Performance Analysis of Cascade Trellis-Block Space–Time Codes

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***Abstract*—This letter concerns the performance assessment of cascade trellis-block space–time codes. We focus on the case where the 2×2 orthogonal block space–time code is used as the inner code. Either bounding or hybrid bounding/simulation techniques can be used. The proposed work provides some insights into the properties of such codes.**

***Index Terms*—Hamming weights, space–time codes, union bounds.**

I. Introduction

In this letter, techniques are derived for bit-error rate (BER) performance assessment of cascade trellis-block space–time (CTBST) codes that combine the high performance of (outer) trellis codes and low complexity of (inner) block space–time (BST) codes. We only consider the 2×2 orthogonal BST code [1] as the inner code. In this case, the BER performance of CTBST codes can be evaluated easily in both fully interleaved and quasi-static Rayleigh fading channels. Such techniques serve as a simple alternative assessment method (compared with the techniques in [2]–[5], applicable for general cases), and provide useful insights into the properties of CTBST codes.

II. Systematic Overview

Fig. 1 illustrates the CTBST coding scheme under consideration. At the transmitter, the information sequence $\boldsymbol{d}$ is first encoded by a channel protection code C (outer code), generating $\boldsymbol{c} = \{c_i\}$. The interleaved version $\pi(\boldsymbol{c})$ of $\boldsymbol{c}$ is mapped to an appropriate constellation, producing $\boldsymbol{x} = \{x_j\}$. Then $\boldsymbol{x}$ is used to drive an inner orthogonal BST encoder, generating the transmitted code matrix $\boldsymbol{\Phi}$. In this letter, we consider the 2×2 orthogonal BST code [1] with Gray mapping and binary phase-shift keying (BPSK) or quaternary phase-shift keying (QPSK) modulation for two-transmit-antenna systems, i.e., $\boldsymbol{\Phi}$ is in the form $\boldsymbol{\Phi} = (\boldsymbol{\Phi}_0 \boldsymbol{\Phi}_1, \ldots, \boldsymbol{\Phi}_n, \ldots)$ with

$$\boldsymbol{\Phi}_n = \begin{pmatrix} x_{2n} & x_{2n+1} \\ -x_{2n+1}^* & x_{2n}^* \end{pmatrix} = (\boldsymbol{\varphi}_{2n} \boldsymbol{\varphi}_{2n+1}), \quad n = 0, 1, \ldots \tag{1}$$

where $\boldsymbol{\varphi}_t$ contains the two symbols transmitted over the two antennas at time t. At the receiver, maximum-likelihood (ML) decoding is carried out for the BST code [1]. The symbol-by-symbol soft outputs $\hat{\boldsymbol{x}}$ of the BST decoder are converted to bit-likelihood values $\pi(\hat{\boldsymbol{c}})$ in the demapper, as in Fig. 1. The outer decoder then decodes $\boldsymbol{c}$ using the deinterleaved version $\hat{\boldsymbol{c}}$ of $\pi(\hat{\boldsymbol{c}})$ as the *a priori* values, producing the estimate $\hat{\boldsymbol{d}}$ of $\boldsymbol{d}$.

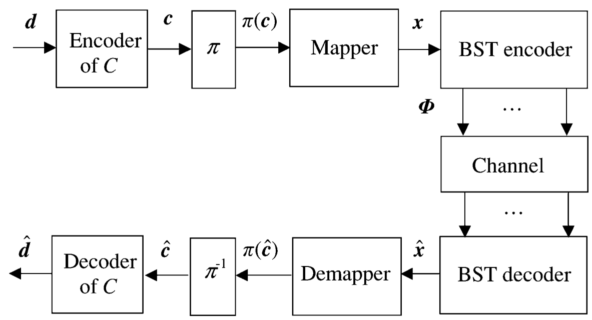

Fig. 1. Transmitter and receiver structures of the CTBST code, where π and π^{-1} denote a pair of interleaver and deinterleaver.

III. BER Performance Assessment

Let m be the number of receive antennas, and $\alpha_{i,j}(t)$ the path gain between the ith transmit antenna and the jth receive antenna at time t. $\{\alpha_{i,j}(t), \forall i, j, t\}$ are modeled as independent complex Gaussian random variables with zero mean and variance $\sigma^2 = 0.5$ per dimension.

