Modified LMMSE Turbo Equalization

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***Abstract*—This letter presents a modified linear minimum mean square error (LMMSE) turbo equalization scheme that uses an augmented real matrix representation for quadrature modulation systems. In the proposed scheme, the estimates of the two quadrature components of the transmitted symbol have both their individual variances and their covariance considered. Hence, the new scheme is able to achieve a significant performance improvement while retaining a complexity similar to that of the existing LMMSE turbo equalization scheme.**

***Index Terms*—Intersymbol interference (ISI), iterative methods, turbo equalization.**

I. INTRODUCTION

INSPIRED by turbo codes [1], turbo equalization was introduced to improve the performance of digital communication systems with intersymbol interference (ISI) [2]. A turbo equalization system consists of two soft-in-soft-out elements: a channel equalizer and a channel decoder. These two elements are operated in an iterative manner. However, the computational complexity of the conventional *maximum a posteriori* probability (MAP) equalizer is very high [2]–[4]. A low-cost approach based on a linear minimum mean square error (LMMSE) equalizer was proposed in [5], [6]. It is assumed implicitly in [5], [6] that the estimates of the two quadrature components of the transmitted symbol have the same conditional variance and zero covariance. This assumption simplifies the derivation but it potentially introduces a performance degradation.

In this letter, we present a modified LMMSE equalizer based on an augmented real matrix representation. The two quadrature components are described by a joint Gaussian distribution and both their conditional variances and covariance are estimated. Compared with [5], [6], the proposed scheme provides a more accurate modeling of the statistics of the two components and can achieve improved performance without increasing receiver cost.

II. SYSTEM DESCRIPTION

The communication system considered in this letter is shown in Fig. 1. A binary input data sequence is encoded using a convolutional code and permuted into a coded sequences $\{c_j\}$ by an interleaver π. The coded sequence is then partitioned into segments. Each segment is of length Q and is mapped into a symbol x_n selected from a quadrature symbol constellation $\boldsymbol{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_{2Q}\}$. The resultant symbol sequences $\{x_n\}$ are then transmitted over an ISI channel.

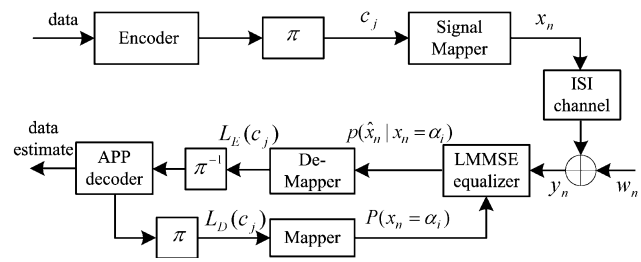

Fig. 1. The block diagram of a turbo equalization scheme, where π and π^{-1} represents interleaver and de-interleaver, respectively.

At the receiver side, there are an *a posteriori* probability (APP) decoder and a LMMSE equalizer operating in an iterative manner, as described in [6]. Based on the received symbol sequences $\{y_n\}$ and *a priori* probabilities $\{P(x_n = \alpha_i)\}$, the LMMSE equalizer first computes the LMMSE estimates $\{\hat{x}_n\}$ of the transmitted symbols and then generates the corresponding likelihood values $\{p(\hat{x}_n \mid x_n = \alpha_i)\}$. Afterwards, the de-mapper computes the extrinsic log-likelihood ratio (LLR) values $\{L_E(c_j)\}$ that are de-interleaved and used as the *a priori* information in the APP decoder [1]–[5]. The outputs of the APP decoder are the extrinsic LLRs $\{L_D(c_j)\}$ that are used to generate the *a priori* information $\{P(x_n = \alpha_i)\}$ for the next iteration.

A detailed discussion of the overall turbo process can be found in [5], [6]. This letter focuses on the LMMSE equalizer.

III. A MODIFIED LMMSE EQUALIZER

A. Existing LMMSE Equalizer—CLMMSE

In [5], the likelihood values $\{p(\hat{x}_n \mid x_n = \alpha_i)\}$ are evaluated under the assumption that the distribution of the conditional estimate $\hat{x}_n$ is complex Gaussian. The real and imaginary parts of $\hat{x}_n$ are assumed to have the same conditional variance and zero covariance. This assumption does not hold in general (see discussion below) which may lead to performance degradation. We denote such an equalizer a complex-number-notation LMMSE (CLMMSE) equalizer. In the following, we will derive an alternative equalizer based on real number notation. We treat the real and imaginary parts of x_n as two correlated real variables and evaluate the conditionally joint distribution of their estimates during the equalization process. We call this new equalizer a real-number-notation LMMSE (RLMMSE) equalizer.

