High-performance filter networks and symmetric matrix systems

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Abstract: A unified investigation is presented for low-sensitivity and limit-cycle-free filter structures. It is shown that important properties like boundedness and pseudopassivity are closely related to the symmetry of the system matrix description. Negative elements can be incorporated in certain prototypes leading to real advantages in switched-capacitor (*SC*) realisations. Stability and noise problems are also discussed. According to the realisation, component variations can result in matrix symmetry being maintained, or more generally, in the introduction of assymmetry. Sensitivity considerations are outlined for both situations. Implementation by passive *RLC*, SC and digital circuits are considered.

1 Introduction

There are a number of attractive features about filter structures derived from passive *RLC* network simulations: they show very low sensitivity in the passband, which is an important factor for active-*RC* and switched-capacitor (*SC*) filter fabrications [1–7]. They can be made limit-cycle free for digital-filter implementation, as shown for wave structures [8–10], and they usually have better dynamic range compared with cascade biquads or other direct-form structures, which can be observed from many practical designs. Limit-cycle suppression and better dynamic range can improve the noise behaviour of the circuits.

Theories have been proposed to analyse and generalise the properties of passive ladders and their simulations [10, 14]. A unified investigation has been proposed in References 17 and 18 for digital circuits. It was shown that general low-sensitivity filters can be constructed by properly connecting LBR (lossless-bounded-real) sections, which include adaptors for wave digital circuits as specific examples. In general, this approach is mainly concerned with the topological point of view.

This paper investigates the problem of high-quality network design based on matrix principles. Attention is given to the properties of the system descriptions of the circuits. It is shown that matrix symmetry is a crucial factor to ensure optimal performance of the systems. Two concepts considered by many other authors, boundedness and pseudopassivity, are proved to be closely related to the matrix symmetry.

A difference between the topological [8–18] and matrix approaches is that the former analyses the behaviour of local building blocks, whereas the latter examines the overall system. The two approaches complement each other to provide insight into the filter design problem.

The matrix system discussed in this paper can be used to produce prototypes for various implementations. Detailed realisation methods are discussed elsewhere [20–22]. It is also shown that the symmetric matrix system is a generalised concept of a passive network, allowing negative elements. Examples will be given to show that advantages can be gained for *SC* and digital simulations.

The problem of sensitivity behaviour for asymmetric deviations is also investigated. In active-*RC* or *SC* implementations the component deviations may destroy the symmetry of the system description. From practical observations the sensitivities of active-*RC* and *SC* ladder simulations are nevertheless very good; this is attributed to their multifeedback nature. Sensitivity formulas are presented which clearly indicate that better performance is assured by more complete symmetry.

2 Basic concepts

The concept of boundedness can be traced back to an observation by Orchard about the low-sensitivity properties of doubly terminated ladders [1].

Definition 1. Boundedness: The transfer function $H(\boldsymbol{P})$ of a system is said to be bounded with respect to the change of a set of parameters, $\boldsymbol{P} = \{p_i\}$, if there is a positive number M and

$$|H(\boldsymbol{P})| \leqslant M \tag{1}$$

is always satisfied when $\boldsymbol{P}$ varies within the allowed range.

When a bounded system is properly designed to make $|H(\boldsymbol{P})|$ attain M at a frequency point in the passband $j\omega_m$, then the deviation of $\boldsymbol{P}$ can only cause $|H(\boldsymbol{P})|$ to decrease. This means that $|H(\boldsymbol{P})|$ must have zero derivative with respect to any parameter p_i at $j\omega_m$, and consequently the sensitivity is also zero, i.e.

$$S_{p_i}^{|H|} = \frac{p_i}{|H(\boldsymbol{P})|}\frac{\partial |H(\boldsymbol{P})|}{\partial p_i} = 0 \quad \text{at } s = j\omega_m \tag{2}$$

and it may be reasonably expected that over the whole passband the sensitivity will remain small, a reassuring argument used by many other authors for ladders as well as various simulation methods [23–25].

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The generalised concept of pseudopassivity has been employed in discussion for wave digital filters [10], which is, in fact, based on the principle of the Lyapunov function. Consider a standard state-space system in the continuous domain

$$sX = AX + BJ \tag{3}$$

or in the discrete domain

$$X = z^{-1}AX + BJ \tag{4}$$

Definition 2. Pseudopassivity: A state-space system eqn. 3 or 4 is said to be pseudopassive if

$$e(t) = x^T(t)x(t) \tag{5}$$

is a monotonically decreasing function for any initial value $x(0) = X_0$ with $J = 0$. (For a discrete system $x(t)$ is examined at a discrete instance, i.e. $t = nT$).

$e(t)$ can be seen as an energy function and it is always decreasing for a pseudopassive system without excitation. The pseudopassive property in a discrete system is important for the suppression of parasitic oscillations. If the input $J = 0$, the state-space variables $x(nT)$, and so all the variables, in a stable digital system (eqn. 4) will approach zero regardless of the initial state in the ideal linear case. However, when the necessary quantisations are adopted in a digital filter, $x(nT)$ may oscillate and take nonzero values due to nonlinear effects, which may even cover the entire number range in the filter when overflow occurs. These parasitic oscillations, so-called limit cycles, can be avoided if the discrete system is pseudopassive and magnitude rounding for quantisation of $x(nT)$ is adopted. In magnitude rounding, a number a is truncated to a finite number of bits $Q[a]$, with $|Q[a]| \leqslant |a|$. Let $Q[x]$ denote the vector of x after magnitude rounding, and suppose in a pseudopassive system (eqn. 4) that these are the only quantisation operations, then according to eqn. 5

$$Q^T[x(nT)]Q[x(nT)] \leqslant x^T(nT)x(nT) \leqslant Q^T[x((n-1)T)] \times Q[x((n-1)T)] \leqslant x((n-1)T)x((n-1)T) \leqslant \cdots \tag{6}$$

Therefore, if $x(nT) \to 0$ in the ideal case, then in the non-ideal case it will still approach zero. This will completely suppress limit cycles [10, 18].

The second norm of a matrix A is given by [27]

$$\|A\| = \max_{x \neq 0} \frac{x^T A^T A x}{x^T x} \tag{7}$$

The time-domain equation (eqn. 4) gives (when $J = 0$)

$$x(n) = Ax(n-1) \tag{8}$$

Hence, from eqns. 7 and 8 a necessary and sufficient condition for pseudopassivity is

$$\|A\| \leqslant 1 \tag{9}$$

In this case

$$x^T(n+k)x(n+k) \leqslant \cdots \leqslant x^T(n)x(n) = x^T(n-1)A^T Ax(n-1) \leqslant x^T(n-1)x(n-1) \tag{10}$$

It has been proved on a topological basis that the condition in eqn. 9 is met by wave, normalised-lattice and LBR structures [18], and the same concept has been used in the design of second-order 'minimum norm' building blocks [19]. In Section 4 it will be shown that higher order networks, based on a symmetric matrix decomposition approach, can also be designed to meet this condition.

3 Continuous symmetric matrix systems

Consider the matrix system of the following form:

$$YV = J \tag{11a}$$

with

$$Y = sC + s^{-1}\Gamma + G \tag{11b}$$

Output functions may be added in the form

$$y = DV + EJ \tag{12}$$

but only the system in eqn. 11 will be considered, since sensitivity and noise problems arise mainly from the feedback loops in eqn. 11.

Eqn. 11 is a generalised form of the standard state-space equation (eqn. 3). Indeed, eqn. 11 is reduced to eqn. 3 when $\Gamma = 0$. Alternatively, eqn. 11 can always be rearranged into the form of eqn. 3 by introducing some intermediate variables. However, the advantage of using the system description of eqn. 11 is that optimal performance can be achieved by imposing some simple conditions (notably symmetry) on the matrix. If the matrices in eqn. 3 are constrained to be symmetric, then the system can only have real poles, which is too restrictive for most applications.

