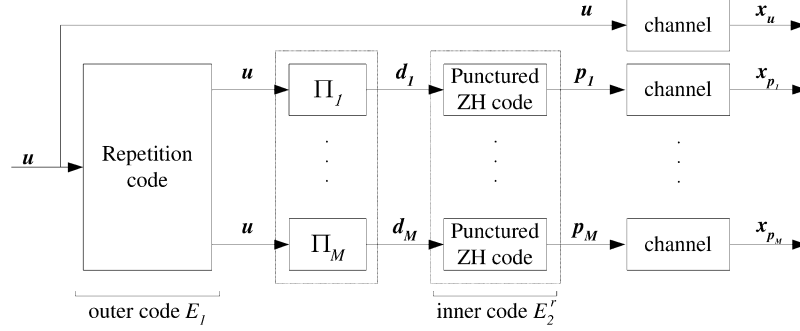


Fig. 9. Parallel implementation of a systematic regular RZH code—PCZH code.

of the turbo-like decoder for systematic RZH codes. With such a structure, an information bit variable node of degree d_v has $d_v + 1$ incoming messages: d_v from the edge interleaver and one from the communication channel. The repetition decoder decodes by computing, for $i = 1, 2, \dots, d_v$

$$L_{\text{rep}}^{\text{out}}(i) = L_{\text{ch}} + \sum_{j \neq i} L_{\text{rep}}^{\text{in}}(j), \quad (22)$$

where L_{ch} is the LLR value about the information bit from the channel. For AWGN channel with noise variance $\sigma^2 = \frac{N_0}{2}$, we have $L_{\text{ch}} = \frac{2x}{\sigma^2}$ where x is the channel observation of the binary phase shift keying (BPSK) modulated information bit. Similar principle can be employed for LDPC-like decoding of systematic RZH codes.

For nonsystematic RZH codes, the information bits are not transmitted. Hence the information bit variable node of degree d_v has only d_v incoming messages from the edge interleaver which is similar to that of unpunctured RZH codes, and the repetition decoder decodes by computing, for $i = 1, 2, \dots, d_v$, $L_{\text{rep}}^{\text{out}}(i) = \sum_{j \neq i} L_{\text{rep}}^{\text{in}}(j)$.

Note that for $r = 2$, the nonsystematic RZH codes are essentially nonsystematic RA codes with a grouping factor [19] of 2, which turn out to be useless due to the start failure of the iterative decoding algorithm [20]. To see why this happens, consider the parity check code embedded in the RA code which produces parity bit p given the two inputs d_1 and d_2 according to $p = d_1 \oplus d_2$, where \oplus denotes modulo-2 summation. For nonsystematic RA codes, no priori information about d_1 and d_2 is available at the beginning of the decoding. Also, it is clear that the map $p \rightarrow \{d_1, d_2\}$ is not injective. Consequently, $\{d_1, d_2\}$ cannot be correctly decoded even p is perfectly known which is the key factor that makes the iterative decoding algorithm fail to start. Such a problem does not exist for nonsystematic RZH codes with $r \geq 4$, since as discussed in Section III-D, the map $\mathbf{p} \rightarrow \mathbf{d}$ is injective for $r \geq 4$.

C. Upper Bound of Regular RZH Under ML Decoding

Similar to regular LDPC codes and RA codes, the outer code of a regular RZH code is a single repetition code of rate $1/d_v$, with d_v being a positive integer. In this section we consider the upper bound of regular RZH codes under ML decoding and compare it to that of the previously proposed PCZH codes [12]. Consider a regular RZH code with $d_v = M$. As shown in Fig. 9, let the length- kM interleaver be implemented by M parallel smaller interleavers of length k , and the inner code be a mixture of M punctured ZH codes. We then obtain the PCZH codes. In this sense, PCZH codes can also be viewed as a specified implementation of the regular RZH codes. To obtain some insight on how the multiple parallel interleavers affect the code performance, we consider the ML BER bounds of both regular RZH codes with single interleaver and PCZH codes with short frame size.