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B. Modified LMMSE Equalizer—RLMMSE

Let a, b, and c be complex scalars. Multiplication of two complex scalars $a = b \cdot c$ can be expressed in vector form as

$$\begin{bmatrix} \mathrm{Re}(a) \\ \mathrm{Im}(a) \end{bmatrix} = \begin{bmatrix} \mathrm{Re}(b) & -\mathrm{Im}(b) \\ \mathrm{Im}(b) & \mathrm{Re}(b) \end{bmatrix} \begin{bmatrix} \mathrm{Re}(c) \\ \mathrm{Im}(c) \end{bmatrix} \tag{1}$$

where $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ denote the real and imaginary parts, respectively.

Assume that we know the channel impulse response (CIR) coefficients $\{h_m\}_{m=0}^{M-1}$. Let $\{w_n\}$ be i.i.d. complex AWGN samples with zero-mean and variance σ^2 per dimension. In general, x_n, y_n, h_m, and w_n are complex numbers. From (1), the equivalent discrete-time model of an ISI channel with AWGN can be described by an augmented real matrix equation

$$\underline{\mathbf{y}}_n = \underline{\mathbf{H}}\underline{\mathbf{x}}_n + \underline{\mathbf{w}}_n \tag{2}$$

where $\underline{\mathbf{H}}$ is a $2N \times 2(N+M-1)$ compound real matrix

$$\underline{\mathbf{H}} \equiv \begin{bmatrix} \mathbf{H}_{M-1} & \mathbf{H}_{M-2} & \cdots & \mathbf{H}_0 & & & \\ & \mathbf{H}_{M-1} & \mathbf{H}_{M-2} & \cdots & \mathbf{H}_0 & \mathbf{0} & \\ \mathbf{0} & \ddots & \ddots & & \ddots & & \\ & & \mathbf{H}_{M-1} & \mathbf{H}_{M-2} & \cdots & \mathbf{H}_0 \end{bmatrix}. \tag{3a}$$

Each $\mathbf{H}_m$ in (3a) is a 2×2 block given by

$$\mathbf{H}_m \equiv \begin{bmatrix} \mathrm{Re}(h_m) & -\mathrm{Im}(h_m) \\ \mathrm{Im}(h_m) & \mathrm{Re}(h_m) \end{bmatrix}. \tag{3b}$$

The other compound vectors in (2) are defined by

$$\underline{\mathbf{x}}_n \equiv \left[\boldsymbol{x}_{n-N_2-M+1}^T, \boldsymbol{x}_{n-N_2-M+2}^T, \ldots, \boldsymbol{x}_{n+N_1}^T\right]^T \tag{4a}$$
$$\underline{\mathbf{y}}_n \equiv \left[\boldsymbol{y}_{n-N_2}^T, \boldsymbol{y}_{n-N_2+1}^T, \ldots, \boldsymbol{y}_{n+N_1}^T\right]^T \tag{4b}$$
$$\underline{\mathbf{w}}_n \equiv \left[\boldsymbol{w}_{n-N_2}^T, \boldsymbol{w}_{n-N_2+1}^T, \ldots, \boldsymbol{w}_{n+N_1}^T\right]^T. \tag{4c}$$

Each entry of the compound vectors in (4) is a 2×1 block. For example, $\boldsymbol{x}_i = [\mathrm{Re}(x_i), \mathrm{Im}(x_i)]^T$, etc. The parameter $N = N_1 + N_2 + 1$ is the width of the observation window for estimating x_n, where N_1 and N_2 indicate the length of the non-causal and the causal parts, respectively.

Using real number notation, a signal point α_i in the constellation can be expressed as a vector $\boldsymbol{\alpha}_i \equiv [\mathrm{Re}(\alpha_i), \mathrm{Im}(\alpha_i)]^T$. Following [5], we assume that the constellation is symmetric with $\sum_{i=1}^{2^Q} \boldsymbol{\alpha}_i = \mathbf{0}_{2\times 1}$ and $2^{-Q}\sum_{i=1}^{2^Q} \boldsymbol{\alpha}_i\boldsymbol{\alpha}_i^T = (1/2)\mathbf{I}_2$, where $\mathbf{0}_{l\times m}$ is an $l \times m$ zero matrix, and $\mathbf{I}_l$ is an $l \times l$ identity matrix. Let $\{P(x_n = \alpha_i), \forall i\}$ be the *a priori* probabilities of the transmitted symbol x_n, initialized to $\{P(x_n = \alpha_i) = 2^{-Q}, \forall i\}$ and updated during the iterative process as described in Section II [5].