The most convenient way to set up the system in eqn. 11 is the formulation of the network equations (nodal, loop or hybrid) of a general passive *RLC* ladder, designed either by a synthesis process or with the help of tables. It can be easily shown that in this case all the matrices of eqn. 11*b* can be made non-negative if nodal or loop formulations are used. For more general cases, an optimisation procedure can be used to adjust the entries of the matrices of eqn. 11 to make the transfer function fit the prescribed specifications. In this case conditions are required for testing the stability of the resulting system.

3.1 Critical stability

It can be shown that the system in eqn. 11 is critically stable if C, Γ and G are all symmetric non-negative. Let $\{s_m = \sigma_m + j\omega_m\}$ be the set of roots of det $Y(s)$ of eqn. 11

$$\det(s_m^2 C + s_m G + \Gamma) = 0 \tag{13}$$

So there is a non-zero vector X which satisfies [27] the equation

$$X^*(s_m^2 C + s_m G + \Gamma)X = 0 \tag{14}$$

(X^* denotes the transposed conjugate of X) or

$$as_m^2 + bs_m + c = 0 \tag{15}$$

with

$$a = X^*CX \quad b = X^*GX \quad c = X^*\Gamma X \tag{16}$$

As C, G and Γ are all definite non-negative, a, b and c are all non-negative numbers [27]. But in this case, eqn. 15 has no roots with

$$\mathrm{Re}\,(s_m) = \sigma_m > 0 \tag{17}$$

That is, the system in eqn. 11 has no poles in the right-half plane if C, Γ and G are all symmetric non-negative.

3.2 Absolute stability

The absolute stability condition for the system in eqn. 11 is that $\sigma_m < 0$ for all m. Therefore, some extra constraints should be added to ensure that no roots lie on the imaginary axis. This can be checked by evaluating $\det|Y(j\omega)|$. In most cases the system in eqn. 11 is designed to realise a transfer function $H(s)$ which has no poles on the imaginary axis. If the system is properly designed without redundancy so that the order of the system is equal to the order of $H(s)$, or in other words it is observable from the output, then it will have no poles on the imaginary axis either, as in this case $H(s)$ and the system have the same set of poles.

The non-negative property of the symmetric matrices $\boldsymbol{C}$, $\boldsymbol{\Gamma}$ and $\boldsymbol{G}$ can be easily tested. For instance, decompose $\boldsymbol{C}$ into symmetric LU form [28]

$$\boldsymbol{C} = \boldsymbol{L}_c \boldsymbol{D}_c \boldsymbol{L}_c^T \tag{18}$$

where $\boldsymbol{D}_c$ is a diagonal matrix. $\boldsymbol{C}$ is non-negative if, and only if, all the entries of $\boldsymbol{D}_c$ are non-negative. The computational requirement for this test is nearly equal to performing Gaussian elimination.

3.3 Boundedness

From network topology it is known that the output power of a doubly terminated ladder is bounded by maximum input power, a reasonable fact since a passive ladder cannot create power within itself. This result can also apply to the system in eqn. 11 in a more abstract sense. Let eqn. 11 be evaluated on the imaginary axis $s = j\omega$ and denote

$$\boldsymbol{Q} = \omega \boldsymbol{C} - \omega^{-1}\boldsymbol{\Gamma} \tag{19}$$

The system can be written as

$$\boldsymbol{Y}\boldsymbol{V} = (j\boldsymbol{Q} + \boldsymbol{G})\boldsymbol{V} = \boldsymbol{J} \tag{20}$$

Suppose matrix $\boldsymbol{G}$ in eqn. 11 can be separated according to input and output parts, respectively,

$$\boldsymbol{G} = \boldsymbol{G}_{in} + \boldsymbol{G}_{out} \tag{21}$$

Then eqn. 20 can be written as

$$j\boldsymbol{Q}\boldsymbol{V} + \boldsymbol{G}_{out}\boldsymbol{V} + \boldsymbol{G}_{in}\boldsymbol{V} = \boldsymbol{J} \tag{22}$$

We first prove a general relation.

Theorem 1: Assume that in eqn. 11
(i) $\boldsymbol{G}_{in}\boldsymbol{X} = \boldsymbol{J}$ has at least one solution
(ii) all matrices are symmetric non-negative.

Then the following inequality holds:

$$\boldsymbol{V}^*\boldsymbol{G}_{out}\boldsymbol{V} \leqslant \tfrac{1}{4}\boldsymbol{J}^*\boldsymbol{R}_{in}\boldsymbol{J} \tag{23}$$

where $\boldsymbol{R}_{in}$ is the Moore-Penrose inverse of $\boldsymbol{G}_{in}$.

Proof: According to Moore-Penrose's theories [28, 29], $\boldsymbol{R}_{in}$ is defined by

$$\begin{aligned} &\boldsymbol{G}_{in}\boldsymbol{R}_{in}\boldsymbol{G}_{in} = \boldsymbol{G}_{in} \qquad \boldsymbol{R}_{in}\boldsymbol{G}_{in}\boldsymbol{R}_{in} = \boldsymbol{R}_{in} \\ &(\boldsymbol{G}_{in}\boldsymbol{R}_{in})^T = \boldsymbol{G}_{in}\boldsymbol{R}_{in} \quad (\boldsymbol{R}_{in}\boldsymbol{G}_{in})^T = \boldsymbol{R}_{in}\boldsymbol{G}_{in} \end{aligned} \tag{24}$$

and $\boldsymbol{X}_s = \boldsymbol{R}_{in}\boldsymbol{J}$ is a solution of $\boldsymbol{G}_{in}\boldsymbol{X} = \boldsymbol{J}$, if it has a solution at all, which means

$$\boldsymbol{G}_{in}\boldsymbol{R}_{in}\boldsymbol{J} = \boldsymbol{J} \tag{25}$$

Now multiply eqn. 11 by $\boldsymbol{V}^*$

$$j\boldsymbol{V}^*\boldsymbol{Q}\boldsymbol{V} + \boldsymbol{V}^*\boldsymbol{G}_{out}\boldsymbol{V} + \boldsymbol{V}^*\boldsymbol{G}_{in}\boldsymbol{V} = \boldsymbol{V}^*\boldsymbol{J} \tag{26}$$

Take the real part of eqn. 26

$$\boldsymbol{V}^*\boldsymbol{G}_{out}\boldsymbol{V} = \mathrm{Re}\,\{\boldsymbol{V}^*\boldsymbol{J}\} - \boldsymbol{V}^*\boldsymbol{G}_{in}\boldsymbol{V} \tag{27}$$

Notice from eqns. 24 and 25

$$\begin{aligned} &[\boldsymbol{J}^*\boldsymbol{R}_{in}\boldsymbol{J} - (\boldsymbol{J}^* - 2\boldsymbol{V}^*\boldsymbol{G}_{in})\boldsymbol{R}_{in}(\boldsymbol{J} - 2\boldsymbol{G}_{in}\boldsymbol{V})]/4 \\ &= [2\boldsymbol{J}^*\boldsymbol{R}_{in}\boldsymbol{G}_{in}\boldsymbol{V} - 2\boldsymbol{V}^*\boldsymbol{G}_{in}\boldsymbol{R}_{in}\boldsymbol{J} - 4\boldsymbol{V}^*\boldsymbol{G}_{in}\boldsymbol{R}_{in}\boldsymbol{G}_{in}\boldsymbol{V}]/4 \\ &= \mathrm{Re}\,\{\boldsymbol{V}^*\boldsymbol{J}\} - \boldsymbol{V}^*\boldsymbol{G}_{in}\boldsymbol{V} \end{aligned} \tag{28}$$

From eqns. 27 and 28

$$\begin{aligned} \boldsymbol{V}^*\boldsymbol{G}_{out}\boldsymbol{V} = [&\boldsymbol{J}^*\boldsymbol{R}_{in}\boldsymbol{J} \\ &- (\boldsymbol{J}^* - 2\boldsymbol{V}^*\boldsymbol{G}_{in})\boldsymbol{R}_{in}(\boldsymbol{J} - 2\boldsymbol{G}_{in}\boldsymbol{V})]/4 \end{aligned} \tag{29}$$

If $\boldsymbol{G}_{in}$ is non-negative, then from eqn. 24 $\boldsymbol{R}_{in}$ is also non-negative, which means that $(\boldsymbol{J}^* - 2\boldsymbol{Y}^*\boldsymbol{G}_{in})\boldsymbol{R}_{in}(\boldsymbol{J} - 2\boldsymbol{G}_{in}\boldsymbol{V})$ is a non-negative number. Theorem 1 follows from eqn. 29 immediately.