Assuming decoded by ML criterion over a binary input additive weight Gaussian noise (BIAWGN) channel, the simple ML bound [21], [22] is a tight upper bound on BER for an (n, k) linear block code with code rate $R_c = k/n$. Let $A_{w,h}$ being the number of codewords with input weight w and output weight h . The refined simple ML bound is given by [22]

$$P_b \leq \sum_{h=h_{\min}}^{h_{\max}} \min \left\{ e^{-nE \left(\frac{E_s}{N_0}, \frac{h}{n}, \beta \right)} \sum_{w=1}^k \frac{w}{k} A_{w,h} Q \left(\sqrt{2 \frac{E_s}{N_0} h} \right) \right\} \quad (23)$$

where

$$E \left(\frac{E_s}{N_0}, \frac{h}{n}, \beta \right) \triangleq \frac{1}{2} \ln \left(1 - \beta + \beta e^{\frac{-2 \ln \left(\sum_{w=1}^k \frac{w}{n} A_{w,h} \right)}{n}} \right) + \frac{\frac{\beta h}{n} \frac{E_s}{N_0}}{1 - \frac{(1-\beta)h}{n} \frac{E_s}{N_0}} \quad (24)$$

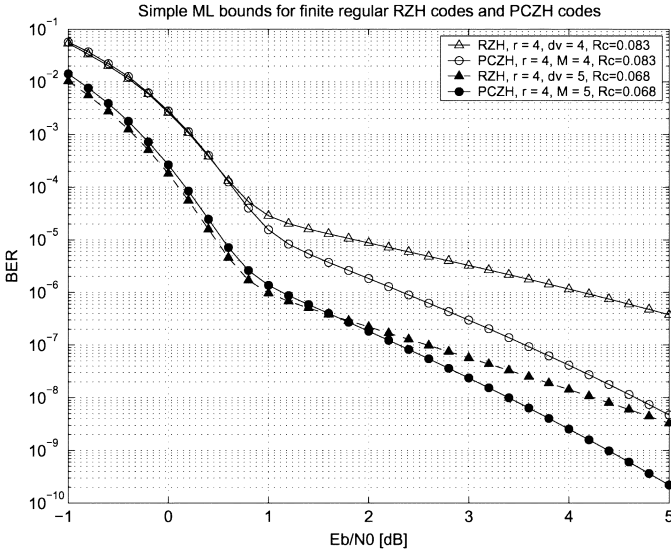


Fig. 10. Simple ML bounds for regular RZH codes and PCZH codes. $k = 200$.

and

$$\beta \triangleq \frac{1 - \frac{h}{n}}{\frac{h}{n}} \left[\sqrt{\frac{E_s}{N_0} \frac{2h}{1 - \frac{h}{n}} - 1 + \left(1 + \frac{E_s}{N_0}\right)^2} - \left(1 + \frac{E_s}{N_0}\right) \right] \quad (25)$$

with $E_s/N_0 = E_b/N_0 R_c$. The bound is valid for $0 \leq \beta \leq 1$, and for other β value, we set $\beta = 1$ which makes the simple ML bound reduce to the union bound.

Fig. 10 shows the simple ML bounds for regular RZH codes and PCZH codes with $r = 4$, $k = 200$ and uniform interleavers. The computing of $A_{w,h}$ for the two codes is given in Appendix A. It is seen from Fig. 10 that, the average performance of PCZH codes exhibits a lower error-floor at the high SNR region and a slightly worse performance at the low SNR region than that of regular RZH codes with the same rate. This shows the effectiveness of the parallel implementation of regular RZH codes. Note that for regular RZH codes, the choice of possible coding rate is limited. For a more flexible coding rate, irregular RZH (IRZH) codes can be used where the outer code is a mixture of repetition codes with different rates. The optimization of IRZH codes will be considered in Section V.

V. IRREGULAR RZH CODE ENSEMBLE OPTIMIZATION

In this section we consider the code design of IRZH codes where the outer code is a mixture of repetition codes with different rates. The design approach closely follows those described for RA codes and LDPC codes [14], [15], [19] where the EXIT chart technique is utilized as a design tool. For simplicity, in what follows we only consider the code design with turbo-like decoding.