The means of $\boldsymbol{x}_n, \underline{\mathbf{x}}_n$, and $\underline{\mathbf{y}}_n$ and the covariance matrix of $\boldsymbol{x}_n$ (denoted by $\mathbf{V}_n \equiv \mathrm{cov}(\boldsymbol{x}_n, \boldsymbol{x}_n)$) are given by

$$E(\boldsymbol{x}_n) = \sum_{\alpha_i \in A} \boldsymbol{\alpha}_i P(x_n = \alpha_i) \tag{5a}$$
$$E(\underline{\mathbf{x}}_n) = \left[E\left(\boldsymbol{x}_{n-N_2-M+1}\right)^T, \ldots, E\left(\boldsymbol{x}_{n+N_1}\right)^T\right]^T \tag{5b}$$
$$E(\underline{\mathbf{y}}_n) = \underline{\mathbf{H}} \times E(\underline{\mathbf{x}}_n) \tag{5c}$$
$$\mathbf{V}_n = \sum_{\alpha_i \in A} \boldsymbol{\alpha}_i\boldsymbol{\alpha}_i^T P(x_n = \alpha_i) - E(\boldsymbol{x}_n)E(\boldsymbol{x}_n)^T. \tag{5d}$$

In the above $\mathbf{V}_n$ is a 2×2 symmetric matrix. The two diagonal entries of $\mathbf{V}_n$ give the variances of $\mathrm{Re}(x_n)$ and $\mathrm{Im}(x_n)$, respectively, and the two (equal) off-diagonal entries of $\mathbf{V}_n$ give the covariance of $\mathrm{Re}(x_n)$ and $\mathrm{Im}(x_n)$. Initially, we assume that there is no *a priori* information, i.e., $P(x_n = \alpha_i) = 2^{-Q}, \forall i$ and set $\mathbf{V}_n = (1/2)\mathbf{I}_2, \forall n$, in (5d).

The main difference between the RLMMSE equalizer discussed in this letter and the CLMMSE equalizer introduced in [5] is as follows. In both approaches, after the first iteration, $\{P(x_n = \alpha_i)\}$ is updated using the feedback LLRs from the APP decoder (see Section II) and so $\{P(x_n = \alpha_i)\}$ may take arbitrary values. Then the covariance matrix $\mathbf{V}_n$ calculated from (5d) for the RLMMSE equalizer may have un-equal diagonal entries and nonzero off-diagonal entries. That means that the variances of $\mathrm{Re}(x_n)$ and $\mathrm{Im}(x_n)$ could be different and their covariance could be nonzero. In comparison, in the CLMMSE approach, only one real number scalar $\mathrm{var}(x_n)$ is used to describe the variance of x_n (which is equivalent to introducing a constraint that $\mathbf{V}_n = (1/2)\,\mathrm{var}(x_n)\mathbf{I}_2$, i.e., $\mathbf{V}_n$ is a scaled unit matrix, in the RLMMSE approach). Clearly, the RLMMSE approach provides a more accurate statistical model of x_n and so it may potentially achieve better performance, as we will see later.

Based on (2)–(5), we can derive the RLMMSE estimator. The general principle follows the discussion in [5], except that the matrices involved here, such as $\underline{\mathbf{H}}, \underline{\mathbf{x}}_n, \underline{\mathbf{y}}_n$, and $\underline{\mathbf{w}}_n$, are all compound, i.e., their entries are blocks. Due to this, we will omit the derivation details and only list the results below. The so-called "extrinsic" LMMSE estimate $\hat{\boldsymbol{x}}_n$ of $\boldsymbol{x}_n$ is given by

$$\hat{\boldsymbol{x}}_n = \mathbf{F}_n^T(\underline{\mathbf{y}}_n - E(\underline{\mathbf{y}}_n) + \mathbf{S}E(\boldsymbol{x}_n)) \tag{6}$$