3.4 Boundedness for terminated reactance network

Eqn. 23 is a general expression which can be applied to multi-input/output systems. To provide some insight of its physical meaning, consider the special case of a single-input/output system. Suppose eqn. 11 has only one input $\boldsymbol{J} = [J_1, 0, \ldots, 0]$ and one output v_n. $\boldsymbol{G}_{in}$ and $\boldsymbol{G}_{out}$ have only one non-zero diagonal entry, respectively, corresponding to the input and output, i.e.

$$\boldsymbol{G}_{in} = \mathrm{diag}\,(g_{11}, 0, 0, \ldots, 0) \tag{30a}$$

$$\boldsymbol{G}_{out} = \mathrm{diag}\,(0, 0, \ldots, 0, g_{nn}) \tag{30b}$$

Then $\boldsymbol{R}_{in}$ can be generated by

$$\boldsymbol{R}_{in} = \mathrm{diag}\,(g_{11}^{-1}, 0, 0, \ldots, 0) \tag{31}$$

Therefore, in this case eqn. 29 is reduced to

$$g_{nn}|v_n|^2 = [1 - |1 - 2g_{11}v_1/J_1|^2]\,\frac{1}{4g_{11}}\,|J_1|^2 \tag{32}$$

so

$$g_{nn}|v_n|^2 \leqslant \frac{1}{4g_{11}}\,|J_1|^2 \tag{33}$$

or

$$|v_n| \leqslant \frac{1}{2(g_{11}g_{nn})^{1/2}}\,|J_1| \tag{34}$$

A typical example of the system constrained by the conditions of eqn. 30 is a doubly terminated ladder (Fig. 1),

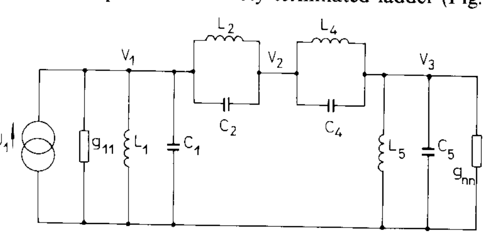


Fig. 1 *Terminated LC ladder passive prototype*

in which case (eqn. 11) are its nodal equations with input and output nodes labelled 1 and n, respectively. The physical meaning of eqn. 32 can be seen by rewriting it as

$$g_{nn}|v_n|^2 = [1 - |\rho|^2]g_{11}^{-1}|J_1|^2/4 \tag{35}$$

with ρ defined by

$$\rho = 1 - 2g_{11}v_1/J_1 \tag{36}$$

Consider a passive ladder with the source resistor being $r_{11} = g_{11}^{-1}$ and input impedance of the two-port ladder including the load is $z_{in} = y_{in}^{-1}$

$$\begin{aligned} \rho &= 1 - 2g_{11}v_1/J_1 = 1 - \frac{2g_{11}}{y_{in} + g_{11}} \\ &= \frac{y_{in} - g_{11}}{y_{in} + g_{11}} = \frac{r_{11} - z_{in}}{r_{11} + z_{in}} \end{aligned} \tag{37}$$

So ρ is just the reflection function and the upper bound of $|v_n|$ is attained at $\rho = 0$. This result is well known in network theory.

In the proof of the boundedness, no condition has been imposed on $\boldsymbol{C}$ and $\boldsymbol{\Gamma}$ except that they must be symmetric and non-negative. Accordingly, zero-sensitivity with respect to symmetric deviation can be achieved at the frequency points where the transfer function attains its upperbound.

4 Sensitivity formulas

The above result provides only an estimation of sensitivity for symmetric deviations. More general sensitivity formulas are now derived. To simplify the problem, only single-input/output system will be considered.

Suppose a single-input/output system meets the conditions of theorem 1 and eqn. 30. Let the system in eqn. 11*a* be excited by another arbitrary input $\boldsymbol{K}$ instead of $\boldsymbol{J}$ and let the response be $\boldsymbol{U}$. The system can be written as

$$\boldsymbol{YU} = \boldsymbol{K} \tag{38}$$

then the output of the new system is related to the old system by

$$u_n = (2g_{nn}\bar{v}_n)^{-1}(\bar{\rho}\boldsymbol{V}^T - \boldsymbol{V}^*)\boldsymbol{K} \tag{39}$$

($\bar{}$ indicates conjugate). The derivation of eqn. 39 is given in Appendix 10.1.

4.1 Sensitivity formulas

Differentiate eqn. 11*a* w.r.t. some network element ξ to get

$$\boldsymbol{Y}\,d\boldsymbol{V}/d\xi + \boldsymbol{V}\,d\boldsymbol{Y}/d\xi = 0 \tag{40}$$

Here the second term can be viewed as the new input vector for eqn. 38 and we have

Theorem 2:

$$dv_n/d\xi = (2g_{nn}\bar{v}_n)^{-1}(-\bar{\rho}\boldsymbol{V}^T + \boldsymbol{V}^*)\,d\boldsymbol{Y}/d\xi\,\boldsymbol{V} \tag{41a}$$

and

$$\begin{aligned} d|v_n|/d\xi &= \mathrm{Re}\,[\bar{v}_n\,dv_n/d\xi]/|v_n| \\ &= (2g_{nn}|v_n|)^{-1}\,\mathrm{Re}\,[(-\bar{\rho}\boldsymbol{V}^T + \boldsymbol{V}^*)\,d\boldsymbol{Y}/d\xi\,\boldsymbol{V}] \end{aligned} \tag{41b}$$

In particular, if the deviation of ξ only perturbs the imaginary part of $\boldsymbol{Y}$, $j\boldsymbol{X}$ say, and $d\boldsymbol{Y}/d\xi = j\,d\boldsymbol{X}/d\xi$ is symmetric, then

$$\begin{aligned} d|v_n|/d\xi &= (2g_{nn}|v_n|)^{-1}\,\mathrm{Re}\,[-j\bar{\rho}\boldsymbol{V}^T\,d\boldsymbol{X}/d\xi\,\boldsymbol{V} \\ &\quad + j|\boldsymbol{V}^*\,d\boldsymbol{X}/d\xi\,\boldsymbol{V}|] \\ &= (2g_{nn}|v_n|)^{-1}\,\mathrm{Re}\,[-j\bar{\rho}\boldsymbol{V}^T\,d\boldsymbol{X}/d\xi\,\boldsymbol{V}] \\ &= -(2g_{nn}|v_n|)^{-1}\,\mathrm{Im}\,[\rho\boldsymbol{V}^T\,d\boldsymbol{X}/d\xi\,\boldsymbol{V}] \end{aligned} \tag{42}$$

So $d|v_n|/d\xi = 0$ when $\rho = 0$. This again confirms the conclusion for single-input/output system, that $|v_n|$ attains its upper bound and has zero-sensitivity at $\rho = 0$, if the deviation is symmetric.