A. EXIT Functions

In order to perform code design, we first need to get the mutual information transfer functions of the outer repetition decoder and the inner ZH code. For an (n, k) IRZH codes, the outer code consists of k repetition codes with variable rates. As

shown in Fig. 6, suppose the rate of the i th repetition code is $1/d_v^{(i)}$, the degree of the i th variable node of information bit is $d_v^{(i)}$. For the outer repetition codes, the EXIT curve is obtained in a similar manner as for RA codes and LDPC codes. Specifically, consider first unpunctured RZH codes. As discussed in Section 4.1, an information bit variable node of degree d_v has d_v incoming messages from the edge interleaver, and the decoder outputs are given by (21). As in [14], [15], we model $L_{\text{rep}}^{\text{in}}(j)$ as the output LLR of an AWGN channel whose input is the j th interleaver bit transmitted using BPSK. The EXIT function of a degree- d_v variable node is then

$$I_{E,\text{VND}}(I_{A,\text{VND}}, d_v) = J \left(\sqrt{(d_v - 1)} J^{-1}(I_{A,\text{VND}}) \right) \quad (26)$$

where

$$J(\sigma) \triangleq 1 - \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{(\zeta - \sigma^2/2)^2}{2\sigma^2}} \log_2[1 + e^{-\zeta}] d\zeta.$$

Since $J(\sigma)$ is a monotonically increasing function of σ , it is invertible.

Given the Hadamard code order r , the IRZH design involves specifying the variable node degrees $d_v^{(i)}$, $i = 1, 2, \dots, k$. Let D be the number of different variable node degrees, and denote these degrees by $d_{v,i}$, $i = 1, 2, \dots, D$. Then the degree profile of a code can be specified by a polynomial $f(x) = \sum_{i=1}^D a_i x^{d_{v,i}}$ where $a_i \geq 0$ is the fraction of nodes having degree $d_{v,i}$. Note that $\{a_i\}$ must satisfy $\sum_i a_i = 1$. The average variable node degree is then given by

$$\bar{d}_v = \sum_{i=1}^D a_i \cdot d_{v,i}. \quad (27)$$

Since for $k \rightarrow \infty$, $R_c = \frac{r}{d_v 2^r}$, we have

$$\bar{d}_v = \frac{r}{R_c 2^r}. \quad (28)$$

Let b_i be the fraction of edges incident to variable nodes of degree $d_{v,i}$. There are $(k a_i) d_{v,i}$ edges involved with such nodes, so we have

$$b_i = \frac{k a_i d_{v,i}}{k \bar{d}_v} = \frac{d_{v,i}}{\bar{d}_v} \cdot a_i. \quad (29)$$

Note that $\{b_i\}$ must satisfy $\sum_i b_i = 1$. It is shown in [23], [24] that the EXIT function of a mixture of codes is an average of the component EXIT functions. With (26), (28), and (29), the effective transfer function of the outer mixture codes is thus given by

$$I_{E,\text{VND}} = g_{\text{VND}}(I_{A,\text{VND}}, R_c, r) \quad (30)$$

$$= \sum_{i=1}^D b_i \cdot I_{E,\text{VND}}(I_{A,\text{VND}}, d_{v,i}) \quad (31)$$

$$= \frac{R_c 2^r}{r} \sum_{i=1}^D d_{v,i} \cdot a_i \cdot I_{E,\text{VND}}(I_{A,\text{VND}}, d_{v,i}). \quad (32)$$

Since $g(\cdot)$ is a monotonically increasing function of $I_{A,\text{VND}}$, it is invertible and we have $I_{A,\text{VND}} = g_{\text{VND}}^{-1}(I_{E,\text{VND}}, R_c, r)$.

For systematic RZH codes, an information bit variable node of degree d_v has $d_v + 1$ incoming messages: d_v from the edge

interleaver and one from the channel. The decoder outputs are given by (22). Then the EXIT function of a degree- d_v variable node is given by

$$I_{E,\text{VND}} \left(I_{A,\text{VND}}, d_v, \frac{E_s}{N_0} \right) = J \left(\sqrt{(d_v - 1)[J^{-1}(I_{A,\text{VND}})]^2 + \sigma_{\text{ch}}^2} \right) \quad (33)$$

where $\sigma_{\text{ch}}^2 = \frac{4}{\sigma^2} = 8 \cdot \frac{E_s}{N_0}$, with $\frac{E_s}{N_0} = R_c \cdot \frac{E_b}{N_0}$ being the symbol SNR.