A. Fully Interleaved Rayleigh Fading Channels

Let $\boldsymbol{\Phi}^{\pi(\boldsymbol{c})}$ be the transmitted code matrix corresponding to a codeword $\boldsymbol{c}$ of C with a particular interleaver π. Denote by $\boldsymbol{\varphi}_t^{\pi(\boldsymbol{c})} = (\varphi_{t,1}^{\pi(\boldsymbol{c})} \varphi_{t,2}^{\pi(\boldsymbol{c})})^T$ the tth column of $\boldsymbol{\Phi}^{\pi(\boldsymbol{c})}$. With Gray mapping and BPSK or QPSK modulation, it can be verified that the CTBST code in Fig. 1 is geometrically uniform [6]. Thus, we can assume that the all-zero codeword $\mathbf{0}$ of C is transmitted. Denote by $\boldsymbol{\Phi}^{\mathbf{0}}$ the transmitted code matrix corresponding to $\mathbf{0}$, and $\boldsymbol{\varphi}_t^{\mathbf{0}} = (\varphi_{t,1}^{\mathbf{0}} \varphi_{t,2}^{\mathbf{0}})^T$ the tth column of $\boldsymbol{\Phi}^{\mathbf{0}}$. Let E_s be the average symbol energy per transmit antenna. For any given π and $\{\alpha_{i,j}(t)\}$, we have

$$\begin{aligned} &\Pr(\boldsymbol{\Phi}^{\mathbf{0}} \to \boldsymbol{\Phi}^{\pi(\boldsymbol{c})} \mid \{\alpha_{i,j}(t)\}) \\ &\quad = Q\left(\sqrt{\frac{E_s}{2N_0} \cdot d^2\left(\boldsymbol{\Phi}^{\mathbf{0}}, \boldsymbol{\Phi}^{\pi(\boldsymbol{c})}\right)}\right) \\ &\quad = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \exp\left(-\frac{E_s \cdot d^2\left(\boldsymbol{\Phi}^{\mathbf{0}}, \boldsymbol{\Phi}^{\pi(\boldsymbol{c})}\right)}{4N_0 \sin^2\theta}\right) d\theta \end{aligned} \tag{2a}$$

where $N_0/2$ is the noise variance per dimension, and

$$d^2\left(\boldsymbol{\Phi}^{\mathbf{0}}, \boldsymbol{\Phi}^{\pi(\boldsymbol{c})}\right) = \sum_j \sum_t \left|\sum_i \alpha_{i,j}(t) \cdot \left(\varphi_{t,i}^{\mathbf{0}} - \varphi_{t,i}^{\pi(\boldsymbol{c})}\right)\right|^2. \tag{2b}$$

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Averaging (2a) over all possible $\{\alpha_{i,j}(t)\}$, we have

$$\Pr\left(\mathbf{\Phi}^{\mathbf{0}} \to \mathbf{\Phi}^{\pi(\boldsymbol{c})}\right) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \left\{ \prod_t \left(1 + \frac{E_s \cdot \left|\phi_t^{\mathbf{0}} - \phi_t^{\pi(\boldsymbol{c})}\right|^2}{4N_0 \sin^2\theta}\right)^{-m} \right\} d\theta. \quad (3)$$

From (1), we can verify that $|\phi_{2n}^{\mathbf{0}} - \phi_{2n}^{\pi(\boldsymbol{c})}|^2 = |\phi_{2n+1}^{\mathbf{0}} - \phi_{2n+1}^{\pi(\boldsymbol{c})}|^2 (n = 0,1,\ldots)$. For simplicity, define $I_n = |\phi_{2n}^{\mathbf{0}} - \phi_{2n}^{\pi(\boldsymbol{c})}|^2 = |\phi_{2n+1}^{\mathbf{0}} - \phi_{2n+1}^{\pi(\boldsymbol{c})}|^2$ and $\boldsymbol{c}' = \pi(\boldsymbol{c})$.

First, consider BPSK modulation $(1 \leftrightarrow -1, 0 \leftrightarrow +1)$. In this case, $I_n (n = 0,1,\ldots)$ has only three possible values, depending on (x_{2n}, x_{2n+1}), i.e., (c'_{2n}, c'_{2n+1}).

Case 1) $(x_{2n}, x_{2n+1}) = (-1,-1)$ [i.e., $(c'_{2n}, c'_{2n+1}) = (1,1)$], then $I_n = 8$.