4.2 Application to passive networks

When the system in eqn. 11 is implemented by a real passive *RLC* network, $\xi \in \{R_i, L_i, C_i\}$ and $d\boldsymbol{Y}/d\xi$ is always symmetric. Then a very simple alternative to the topological derivation of sensitivity follows: let the contribution to $\boldsymbol{Y}$ of a branch admittance jq_k between nodes a, b be $jq_k\boldsymbol{M}_{ab}$, where

$$\boldsymbol{M}_{ab} = \begin{bmatrix} 0 & & \vdots & \vdots & & 0 \\ \cdots & \ddots & 1 & -1 & \cdot^{\cdot^{\cdot}} & \cdots \\ \cdots & \cdot^{\cdot^{\cdot}} & -1 & 1 & \ddots & \cdots \\ 0 & & \vdots & \vdots & & 0 \end{bmatrix} \begin{matrix} \\ a \\ b \\ \\ \end{matrix} \tag{43}$$
$$\qquad\qquad a \quad b$$

So

$$d\boldsymbol{Y}/dq_k = j\boldsymbol{M}_{ab} \tag{44}$$

It is easily seen that

$$\boldsymbol{V}^T\boldsymbol{M}_{ab}\boldsymbol{V} = v_k^2 \quad \text{and} \quad \boldsymbol{V}^*\boldsymbol{M}_{ab}\boldsymbol{V} = |v_k|^2 \tag{45}$$

where v_k is the voltage across jq_k. Then eqns. 41*a* and *b* are reduced to

$$dv_n/dq_k = (2g_{nn}|v_n|)^{-1}(-j\bar{\rho}v_k^2 + |v_k|^2) \tag{46a}$$

and

$$d|v_n|/dq_k = (2g_{nn}|v_n|)^{-1}\,\mathrm{Im}\,[\bar{\rho}v_k^2] \tag{46b}$$

Eqn. 46*b* is zero at the frequency points where $\rho = 0$ or equivalently $|v_n|$ attains maximum bound. This is just the well known zero-sensitivity property for doubly terminated ladders. An alternative derivation of eqn. 46 based on a topological approach is given elsewhere [31].

4.3 Application to digital and active networks

In the following Sections it will be shown that the system in eqn. 11 can be simulated by digital or active networks. For digital simulations, even in non-ideal cases, it is still possible to keep deviations in $\boldsymbol{Y}$ symmetric by carefully selecting the coefficient quantisations, so the zero-sensitivity property can be preserved. For active-*RC* and *SC* simulations, it is difficult to keep deviations of $\boldsymbol{Y}$ symmetric, since the element value drift is a random phenomenon. The component drift may cause the equivalent system description (eqn. 11) to become nonsymmetric, so that the output may exceed the bound given by eqn. 23 or 34. However, in practical active-*RC* or *SC* implementations, low sensitivity is still observed, a property due to the multifeedback nature of the structures. Eqns. 41*a* and *b* are valid for these general cases.

5 Discrete symmetric matrix systems

The results in the previous Section can be readily extended to the discrete domain if a bilinear transformation is applied to the system in eqn. 11

$$\boldsymbol{YV} = \boldsymbol{J} \tag{47a}$$

with

$$\boldsymbol{Y} = \Psi\boldsymbol{C} + \Psi^{-1}\boldsymbol{\Gamma} + \boldsymbol{G} \tag{47b}$$

where

$$\Psi = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{47c}$$

Eqn. 47 can be rearranged as

$$(\boldsymbol{P} + z^{-1}\boldsymbol{Q} + z^{-2}\boldsymbol{R})\boldsymbol{V} = (1 - z^{-2})\boldsymbol{J} \tag{48a}$$

with

$$P = \left(\frac{2}{T}C + \frac{T}{2}\Gamma + G\right)$$

$$Q = -2\left(\frac{2}{T}C - \frac{T}{2}\Gamma\right) \qquad (48b)$$

$$R = \left(\frac{2}{T}C + \frac{T}{2}\Gamma - G\right)$$

which can be seen as a generalised form of the standard state-space equation (eqn. 4), by introducing a second-order term.

As bilinear transformation will keep the stability property and map the imaginary axis in the s domain to the unit circle in the z domain, it is easy to show:

Remark: The system in eqn. 47 has no poles outside the unit circle $z = e^{j\omega T}$ if C, Γ and G are all symmetric non-negative and has the same boundedness and sensitivity properties as indicated by theorems 1 and 2, except that eqn. 47 is evaluated on the unit circle $z = e^{j\omega T}$.

As indicated in Section 2, the pseudopassive property is of particular interest for discrete systems in order to suppress parasitic oscillations. Consider the problem of constructing a pseudopassive state-space system from eqn. 47, which can be written in an equivalent form

$$\left(\frac{2}{T}C + \frac{T}{2}\Gamma + G\right)V + \left(\frac{2}{T}\frac{-2z^{-1}}{1+z^{-1}}C + \frac{T}{2}\frac{2z^{-1}}{1-z^{-1}}\Gamma\right)V = J \qquad (49)$$

Let C and Γ be decomposed into symmetric forms.

$$\frac{2}{T}C = L_C L_C^T \quad \frac{T}{2}\Gamma = L_\Gamma L_\Gamma^T \qquad (50a)$$

Define

$$X = \begin{bmatrix} X_C \\ X_\Gamma \end{bmatrix} = \begin{bmatrix} \dfrac{2z^{-1}}{1+z^{-1}} L_C \\ \dfrac{-2z^{-1}}{1-z^{-1}} L_\Gamma \end{bmatrix} V \qquad (50b)$$

From eqns. 49 and 50

$$V = \left(\frac{2}{T}C + \frac{T}{2}\Gamma + G\right)^{-1}\left\{[L_C L_\Gamma]\begin{bmatrix} X_C \\ X_\Gamma \end{bmatrix} + J\right\} \qquad (51)$$

Substituting eqn. 51 into eqn. 50 we get a state-space description

$$X = z^{-1}AX + z^{-1}BU \qquad (52)$$

with

$$A = 2\begin{bmatrix} L_C^T \\ -L_\Gamma^T \end{bmatrix}\left(\frac{2}{T}C + \frac{T}{2}\Gamma + G\right)^{-1}[L_C L_\Gamma] + \begin{bmatrix} -I & \\ & I \end{bmatrix} \qquad (53a)$$

$$B = \begin{bmatrix} L_\Gamma^T \\ -L_\Gamma^T \end{bmatrix}\left(\frac{2}{T}C + \frac{T}{2}\Gamma + G\right)^{-1} \qquad (53b)$$

Theorem 3: $\|A\| \leqslant 1$ if C, Γ and G are non-negative.

Proof: First, only if C and Γ are both non-netative can the decompositions of eqn. 5 be carried out. Substantial manipulation of eqns. 50 and 53a gives

$$A^T A = \begin{bmatrix} I & \\ & I \end{bmatrix} - 4\begin{bmatrix} L_C^T \\ L_\Gamma^T \end{bmatrix}\left(\frac{2}{T}C + \frac{T}{2}\Gamma + G\right)^{-T} G \times \left(\frac{2}{T}C + \frac{T}{2}\Gamma + G\right)^{-1}[L_C L_\Gamma] \qquad (54)$$

From eqns. 51 and 54 it can be seen that the following relationships hold when $J = [0]$:

$$x^T(n)A^T A x(n) = x^T(n)x(n) - v^T(n)Gv(n) \qquad (55a)$$

If G is non-negative, then $v^T(n)Gv(n)$ is a non-negative number, and therefore

$$x^T(n)A^T A x(n) \leqslant x^T(n)x(n) \qquad (55b)$$

no matter what the value of $x(n)$. The theorem follows from eqns. 7 and 9. Incidentally, from eqn. 54 it can be seen that matrix A is orthogonal if $G = [0]$.

Conditions for the continuous time-domain systems have no direct practical applications; however, for completeness a derivation is given in 10.2 Appendix.

6 Circuit implementations

In the previous Sections effort has been given to the investigation of the theoretical properties of symmetric matrix systems (eqn. 11) and their discrete form (eqn. 47). Now the circuit implementations of these systems by various techniques, general *RLC* networks, active-*RC*, *SC* and digital networks, are considered. The discussion is brief as *RLC* and wave circuit designs are well known and matrix methods for circuit design have presented in other recent publications.

6.1 General passive RLC implementations

Eqn. 11 can always be derived from a nodal-voltage or loop-current formulation of a passive *RLC* network. In a nodal-voltage approach the contribution of every component is indicated by eqn. 43. The reverse procedure, from an equation with symmetric non-negative matrices to a network, is not always possible unless negative element values are allowed. Advantage can be gained by permitting negative elements [7], as regular structures easier for fabrication may result. According to the last Section, negative elements would not appear to cause any special sensitivity problems.