For systematic RZH codes, we have $R_c = \frac{r}{d_v(2^r - r - 1) + r}$, hence

$$\bar{d}_v = \frac{r}{2^r - r - 1} \left(\frac{1}{R_c} - 1 \right). \quad (34)$$

Then with (29), (33) and (34), we have the effective transfer function of the outer mixture codes as follows:

$$I_{E,\text{VND}} = g_{\text{VND}} \left(I_{A,\text{VND}}, \frac{E_s}{N_0}, R_c, r \right) \quad (35)$$

$$= \sum_{i=1}^D b_i \cdot I_{E,\text{VND}} \left(I_{A,\text{VND}}, d_{v,i}, \frac{E_s}{N_0} \right) \quad (36)$$

$$= \frac{2^r - r - 1}{r} \cdot \frac{R_c}{1 - R_c} \sum_{i=1}^D d_{v,i} \cdot a_i \cdot I_{E,\text{VND}} \left(I_{A,\text{VND}}, d_{v,i}, \frac{E_s}{N_0} \right). \quad (37)$$

Again, $g(\cdot)$ is invertible, and we have $I_{A,\text{VND}} = g_{\text{VND}}^{-1}(I_{E,\text{VND}}, \frac{E_s}{N_0}, R_c, r)$.

Now consider nonsystematic codes. In this case, the information bit node of degree d_v has d_v incoming messages which is the same as that of unpunctured case. Hence the EXIT function of a degree- d_v variable node is given by (26). For nonsystematic RZH codes, we have $R_c = \frac{r}{d_v(2^r - r - 1)}$, hence

$$\bar{d}_v = \frac{r}{R_c(2^r - r - 1)}. \quad (38)$$

Then with (29), (26) and (38), we have the effective transfer function of the outer mixture codes as follows:

$$I_{E,\text{VND}} = g_{\text{VND}}(I_{A,\text{VND}}, R_c, r) \quad (39)$$

$$= \sum_{i=1}^D b_i \cdot I_{E,\text{VND}}(I_{A,\text{VND}}, d_{v,i}) \quad (40)$$

$$= \frac{R_c(2^r - r - 1)}{r} \sum_{i=1}^D d_{v,i} \cdot a_i \cdot I_{E,\text{VND}}(I_{A,\text{VND}}, d_{v,i}). \quad (41)$$

Again, $g_{\text{VND}}(\cdot)$ is invertible, and we have $I_{A,\text{VND}} = g_{\text{VND}}^{-1}(I_{E,\text{VND}}, R_c, r)$.

In order to perform code ensemble optimization, we also need the EXIT transfer function of the inner ZH decoder. In general, there is no closed-form formula, hence we compute the EXIT function by simulation and denote it by

$$I_{E,\text{ZH}} = g_{\text{ZH}} \left(I_{A,\text{ZH}}, \frac{E_s}{N_0}, r \right). \quad (42)$$

The exchange of the mutual information between the inner ZH decoder and the outer repetition decoder is specified in Figs. 7 and 8.

B. Code Ensemble Optimization

For given r and $\frac{E_s}{N_0}$, the optimal IRZH ensemble parameters $\{a_i\}$ that maximize R_c subject to vanishing BER $P_b(\frac{E_s}{N_0})$, are solution of the following optimization problem

$$\max R_c \text{ s.t. } \sum_i a_i = 1, a_i \geq 0, \forall i \text{ and } P_b \left(\frac{E_s}{N_0} \right) = 0. \quad (43)$$

As shown in [14], [15], EXIT charts can be used to analyze the convergence properties of RA codes and LDPC codes. The same principle can be applied to RZH codes. Specifically, if there is an open tunnel between the EXIT functions of the inner code and outer code for given r and $\frac{E_s}{N_0}$, the iterative decoding algorithm can asymptotically converge and we have $P_b(\frac{E_s}{N_0}) = 0$ with a sufficient large number of iteration. Mathematically, the convergence condition is given by

$$g_{\text{ZH}} \left(I_{A,\text{ZH}}, \frac{E_s}{N_0}, r \right) > g_{\text{VND}}^{-1}(I_{A,\text{ZH}}, R_c, r) \quad \text{for } I_{A,\text{ZH}} \in [0, 1) \quad (44)$$