Case 2) $(x_{2n}, x_{2n+1}) = (+1,-1)$ or $(-1,+1)$ [i.e., $(c'_{2n}, c'_{2n+1}) = (0,1)$ or $(1,0)$], then $I_n = 4$.

Case 3) $(x_{2n}, x_{2n+1}) = (+1,+1)$ [i.e., $(c'_{2n}, c'_{2n+1}) = (0,0)$], then $I_n = 0$.

Let h be the Hamming weight of $\boldsymbol{c}$ and so $\boldsymbol{c}'$. Assume that there are i pairs of (c'_{2n}, c'_{2n+1}) in $\boldsymbol{c}'$ belonging to Case 1 and j pairs belonging to Case 2. (The values of i and j depend on h and the interleaver π.) Then we have $2i + j = h$ and, from (3)

$$\Pr\left(\mathbf{\Phi}^{\mathbf{0}} \to \mathbf{\Phi}^{\pi(\boldsymbol{c})}\right) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \left\{ \left(1 + \frac{8E_s}{4N_0\sin^2\theta}\right)^{-2mi} \cdot \left(1 + \frac{4E_s}{4N_0\sin^2\theta}\right)^{-2mj} \right\} d\theta. \quad (4)$$

The pairwise error probability of transmitting $\mathbf{0}$ and deciding in favor of $\boldsymbol{c}$ can be calculated by averaging (4) over all possible interleavers, i.e., all possible (i,j) pairs

$$\Pr(\mathbf{0} \to \boldsymbol{c}) = \sum_{i,j} \Pr(i,j \,|\, 2i+j=h) \cdot \Pr\left(\mathbf{\Phi}^{\mathbf{0}} \to \mathbf{\Phi}^{\pi(\boldsymbol{c})}\right) = \frac{1}{\pi}\sum_{i,j} \left\{ \Pr(i,j \,|\, 2i+j=h) \cdot \int_0^{\frac{\pi}{2}} \left[\left(1 + \frac{8E_s}{4N_0\sin^2\theta}\right)^{-2mi} \cdot \left(1 + \frac{4E_s}{4N_0\sin^2\theta}\right)^{-2mj} \right] d\theta \right\} \quad (5)$$

where $\Pr(i,j|2i+j=h)$ is the occurrence probability of a particular pair of (i,j) conditioned on $2i+j=h$. With the uniform interleaver [7], we have

$$\Pr(i,j \,|\, 2i+j=h) = \binom{N/2}{i} \times \binom{N/2 - i}{j} \times 2^j \Big/ \binom{N}{h} \quad (6)$$

where N is the length of the interleaver π. We then obtain the union bound on BER as follows [7]:

$$P_b \le \sum_{\boldsymbol{c}\neq\mathbf{0}} \frac{w}{K}\Pr(\mathbf{0} \to \boldsymbol{c}) = \frac{1}{\pi}\sum_{w,h} \left\{ A_{w,h}\frac{w}{K}\sum_{i,j}\Pr(i,j|2i+j=h) \cdot \int_0^{\frac{\pi}{2}} \left[\left(1 + \frac{8E_s}{4N_0\sin^2\theta}\right)^{-2mi} \cdot \left(1 + \frac{4E_s}{4N_0\sin^2\theta}\right)^{-2mj} \right] d\theta \right\} \quad (7)$$

where w is the information weight of $\boldsymbol{c}$, $A_{w,h}$ the number of codewords of C with information weight w and overall weight h, and K the information length.

For QPSK modulation, the union bound on BER of the CTBST code can be obtained similarly. In this case, I_n has five possible values: 8, 6, 4, 2, and 0

$$P_b \le \frac{1}{\pi}\sum_{w,h} \left\{ A_{w,h}\frac{w}{K}\sum_{i,j,k,l}\Pr(i,j,k,l \,|\, 4i+3j+2k+l=h) \cdot \int_0^{\frac{\pi}{2}} \left[\left(1+\frac{8E_s}{4N_0\sin^2\theta}\right)^{-2mi} \cdot \left(1+\frac{6E_s}{4N_0\sin^2\theta}\right)^{-2mj} \cdot \left(1+\frac{4E_s}{4N_0\sin^2\theta}\right)^{-2mk} \cdot \left(1+\frac{2E_s}{4N_0\sin^2\theta}\right)^{-2ml} \right] d\theta \right\} \quad (8)$$

where

$$\Pr(i,j,k,l|4i+3j+2k+l=h) = \binom{\frac{N}{4}}{i} \cdot \binom{\frac{N}{4} - i}{j} \cdot 4^j \cdot \binom{\frac{N}{4} - i - j}{k} \cdot 6^k \cdot \binom{\frac{N}{4} - i - j - k}{l} \cdot 4^l \Big/ \binom{N}{h}. \quad (9)$$