6.2 Active-RC circuit implementations

Let matrices C and Γ in eqn. 11 be decomposed into the form, $C = L_C L_C^T$, $\Gamma = L_\Gamma L_\Gamma^T$. Then eqn. 11 can be written as the left-LUD form

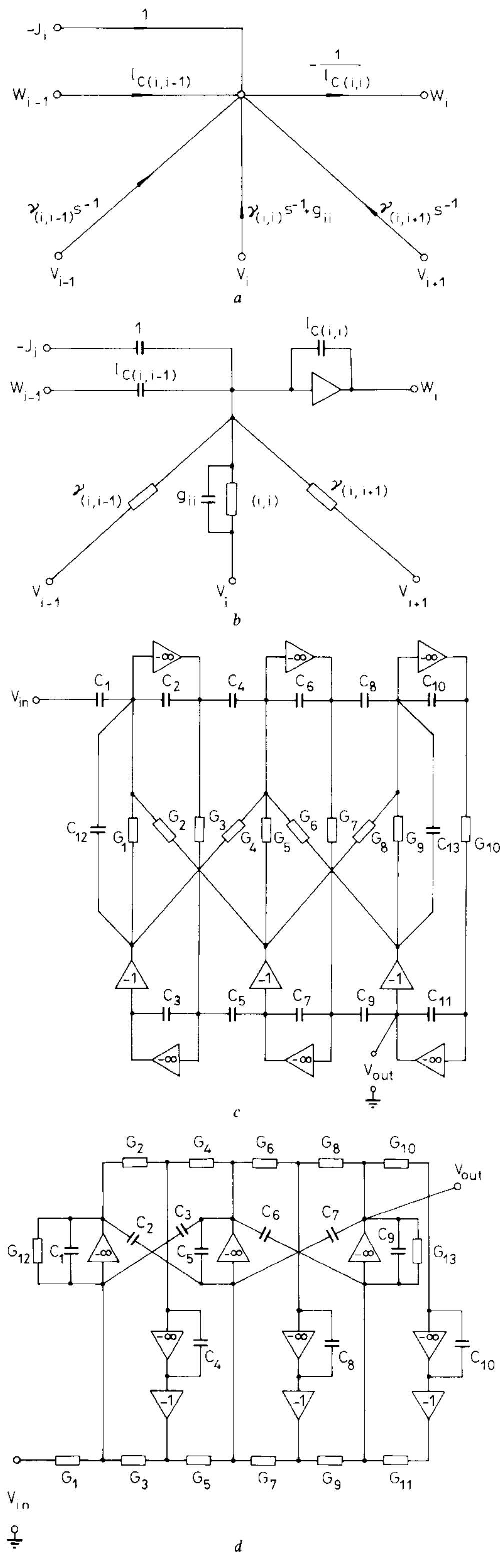
$$\begin{cases} L_C W = -(s^{-1}\Gamma + G)V + J & (56a) \\ L_C^T V = s^{-1}W & (56b) \end{cases}$$

or the right-LUD form

$$\begin{cases} (sC + G)V = -L_\Gamma U + J & (57a) \\ U = s^{-1}L_\Gamma^T V & (57b) \end{cases}$$

Both eqns. 56 and 57 can be realised directly by active-*RC* circuits. Every non-zero entry is realised by a corresponding element. To illustrate the implementation technique, consider a typical row equation in eqn. 56a

$$(l_{c(i,i-1)} + l_{c(i,i)})w_i = -s^{-1}[\gamma_{(i,i-1)}v_{i-1} + \gamma_{(i,i)}v_i + \gamma_{(i,i+1)}v_{i+1}] + g_{ii}v_i + J_i \qquad (58)$$



This can be represented by a signal-flow graph (SFG) (Fig. 2*a*) and the corresponding active circuit, (Fig. 2*b*). The overall active-*RC* simulations of the passive prototype (Fig. 1) are shown in Figs. 2*c* and *d*.

In an example of sensitivity analysis for a right-LUD form (eqn. 57), ξ corresponds to a single entry c_{ij} in $\boldsymbol{C}$, then from eqn. 41*a*

$$\begin{aligned} dv_n/dc_{ij} &= (2g_{nn}\bar{v}_n)^{-1}(-\bar{p}\boldsymbol{V}^T + \boldsymbol{V}^*)j\omega\, d\boldsymbol{C}/d\xi\, \boldsymbol{V} \\ &= (2g_{nn}\bar{v}_n)^{-1} j\omega(-\bar{p}v_i v_j + \bar{v}_i v_j) \end{aligned} \quad (59)$$

Similar formulas can be derived in the same way for other elements.

For the lowpass case, the right-LUD system (eqn. 57) results in identical circuit structures to those derived from a leapfrog approach. The left-LUD simulation (eqn. 56) has a significant advantage for elliptic transfer functions in that there are no capacitor-coupled op amp loops in the circuit implementation. The right-LUD simulation may produce such loops, and high-frequency oscillations may exist in these loops, resulting in undesirable noise problems.

6.3 Switched capacitor circuit implementations

Eqn. 47 is equivalent to

$$\left(\boldsymbol{P} + 4\frac{-1}{(1-z^{-1})}\frac{z^{-1}}{(1-z^{-1})}\frac{T}{2}\,\boldsymbol{\Gamma} + 2\frac{z^{-1}}{1-z^{-1}}\boldsymbol{G}\right)\boldsymbol{V} = \frac{1+z^{-1}}{1-z^{-1}}\boldsymbol{J} \quad (60)$$

As matrix $\boldsymbol{P}$ is always symmetric, it can be decomposed into the form

$$\boldsymbol{P} = \boldsymbol{L}_p \boldsymbol{L}_p^T \quad (61)$$

Similar to active-*RC* design, eqn. 60 can be written in the left-LUD form

$$\begin{cases} \boldsymbol{L}_p\boldsymbol{W} = -\left(\dfrac{2}{1-z^{-1}}\dfrac{T}{2}\boldsymbol{\Gamma} + 2\boldsymbol{G}\right)\boldsymbol{V} + (1+z)\boldsymbol{J} & (62a) \\ \boldsymbol{L}_p^T\boldsymbol{V} = \dfrac{2z^{-1}}{1-z^{-1}}\boldsymbol{W} & (62b) \end{cases}$$

or the right-LUD form (with $\boldsymbol{L}_\Gamma \boldsymbol{L}_\Gamma^T = T/2\,\boldsymbol{\Gamma}$)

$$\begin{cases} \boldsymbol{P}\boldsymbol{V} = -\dfrac{2}{1-z^{-1}}(\boldsymbol{L}_\Gamma \boldsymbol{U} + \boldsymbol{G}\boldsymbol{V}) + \dfrac{1+z^{-1}}{1-z^{-1}}\boldsymbol{J} & (63a) \\ \boldsymbol{U} = \dfrac{2z^{-1}}{1-z^{-1}}\boldsymbol{L}_\Gamma^T\boldsymbol{V} & (63b) \end{cases}$$

A number of new realisations have been derived in References 20–21. Here only an application of adopting negative elements in the prototype design is discussed. When the prototype relation (eqn. 11) is produced by nodal-

Fig. 2

a SFG representation of a continuous domain equation
b Corresponding active-RC realisation
c Left-LUD type active-RC realisation
d Right-LUD type active-RC realisation

voltage simulation of a passive ladder, matrix $\boldsymbol{P}$, given by eqn. 48, is generally not diagonal. The design procedures according to eqn. 62 or 63 require cross-coupling capacitors to realise the non-zero off-diagonal entries of $\boldsymbol{P}$. However, in an all-pole case, negative elements can be introduced to simplify the circuit structure. If every series-inductor branch L_i in a Chebyshev ladder has a parallel negative capacitor, C_i, added according to the relation

$$\frac{2}{T} C_i = -\frac{T}{2} \frac{1}{L_i} \tag{64}$$

This is shown by dotted lines in Fig. 3*a*. Then, from eqn. 48*b*, it can be verified that the off-diagonal entries of $\boldsymbol{C}$