for unpunctured RZH codes and punctured nonsystematic codes, and

$$g_{\text{ZH}} \left(I_{A,\text{ZH}}, \frac{E_s}{N_0}, r \right) > g_{\text{VND}}^{-1} \left(I_{A,\text{ZH}}, \frac{E_s}{N_0}, R_c, r \right) \quad \text{for } I_{A,\text{ZH}} \in [0, 1) \quad (45)$$

for systematic codes. With the help of the EXIT chart analysis, we can simplify the above optimization problem by replacing the constraint $P_b(\frac{E_s}{N_0}) = 0$ in (43) with condition (44) or (45). To design an IRZH code for a particular rate R_c , we simply choose proper Hadamard degree and change $\frac{E_s}{N_0}$ in (43) until the resulting optimization result $R_c^* = R_c$.

C. Design Examples

In what follows, we provide design examples for both unpunctured IRZH codes and IRZH codes in BIAWGN channel.

Example 1: Performance comparison of regular and irregular unpunctured RZH codes

Consider first an unpunctured regular RZH code with $r = 4$ and $d_v = 14$ which results in a coding rate of around 0.018. Fig. 11 plots the EXIT chart of this code, from which it can be seen that the corresponding SNR threshold (in terms of E_b/N_0) is around -0.05 dB. Now consider the irregular code with $\bar{d}_v = 14$. With code ensemble optimization, we are able to obtain an irregular code at $\frac{E_b}{N_0} = -0.98$ dB where the degree profile polynomial is given by $f(x) = 0.323x^3 + 0.175x^8 + 0.026x^{22} + 0.476x^{23}$. The corresponding EXIT chart for $\frac{E_b}{N_0} = -0.98$ dB is also plotted in Fig. 11, where the solid line denotes the EXIT function of the outer mixture repetition codes with the optimized profile and diamond denotes the EXIT function of the inner ZH code.

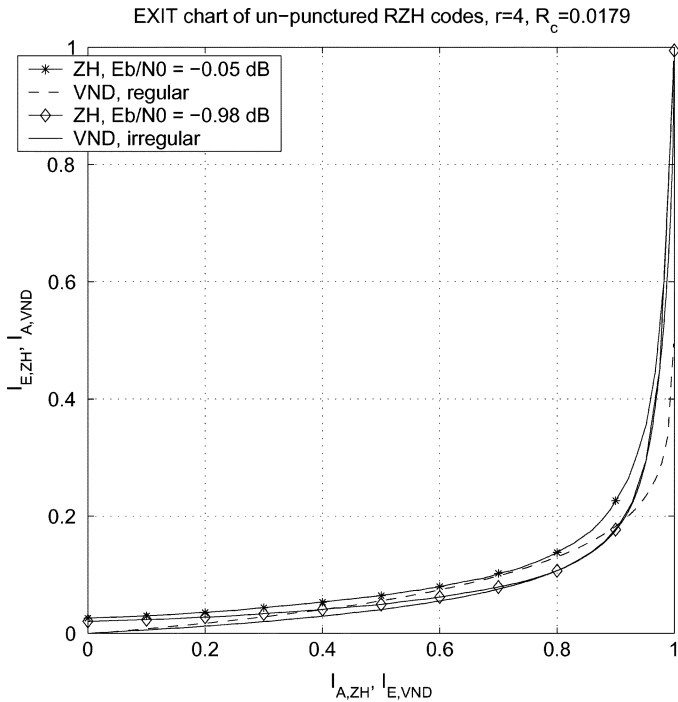


Fig. 11. EXIT chart of un-punctured RZH codes with $r = 4$ and $R_c = 0.0179$.