Fig. 2 shows the union bounds and simulation results of CTBST codes with uniform interleavers in fully interleaved Rayleigh fading channels. We use two trellis codes as outer codes: a zigzag code [8] with four constituent codes and four information bits per segment, and a convolutional code with generators $(554, 744)_8$. It is seen that the bound results agree well with the simulation results in the error-floor range for both codes. We also observe that the error floor of the zigzag-based CTBST code is dominated by those $A_{w,h}$ with small w and h. This property is very similar to that of the outer zigzag code in AWGN channels [8].

B. Quasi-Static Rayleigh Fading Channels

With quasi-static fading, $\{\alpha_{i,j}(t), \forall t\}$ remain unchanged within a frame $\mathbf{\Phi}$, so we simply denote them by $\alpha_{i,j}$. After the

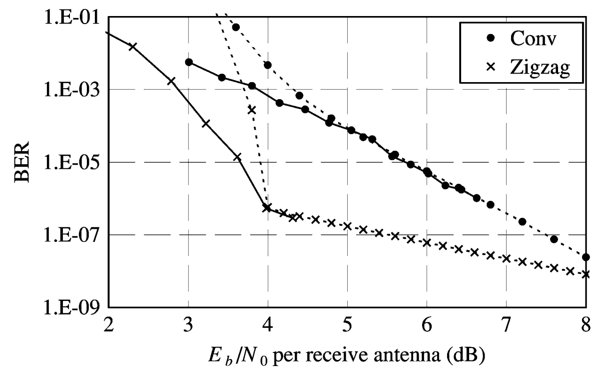

Fig. 2. Union bounds on BER of the CTBST codes in fully interleaved Rayleigh fading channels. Zigzag [8] and convolutional codes are used as the outer code C. Both codes have information length of 512 bits and coding rate of $1/2$. The number of receive antennas $= 1$. QPSK modulation is used. Solid lines are for the simulation results and dashed lines for the union bound results. The zigzag-based CTBST code uses 20 iterations for the outer code.

orthogonal translation, the received signal of the CTBST code at time t can be represented as [1]

$$\begin{aligned} y_t &= \sum_{j=1}^{m} (|\alpha_{1,j}|^2 + |\alpha_{2,j}|^2)\sqrt{E_s}x_t \\ &\quad + \sum_{j=1}^{m} (\alpha_{1,j}^{*}\eta_{t,1,j} + \alpha_{2,j}\eta_{t,2,j}) \\ t &= 0, 1, 2, \ldots \end{aligned} \tag{10}$$

where $\eta_{t,1,j}$ and $\eta_{t,2,j}$ are independent complex Gaussian noises with zero mean and variance $\sigma^2 = N_0/2$ per dimension. This is equivalent to transmitting $\boldsymbol{x}$ (the modulated version of $\pi(\boldsymbol{c})$) in a single-antenna system in AWGN channels with symbol signal-to-noise ratio (SNR)

$$\begin{aligned} \mathrm{SNR} &= \frac{\left(\sum_{j=1}^{m}\left(|\alpha_{1,j}|^2 + |\alpha_{2,j}|^2\right)\right)^2 \cdot E_s}{\sum_{j=1}^{m}\left(|\alpha_{1,j}|^2 + |\alpha_{2,j}|^2\right) \cdot N_0} \\ &= \sum_{j=1}^{m}\left(|\alpha_{1,j}|^2 + |\alpha_{2,j}|^2\right) \cdot \frac{E_s}{N_0} = g \cdot \frac{E_s}{N_0} \end{aligned} \tag{11}$$

where g is a chi-square distributed random variable with probability density function (pdf) $p(g) = g^{2m-1}e^{-g}/(2m-1)!$. Therefore, the BER performance of the CTBST code in quasi-static Rayleigh fading channels can be assessed by averaging that of C (after proper modulation) in AWGN channels over g, i.e.,