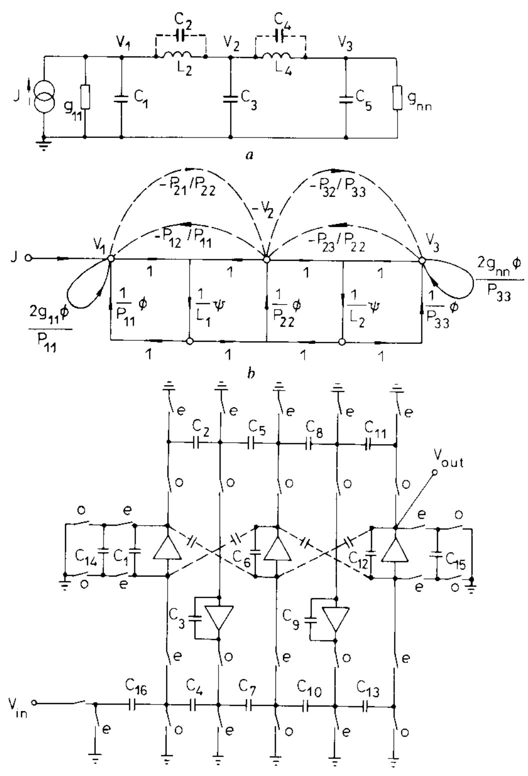


Fig. 3

a Terminated all-pole LC ladder prototype
Component values for normalised ladder
$C_1 = 0.93106$ $C_2 = -0.073694$ $L_2 = 1.4326$
$C_3 = 1.9453$ $C_4 = -0.071313$ $L_4 = 1.4804$
$C_5 = 1.5760$ $g_{11} = g_{nn} = 1$
b Leapfrog (or right-LUD) simulation

$$\psi = \frac{z^{-1}}{1 - z^{-1}}$$

$$\phi = \frac{-2T}{1 - z^{-1}}$$

c Leapfrog SCF simulation of the circuit in Fig. 3A
Capacitance values for the SC ladder
$C_1 = C_2 = C_{10} = C_{11} = C_{15} = 1$
$C_3 = 3.410$ $C_4 = 1.658$ $C_5 = 1.352$
$C_6 = 3.125$ $C_7 = 1.194$ $C_8 = 1.846$
$C_9 = 4.029$ $C_{12} = 1.925$ $C_{13} = 1.769$
$C_{14} = 1.072$ $C_{16} = 1.547$

and Γ will cancel each other and make $\boldsymbol{P}$ diagonal. This will remove the coupling capacitor in the simulation and save on the fabrication cost. However, zeros are introduced by these series branches and are given by

$$s^2 = \frac{-1}{L_i C_i} = \frac{4}{T^2} \tag{65}$$

A response error is thereby incurred, but this can be eliminated in the approximation procedure, by placing transmission zeros on the real axis. Examples of a right-LUD SFG and *SC* circuit are shown in Fig. 3*b* and *c*. The input stage is modified to realise $z^{-1}/(1 - z^{-1})$. The resulting distortion is compensated, together with that caused by real zeros. The response is shown in Fig. 4. If

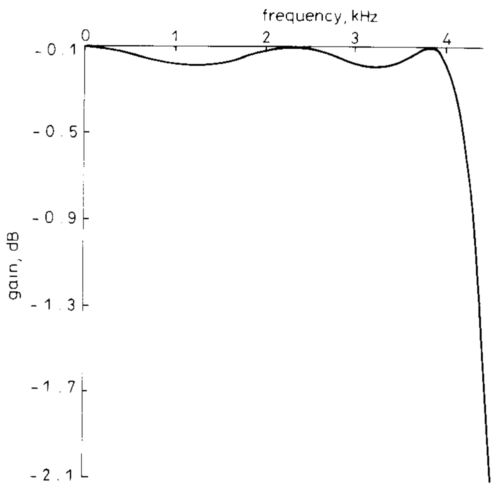


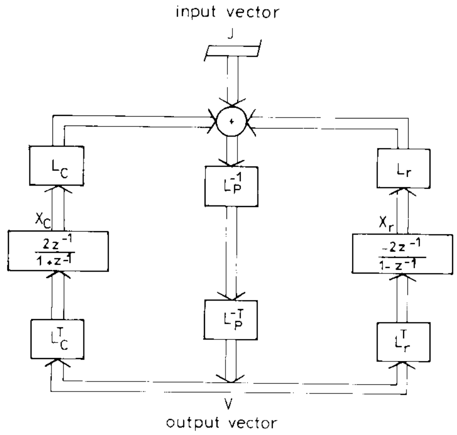
Fig. 4 *Simulated response of the circuit in Fig. 3C*

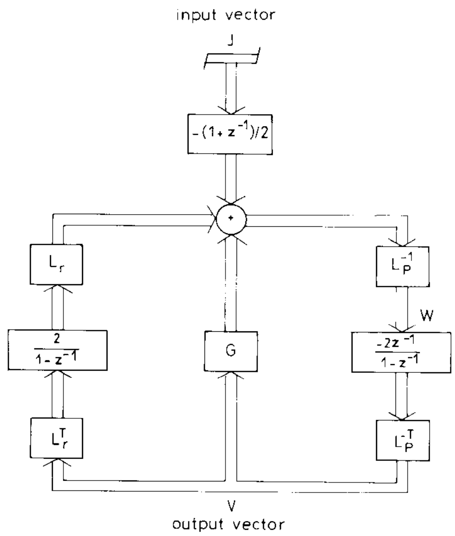
the off-diagonal entries of $\boldsymbol{P}$ are not zero, then the dotted branches in the SFG and the *SC* circuit would be required. Notice that in this lowpass case the right-LUD structure corresponds with that produced by a leapfrog approach.

6.4 Digital implementations

6.4.1 Matrix method: Eqn. 52 has already given one realisation of the system (eqn. 47). Direct implementation of the multiplication of $\boldsymbol{X}$ and $\boldsymbol{U}$ by $\boldsymbol{A}$ and $\boldsymbol{B}$ can require a high number of multiplications and additions, as usually $\boldsymbol{A}$ and $\boldsymbol{B}$ are full matrices. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are decomposed according to eqn. 53, the multiplications of $\boldsymbol{A}$ and $\boldsymbol{B}$ are then carried out by a sequence of multiplications by factors. Efficiency is achieved by taking into consideration matrix sparsity. Numerical methods, such as LU factorisation, can be employed in the multiplications by $(2/T\,\boldsymbol{C} + T/2\,\boldsymbol{\Gamma} + \boldsymbol{G})^{-1}$ (Fig. 5*a*). This method could have advantages for implementation on concurrent array processors. However, for conventional single-processor implementation an excessive number of multiplications are required.

Digital structures can also be derived from eqn. 62, since there are no delay-free loops in the corresponding signal-flow graph (Fig. 5*b*). This approach has advantages concerning the properties of sensitivity behaviour, parallelism and the ability to be pipelined [22]. However, no efficient way to suppress parasitic oscillation for this type of structure has yet been found. As pseudopassivity is only a sufficient condition for limit-cycle suppression, a possible solution should not be ruled out.





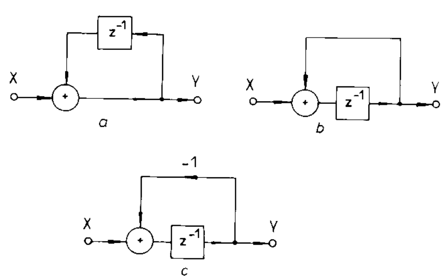


Fig. 5

a Block realisation of wave-related structure
b Block realisation of LDI-related structure
c Realisation of frequency-dependent factors

$$a\quad Y = \frac{1}{1 - z^{-1}} X$$

$$b\quad Y = \frac{z^{-1}}{1 - z^{-1}} X$$

$$c\quad Y = \frac{z^{-1}}{1 + z^{-1}} X$$

6.4.2 Topological method: As indicated above, the prototype system (eqn. 11) can always be realised by a general passive *RLC* ladder. Consequently, a indirect realisation of system (eqn. 47) can be

(i) determine eqn 11 from eqn. 47 according to the bilinear transformation relationship

(ii) construct a general passive circuit from eqn. 11

(iii) design a wave digital circuit from the general passive circuit.