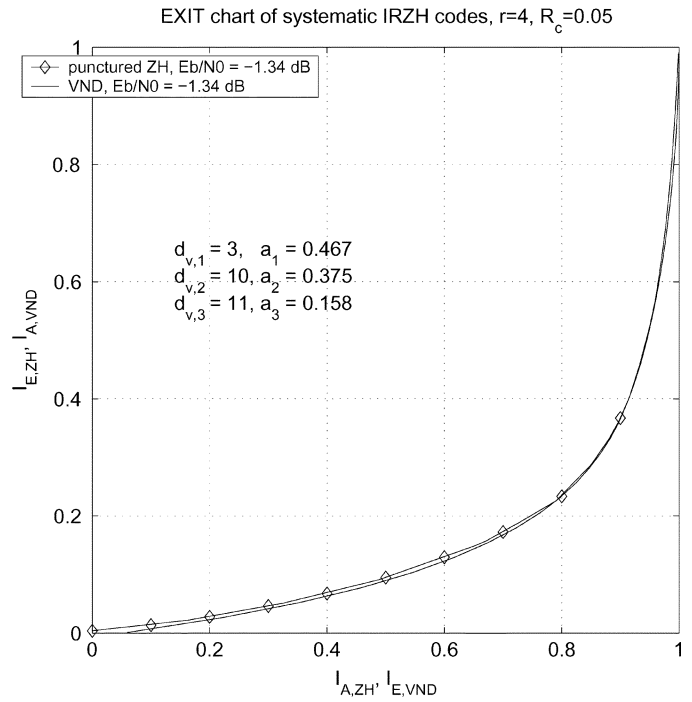


Fig. 13. EXIT chart of systematic IRZH codes with $r = 4$ and $R_c = 0.05$.

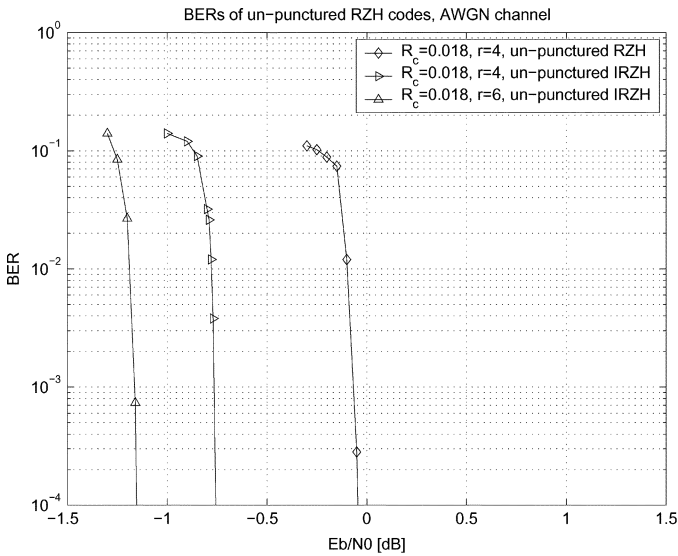


Fig. 12. BERs for un-punctured RZH codes, with an information block length of 65 536 and a maximum iteration number of 150.

Fig. 12 plots the simulated BER results of the regular and irregular un-punctured RZH codes with an information block length of $k = 65536$ and a maximum iteration number of 150. Also note that since the code rate R_c is very low, the codeword length $n = k/R_c$ is quite large. It is seen that with $r = 4$, the optimized un-punctured IRZH code has a gain of 0.71 dB (measured at $P_b = 10^{-4}$) over the regular code, whereas the gap to the capacity is around 0.8 dB. With larger r , better codes can be obtained. For instance, for $r = 6$, we obtain a code which is optimized at $\frac{E_b}{N_0} = -1.30$ dB with $f(x) = 0.363x^3 + 0.338x^6 + 0.299x^7$, and the simulated threshold (measured at $P_b = 10^{-4}$) is -1.13 dB, which is only 0.41 dB away from the capacity.

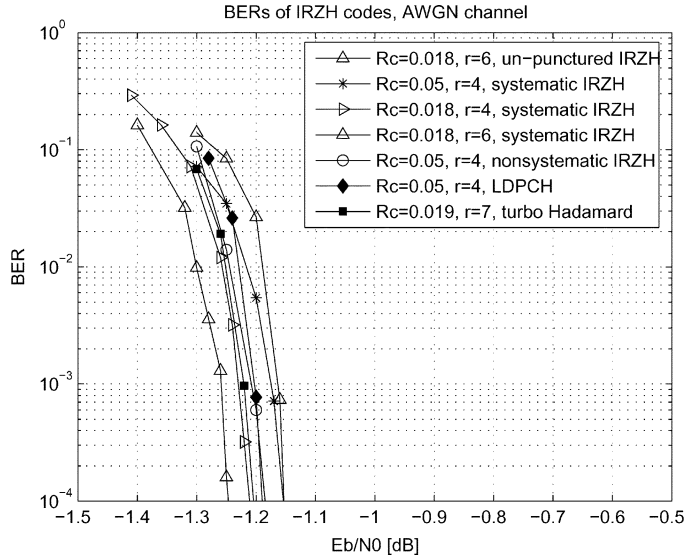


Fig. 14. BERs for IRZH codes, with an information block length of 65 536 and a maximum iteration number of 150.