$$\begin{aligned} &\mathrm{BER}_{\mathrm{quasi-static}}(E_s/N_0) \\ &\qquad = \int p(g) \times \mathrm{BER}_{\mathrm{AWGN}}(g \cdot E_s/N_0)\, dg \end{aligned} \tag{12}$$

where $\mathrm{BER}_{\mathrm{AWGN}}(\cdot)$ is the function of BER versus SNR for C (after modulation) in AWGN channels. Various techniques can be used to upper bound $\mathrm{BER}_{\mathrm{AWGN}}(\cdot)$, such as 1) the union bound [7] and 2) the tangential-sphere bound [9]. Fig. 3 shows the results of these two methods. We can see that both methods fail to provide tight bounds. This is due to the fact that both of these bounds on $\mathrm{BER}_{\mathrm{AWGN}}(\cdot)$ are loose or even diverge in the low SNR range. (We have clipped both bounds to 0.5 in the divergent parts). A simple (but not strictly a bounding) technique to get round the problem is to approximate $\mathrm{BER}_{\mathrm{AWGN}}(\cdot)$ using the simulated performance of C in AWGN channels for the low SNR range. This method is referred to as method 3. Since the simulation and union bound results usually merge in the error-floor range in AWGN channels, we can use the former before the merging point and the latter afterwards. Method 3 is useful if one wants to predict the performance of a CTBST code based on C, and the simulation and bound results of C in AWGN channels are readily available. The accuracy of this method is also illustrated in Fig. 3.

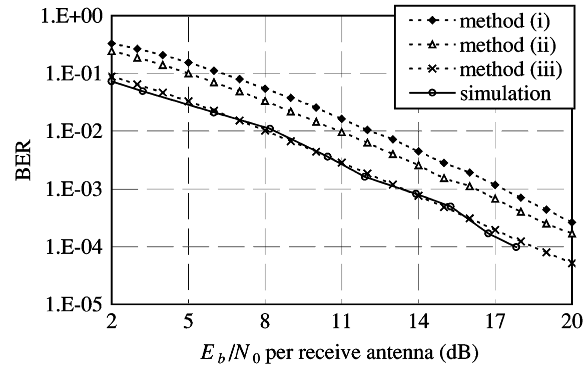

Fig. 3. Comparison of various performance assessment methods in quasi-static Rayleigh fading channels. The same zigzag-based CTBST code as that in Fig. 2 is used. The number of receive antennas $= 1$.

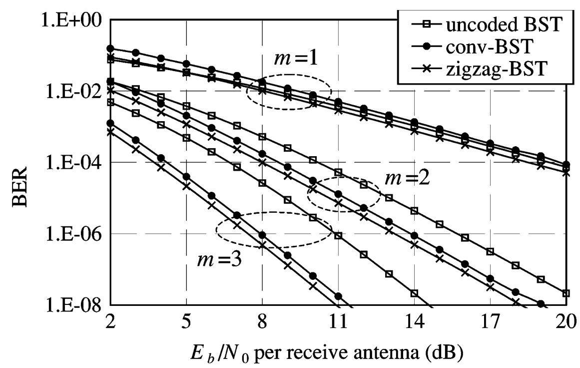

Fig. 4. Performance comparison among the two CTBST codes in Fig. 2 and the uncoded orthogonal 2×2 BST code [1] in quasi-static Rayleigh fading channels, based on method 3. Numbers of receive antennas are marked on the figure.

Fig. 4 illustrates the effect of the receive antenna number m. Comparison is made between the two CTBST codes in Fig. 2 and the uncoded 2×2 orthogonal BST code [1]. For clarity, we only include the results based on method 3, whose accuracy has been validated in Fig. 3. It is seen that the outer code brings very little benefit when $m = 1$. This is because with one receive antenna, the performance is dominated by deep fades, where systems with or without channel coding perform equally badly. With more receive antennas, the advantage of the CTBST codes over the uncoded BST code becomes significant. This is expected, since the probability of simultaneous deep fades on all receive antennas reduces rapidly when m increases.

IV. Conclusion

The techniques discussed in this letter can provide simple and accurate performance assessments for CTBST codes. They are especially useful in the high SNR range, where simulation can be very time consuming.

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