In Appendix 10.3 it is shown that the rounding of wave variables is equivalent to the rounding of the state variables given by eqn. 50 to keep the pseudopassivity property of the system. This agrees with the early theories of wave circuits using the property of adaptors. It can also be shown that the wave approach represents a special decomposition of matrices $\boldsymbol{A}$, $\boldsymbol{B}$ in a topological way.

The results of this paper can also be applied to the circuits containing not only adaptors. For example, a standard wave circuit may require a excessive number of delays [25]. Techniques have been proposed for the design of wave filters with a canonical number of delays [9], resulting in a circuit structure which does not follow the rules of the connection of adaptors [25]. But, according to the discussion of Section 5 and Appendix 10.3, provided rounding is carried out at the points where wave variables can be accessed, limit cycles can still be suppressed.

It is interesting to point out that the discrete system (eqn. 47) can be rearranged into two equivalent forms, that of eqn. 49 and that of eqn. 60. Using these two different forms as the basis of derivation produces two major families of circuits, wave [8] and LDI [11]. In wave approaches the basic frequency-dependent factors are $-z^{-1}/(1 - z^{-1})$ and $z^{-1}/(1 + z^{-1})$ (Appendix 10.3). On the other hand, in LDI approaches the basic frequency-dependent factors are $-1/(1 - z^{-1})$ and $z^{-1}/(1 - z^{-1})$, as shown in eqn. 60 which can be seen as a stable and exact LDI-type realisation. The system expression of eqn. 49 involves matrices $\boldsymbol{P}$, $\boldsymbol{C}$ and $\boldsymbol{\Gamma}$, whereas that of eqn. 60 involves matrices $\boldsymbol{P}$, $\boldsymbol{\Gamma}$ and $\boldsymbol{G}$. Usually, matrix $\boldsymbol{C}$ is more complicated than $\boldsymbol{G}$. Therefore, implementation based on eqn. 60 may result in simpler structures. However, no simple method for oscillation suppression has been found for standard LDI-type [11–12] digital structures nor for the exact structures based on eqn. 60. Therefore, structures based on eqn. 49 are suitable for digital design, if both sensitivity and noise due to number-truncation-induced oscillations are considered. Realisations based on eqn. 60 are interesting for other discrete systems, in particular for SC filter design, due to the fact that the two LDI factors coincide with the transfer functions of a pair of stray-insensitive integrators, resulting in efficient fabrications [6].

7 Conclusions

A family of low-sensitivity/noise systems have been studied for both continuous and discrete filter design. It is shown that symmetric matrix systems can be designed with optimal performance. Sensitivity can be minimised if the deviation of component values is kept symmetric, which is possible for a digital-filter design by carefully selecting the coefficient truncations. In fact, it can be proved that for some structures like wave circuits the coefficient truncations will always result in symmetric deviations.

For active-RC and SC circuits, asymmetric deviations may occur. Efficient sensitivity analysis and optimisation can thus be carried out utilising theorem 2.

The matrix decomposition method for filter design is the subject of continuing research and other articles will describe the application of these theoretical studies to the practical synthesis of traditional and novel circuit structures.

8 Acknowledgments

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9 References

1 ORCHARD, H.J.: 'Inductorless filters', *Electron. Lett.*, June 1966, **2**, pp. 224–225
2 ANTONIOU, A.: 'Realisation of gyrators using operational amplifiers and their use in RC-active network synthesis', *Proc. IEE*, 1969, **126**, (11), pp. 1838–1850
3 GIRLING, F.E.J., and GOOD, E.F.: 'Active filters: part 12. The leapfrog or active ladder synthesis', *Wireless World*, July 1970, **76**, pp. 341–345
4 GIRLING, F.E.J., and GOOD, E.F.: 'Active filters: part 13. Application of active ladder synthesis', *Wireless World*, September 1970, **76**, pp. 445–450
5 LEE, M.S., and CHANG, C.: 'Switched-capacitor filters using the LDI and bilinear transformations', *IEEE Trans.*, 1981, **CAS-28**, (4), pp. 265–270
6 GREGORIAN, R., MARTIN, K.W., and TEMES, G.C.: 'Switched-capacitor circuit design', *Proc. IEEE*, 1983, **71**, (8), pp. 941–966
7 SIGG, R.P., KAELIN, A., MURALT, A., BLACK, W.C., and MOSCHYTZ, G.S. JR.: 'Automated SC generation with a CMOS gate-array type filter chip'. Proceedings of ISCAS, Espoo, Finland, June 1988, pp. 1725–1728
8 FETTWEIS, A.: 'Digital filter structure related to classical filter networks', *Arch. Elektr. Uebertrag.*, 1971, **25**, (2), pp. 78–89
9 FETTWEIS, A.: 'Canonic realisation of wave digital filters', *Int. J. Circuit Theory Appl.*, 1975, **3**, (4), pp. 321–332
10 FETTWEIS, A.: 'Wave digital filters: theory and practice', *Proc. IEEE*, February 1986, **74**, (2), pp. 270–326
11 BRUTON, L.T.: 'Low sensitivity digital ladder filters', *IEEE Trans.*, 1975, **CAS-22**, (3), pp. 168–176
12 TURNER, L.E., and RAMESH, B.K.: 'Low sensitivity digital LDI ladder filter with elliptic magnitude response', *IEEE Trans.*, 1986, **CAS-33**, (7), pp. 697–706
13 GRAY, A.H., JR., and MARKEL, J.D.: 'Digital lattice and ladder filter synthesis', *IEEE Trans.*, **AU-21**, (12), pp. 491–500
14 GRAY, A.H., JR.: 'Passive cascaded lattice digital filters', *IEEE Trans.*, 1980, **CAS-27**, (5), pp. 337–344
15 DEWILDE, P., and DEPRETTERE, E.: 'Orthogonal cascade realization of real multiport digital filters', *Int. J. Circuit Theory Appl.*, 1980, **8**, pp. 245–277
16 BUTTERWECK, H.J., VAN MEER, A.C.P., and VERKROOST, G.: 'New second-order digital filter section without limit cycles', *IEEE Trans.*, 1984, **CAS-31**, (2), pp. 141–146
17 VAIDYANATHAN, P.P., and MITRA, S.K.: 'Low passband sensitivity digital filters: a general viewpoint and synthesis procedure', *Proc. IEEE*, 1984, **72**, (4), pp. 404–423
18 VAIDYANATHAN, P.P., and MITRA, S.K.: 'Passivity properties of low-sensitivity digital filter structures', *IEEE Trans.*, 1985, **CAS-32**, (3), pp. 217–224
19 BARNES, C.W., and FAM, A.T.: 'Minimum norm recursive digital filters that are free of overflow limit cycles', *IEEE Trans.*, 1977, **CAS-24**, (10), pp. 569–574
20 PING, L., and SEWELL, J.I.: 'The LUD approach to switched capacitor network design', *IEEE Trans.*, 1987, **CAS-34**, (12), pp. 1611–1614
21 PING, L., HENDERSON, R.K., and SEWELL, J.I.: 'Matrix methods for switched-capacitor filter design'. Proceedings of IEEE ISCAS, Espoo, Finland, June 1988, pp. 1021–1024
22 PING, L., and SEWELL, J.I.: 'Filter realisation by passive circuit simulations', *Proc. IEE G*, 1988, **135**, (4), pp. 167–176
23 SEDRA, A.S., and BRACKETT, P.O.: 'Filter theory and design: active and passive' (Portland, OR, USA, Matrix, 1978)
24 GHAUSI, M.S., LAKER, K.R.: 'Modern filter design: active RC and switched-capacitor' (Prentice-Hall, Englewood Cliffs, New Jersey, USA, 1981)
25 ANTONIOU, A.: 'Digital filters: analysis and design' (McGraw-Hill Inc., New York, 1979)
26 CHEN, W.K.: 'Applied graph theory' (North-Holland, Amsterdam, The Netherlands, 1976)
27 WILKINSON, J.H.: 'The algebraic eigenvalue problem' (Oxford University Press, 1965)
28 MORRIS, J.L.: 'Computational methods in elementary numerical analysis' (John Wiley & Sons Ltd., 1983)
29 ALBERT, A.: 'Regression and the Moore-Penrose pseudoinverse' (Academic Press, New York, 1972)
30 RAO, C.R., and MITRA, S.K.: 'Generalized inverse of matrices and its application' (John Wiley & Sons Ltd., New York, 1971)
31 ORCHARD, H.J., TEMES, G.C., and CAEALTEPE, T.: 'Sensitivity formulas for terminated lossless two-ports', *IEEE Trans.*, 1985, **CAS-32**, (5), pp. 459–466