Example 2: Optimization of systematic RZH codes

Next we consider the ensemble optimization of the systematic IRZH codes. Since the repetition bits are punctured, much better performance can be expected. Indeed, for $r = 4$ and $R_c = 0.05$, we are able to obtain an irregular code at $\frac{E_b}{N_0} = -1.34$ dB with $f(x) = 0.467x^3 + 0.375x^{10} + 0.158x^{11}$, which is only 0.1 dB away from the capacity. The EXIT chart of the curve-fitting result is shown in Fig. 13 and the BER performance is shown in Fig. 14. Results show that the simulated SNR threshold is -1.14 dB (only around 0.31 dB away from the capacity and 0.05 dB away from the BER threshold of an LDPC Hadamard code with the same Hadamard order and code rate) which is

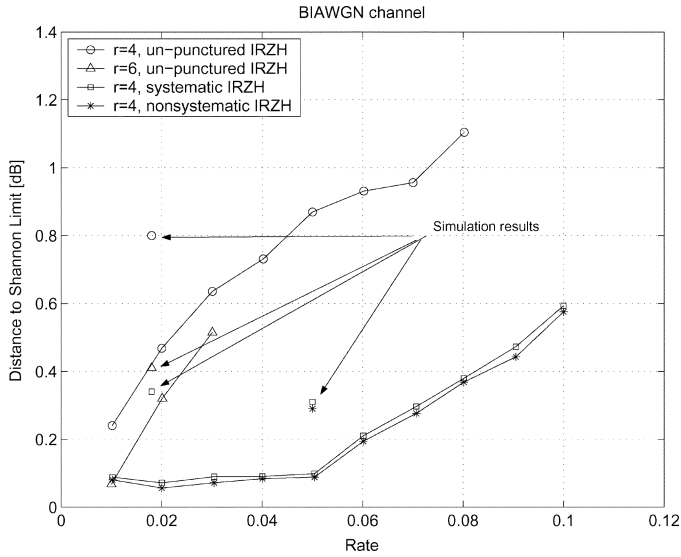


Fig. 15. Gap to Shannon limit (obtained by EXIT chart) versus rate for BIAWGN channel. $3 \leq d_{v,i} \leq 200$. Simulation results for some rates are also plotted in the figure.

much better than that of the optimized unpunctured code with the same r and a much lower (0.0179) coding rate. Fig. 14 also depicts the BER performance of a lower rate code with $r = 4$ and $R_c = 0.018$ which is optimized at $\frac{E_b}{N_0} = -1.42$ dB with $f(x) = 0.364x^3 + 0.014x^{11} + 0.192x^{12} + 0.074x^{13} + 0.095x^{36} + 0.136x^{37} + 0.1087x^{38} + 0.004x^{39} + 0.013x^{200}$, and the simulated SNR threshold is around -1.2 dB, which is 0.34 dB away from the capacity and around 0.08 dB better than the same rate unpunctured IRZH code with $r = 6$. Also note that with $r = 6$, a punctured code can be optimized at $\frac{E_b}{N_0} = -1.43$ dB with $f(x) = 0.401x^3 + 0.573x^7 + 0.026x^{20}$, and the simulated SNR threshold is -1.24 dB, which is slightly better than that of $r = 4$. Also plotted in Fig. 14 is the simulated BER performance of a rate 0.019 turbo Hadamard code with $r = 7$. It is seen that IRZH code can achieve a similar performance with a smaller Hadamard order.

Example 3: Difference between systematic and nonsystematic codes.

To see the difference between the systematic and nonsystematic codes, for $r = 4$ and $R_c = 0.05$, a nonsystematic code can be optimized at around $\frac{E_b}{N_0} = -1.36$ dB with $f(x) = 0.456x^3 + 0.147x^9 + 0.343x^{10} + 0.054x^{20}$, which is slightly better than that of systematic code. The BER performance of this code is shown in Fig. 14 which shows a simulated SNR threshold of around -1.16 dB, with a small coding gain of 0.02 dB over its systematic counterpart. This is as expected since for nonsystematic IRZH codes, the systematic bits are further punctured and for the same rate, more parity bits can be transmitted.

Example 4: Performance comparison between punctured and unpunctured IRZH codes

Fig. 15 compares the performance of IRZH ensembles with that of the unpunctured codes on the BIAWGN channel for different rates. As can be seen, for $r = 4$, the performance of opti-

mized IRZH ensembles is much better than that of unpunctured ensembles in the considered rate range, and the difference becomes larger with the increasing rate. Although with increasing Hadamard order, the performance of unpunctured RZH codes can be significantly improved and becomes comparable to that of RZH codes in the extremely low-rate regime, this comes at a cost of increasing decoding complexity since the complexity of the APP Hadamard decoding is proportional to 2^r . Also, as expected, it is seen from Fig. 15 that the nonsystematic codes have a small gain over the systematic codes.

VI. CONCLUSION

We have proposed a new class of low-rate codes called RZH codes. By analyzing the code structure of RZH codes, we have proved that RZH codes are good codes. Both serial and parallel decoding algorithms are developed. Irregular codes are also considered and a technique based on EXIT chart is introduced for the code ensemble optimization. Our results have shown that the optimized irregular codes work very well in the low-rate regime. It is also evident that, the optimized IRZH codes have the capability of approaching capacity with a relatively smaller Hadamard order than that of unpunctured codes. With its capacity approaching performance, simple encoding/decoding structure and flexibility in terms of parameter selection, RZH code is a promising low-rate coding candidate for systems operating in the low-SNR regime, e.g., the code-spread communication systems and the power-critical sensor networks.

APPENDIX A

INPUT-OUTPUT WEIGHT ENUMERATORS FOR RZH CODES AND PCZH CODES

In this Appendix, we introduce the calculation of input-output weight enumerator function (IOWEF) for RZH codes and PCZH codes with finite length.

Consider an (n, k) ZH code. Then the IOWEF of this code is

$$A^{[\text{zh}]}(W, H) = \sum_{w=0}^k \sum_{h=0}^n A_{w,h}^{[\text{zh}]} W^w H^h \quad (46)$$

As shown in Fig. 5, a ZH code can be represented by a two-state diagram. For simplicity, consider a punctured ZH code with $r = 4$ (here, both information bits and common bits are punctured). The state-transition matrix for this code is

$$C(W, H) = \begin{bmatrix} 1 + 6W^2H^6 + W^4H^4 & 4WH^7 + 4W^3H^5 \\ 4WH^6 + 4W^3H^4 & H^7 + 6W^2H^5 + W^4H^{11} \end{bmatrix} \quad (47)$$

and the IOWEF is

$$A^{[\text{zh}]}(W, H) = [1 \ 0] C^J(W, H) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (48)$$

where $J = k/r$ is the total number of Hadamard segments.

For an (n, k) regular systematic RZH code with the coding rate of the outer code being $1/d_v$, based on the uniform interleaver assumption [16], the IOWE is given by

$$A_{w,h} = \sum_{d=0}^N \frac{\binom{k}{w} A_{d,h-w}^{[\text{zh}]}}{\binom{N}{wd_v}} \quad (49)$$

where $N = kd_v$ and $A_{d,h-w}^{[\text{zh}]}$ is the IOWE of the ZH code which can be calculated by (46) and (48).

For an (n, k) code, the conditional weight enumerating function (CWEF) is defined as

$$A_w(H) \triangleq \sum_{h=0}^n A_{w,h} H^h, \quad w = 0, \dots, k. \quad (50)$$

Then based on the uniform interleaver assumption, the CWEF of the PCZH code is given by

$$A_w(H) = \frac{[A_w^{[\text{zh}]}(H)]^M H^w}{\binom{k}{w}^{M-1}}, \quad w = 0, \dots, k \quad (51)$$

where $A_w^{[\text{zh}]}(H)$ is the CWEF of the punctured ZH code and M is the number of parallel components as shown in Fig. 9.

With $\{A_{w,h}\}_{w,h}$, the simple ML bound can be obtained by (23).

Note that for other Hadamard order r , it is easy to obtain $C(W, H)$ from the corresponding Hadamard matrix in case that r is not very large, and a similar procedure can be employed to compute the IOWEF.

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