10 Appendix

10.1 Derivation of the perturbed system output equation

Commence with

$$\boldsymbol{YU} = \boldsymbol{K} \tag{66}$$

Left multiply by $\boldsymbol{V}^*$ gives

$$\boldsymbol{V}^*\boldsymbol{YU} = \boldsymbol{V}^*\boldsymbol{K} \tag{67}$$

Note that when $\boldsymbol{J}$ is a real vector

$$\begin{aligned}(\boldsymbol{V}^*\boldsymbol{Y})^T &= \boldsymbol{Y}\bar{\boldsymbol{V}} = (j\boldsymbol{Q} + \boldsymbol{G})\bar{\boldsymbol{V}}\\ &= -(-j\boldsymbol{Q} + \boldsymbol{G})\bar{\boldsymbol{V}} + 2\boldsymbol{G}\bar{\boldsymbol{V}}\\ &= 2\boldsymbol{G}\bar{\boldsymbol{V}} - \boldsymbol{J}\\ &= (2g_{11}\bar{v}_1 - J_1, 0, \ldots, 0, 2g_{nn}\bar{v}_n)^T\\ &= (\bar{\rho}J_1, 0, \ldots, 0, 2g_{nn}\bar{v}_n)^T\end{aligned} \tag{68}$$

Substitute eqn. 68 into eqn. 67 and make some rearrangement to get

$$2g_{nn}\bar{v}_n u_n - \bar{\rho}J_1 u_1 + \boldsymbol{V}^*\boldsymbol{K} = 0 \tag{69}$$

Again left multiplying eqn. 66 by $\boldsymbol{V}^T$ and noticing that $\boldsymbol{V}^T\boldsymbol{Y} = \boldsymbol{J}^T = [J_1, 0, \ldots, 0]$ we have

$$\boldsymbol{V}^T\boldsymbol{K} = \boldsymbol{V}^T\boldsymbol{YU} = \boldsymbol{J}^T\boldsymbol{U} = J_1 u_1 \tag{70}$$

Eqn. 39 follows by substituting eqn. 70 into eqn. 69.

10.2 Continuous time-domain pseudopassive systems

We first show that a continuous time system (eqn. 3) is pseudopassive if $-(\boldsymbol{A} + \boldsymbol{A}^T)$ is non-negative. Set $\boldsymbol{J} = \boldsymbol{0}$ in eqn. 11. Then the time-domain solution is given by

$$\boldsymbol{x}(t) = \exp(\boldsymbol{A}t)\boldsymbol{x}_0 \tag{71}$$

where $\boldsymbol{x}_0$ is the initial value vector. Take the derivative of $e(t) = \boldsymbol{x}^T(t)\boldsymbol{x}(t)$

$$de/dt = \boldsymbol{x}_0^T \exp(\boldsymbol{A}t)^T(\boldsymbol{A} + \boldsymbol{A}^T)t \exp(\boldsymbol{A}^T t)\boldsymbol{x}_0 \tag{72}$$

$e(\boldsymbol{x}(t))$ is monotonically decreasing if $de/dt \leqslant 0$, or equivalently, the system in eqn. 11 is pseudopassive if $-(\boldsymbol{A} + \boldsymbol{A}^T)$ is non-negative.

Let a state space system $\boldsymbol{X} = s^{-1}\boldsymbol{AX} + \boldsymbol{L}_C^{-1}\boldsymbol{J}$ be constructed from eqn. 11 by

$$\boldsymbol{X} = \begin{pmatrix}\boldsymbol{X}_C\\ \boldsymbol{X}_L\end{pmatrix} = \begin{pmatrix}\boldsymbol{L}_C^T\\ \boldsymbol{L}_\Gamma^T\end{pmatrix}\boldsymbol{V} \tag{73}$$

with

$$\boldsymbol{C} = \boldsymbol{L}_C\boldsymbol{L}_C^T \quad \Gamma = \boldsymbol{L}_\Gamma\boldsymbol{L}_\Gamma^T \tag{74a}$$

and

$$\boldsymbol{A} = \begin{bmatrix}-\boldsymbol{L}_C^{-1}\boldsymbol{G}\boldsymbol{L}_C^{-T} & -\boldsymbol{L}_C^{-1}\boldsymbol{L}_\Gamma\\ \boldsymbol{L}_\Gamma^T\boldsymbol{L}_C^{-T} & \boldsymbol{0}\end{bmatrix} \tag{74b}$$

It is easily seen that

$$-(A + A^T) = 2\begin{bmatrix} L_C^{-1} G L_C^{-T} & 0 \\ 0 & 0 \end{bmatrix} \tag{75}$$

is non-negative. The pseudopassivity of eqn. 73 follows immediately.

10.3 Wave variables

When eqn. 11 is derived from a passive ladder by nodal formulation, the matrices can be generated by topological means [26]

$$C = A_C D_C A_C^T \tag{76a}$$

$$\Gamma = A_\Gamma D_\Gamma A_\Gamma^T \tag{76b}$$

where D_C and D_Γ are diagonal branch-admittance matrices with entries consisting of the corresponding capacitance or inverse inductance values. A_C and A_Γ are the corresponding incidence matrices. Let V_C, I_C, V_Γ and I_Γ be vectors of the voltages and currents of the capacitance and inductance branches, respectively, then the voltage vectors are related to the nodal-voltage vector V by

$$V_C = A_C^T V \tag{77a}$$

$$V_\Gamma = A_\Gamma^T V \tag{77b}$$

The current vectors are related to the nodal-voltage vector by

$$I_C = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} D_C A_C^T V \tag{78a}$$

$$I_\Gamma = \frac{T}{2}\frac{1 + z^{-1}}{1 - z^{-1}} D_\Gamma A_\Gamma^T V \tag{78b}$$

According to the definition of wave variables; incident wave vectors are

$$W_{CI} = V_C + \left(\frac{2}{T} D_C\right)^{-1} I_C = \frac{2}{1 + z^{-1}} A_C^T V \tag{79a}$$

$$W_{\Gamma I} = V_\Gamma + \left(\frac{T}{2} D_\Gamma\right)^{-1} I_\Gamma = \frac{2}{1 - z^{-1}} A_\Gamma^T V \tag{79b}$$

and reflected wave vectors are

$$W_{CR} = V_C - \left(\frac{2}{T} D_C\right)^{-1} I_C$$

$$= \frac{2z^{-1}}{1 + z^{-1}} A_C^T V = z^{-1} W_{CI} \tag{80a}$$

$$W_{\Gamma R} = V_\Gamma - \left(\frac{T}{2} D_\Gamma\right)^{-1} I_\Gamma$$

$$= \frac{-2z^{-1}}{1 - z^{-1}} A_\Gamma^T V = -z^{-1} W_{\Gamma I} \tag{80b}$$

By comparing eqns. 50, 76 and 80 it can be found that

$$x^T(n)x(n) = w_{CR}^T(n)\frac{2}{T} D_C w_{CR}(n) + w_{\Gamma R}^T(n)\frac{T}{2} D_\Gamma w_{\Gamma R}(n) \tag{81}$$

When the branch admittance matrices D_C and D_Γ are diagonal with positive element values, the magnitude rounding of w_{CR} and $w_{\Gamma R}$ will have the same effect as magnitude rounding on $x(n)$ to cause a reduction of $x^T(n)x(n)$.