where

$$\mathbf{S} \equiv \underline{\mathbf{H}} \times \left[\mathbf{0}_{2\times 2(N_2+M-1)}, \mathbf{I}_2, \mathbf{0}_{2\times 2N_1}\right]^T \tag{7a}$$
$$\mathbf{F}_n \equiv \frac{1}{2}\left(\mathbf{\Sigma}_n + \mathbf{S}\left(\frac{1}{2}\mathbf{I}_2 - \mathbf{V}_n\right)\mathbf{S}^T\right)^{-1}\mathbf{S} \tag{7b}$$
$$\mathbf{\Sigma}_n \equiv \mathrm{cov}(\underline{\mathbf{y}}_n, \underline{\mathbf{y}}_n) = \sigma^2\mathbf{I}_{2N} + \underline{\mathbf{H}}\underline{\mathbf{V}}_n\underline{\mathbf{H}}^T \tag{7c}$$
$$\underline{\mathbf{V}}_n \equiv \mathrm{diag}\left(\mathbf{V}_{n-N_2-M+1}, \ldots, \mathbf{V}_n, \ldots, \mathbf{V}_{n+N_1}\right). \tag{7d}$$

In (6), $\hat{\boldsymbol{x}}_n$ is a vector whose two entries are the LMMSE estimators for the real and imaginary parts of x_n respectively. It can be verified that the contribution of the *a priori* information related to x_n, i.e., $\mathbf{V}_n$ and $E(\boldsymbol{x}_n)$, is cancelled in (6). This indicates that $\hat{\boldsymbol{x}}_n$ is "extrinsic", which conforms to the fundamental turbo principle developed in [1].

We assume that the conditional PDFs of $\hat{\boldsymbol{x}}_n$ (given $\boldsymbol{x}_n = \boldsymbol{\alpha}_i$) are joint Gaussian with a 2×1 mean vector

$$\mathbf{m}_{n,i} \equiv E(\hat{\boldsymbol{x}}_n \,|\, \boldsymbol{x}_n = \boldsymbol{\alpha}_i) = \mathbf{F}_n^T\mathbf{S}\boldsymbol{\alpha}_i \tag{8a}$$

and a 2×2 covariance matrix

$$\mathbf{R}_{n,i} \equiv \mathrm{cov}(\hat{\boldsymbol{x}}_n, \hat{\boldsymbol{x}}_n \,|\, \boldsymbol{x}_n = \boldsymbol{\alpha}_i) = \mathbf{F}_n^T\left(\mathbf{\Sigma}_n - \mathbf{S}\mathbf{V}_n\mathbf{S}^T\right)\mathbf{F}_n. \tag{8b}$$

According to the joint Gaussian assumption and (6), the likelihood value of $\hat{\boldsymbol{x}}_n$ is given by

$$
\begin{aligned}
p(\hat{\boldsymbol{x}}_n \,|\, \boldsymbol{x}_n = \boldsymbol{\alpha}_i) &= K_1 \exp\left(-\frac{1}{2}(\hat{\boldsymbol{x}}_n - \mathbf{m}_{n,i})^T \mathbf{R}_{n,i}^{-1}(\hat{\boldsymbol{x}}_n - \mathbf{m}_{n,i})\right) \\
&= K_2 \exp\left(\boldsymbol{\alpha}_{n,i}^T \left(\boldsymbol{\Phi}\left(\boldsymbol{\Phi} - \mathbf{V}_n^{-1}\right)^{-1} - \mathbf{I}_2\right) \right. \\
&\quad \left. \times \left(\boldsymbol{\Gamma} + \frac{1}{2}\boldsymbol{\Phi}\boldsymbol{\alpha}_{n,i}\right)\right) \qquad (9)
\end{aligned}
$$

where K_1, K_2 are constants for every $\hat{x}_n$ and

$$\boldsymbol{\alpha}_{n,i} \equiv E(\boldsymbol{x}_n) - \boldsymbol{\alpha}_i \qquad (10a)$$

$$\boldsymbol{\Gamma} \equiv \mathbf{S}^T \boldsymbol{\Sigma}_n^{-1}(\underline{\mathbf{y}}_n - E(\underline{\mathbf{y}}_n)) \qquad (10b)$$

$$\boldsymbol{\Phi} \equiv \mathbf{S}^T \boldsymbol{\Sigma}_n^{-1}\mathbf{S}. \qquad (10c)$$

In (9), the matrix inversion lemma is applied to avoid direct computation of $\mathbf{R}_{n,i}^{-1}$ but we have omitted the details. The derivation is very similar to that in [5]. The likelihood values $\{p(\hat{\boldsymbol{x}}_n \,|\, \boldsymbol{x}_n = \boldsymbol{\alpha}_i)\}$ constitute the outputs of the RLMMSE equalizer.

C. Calculation of $\boldsymbol{\Sigma}_n^{-1}$ and Complexity

Directly computing $\boldsymbol{\Sigma}_n^{-1}$ in (10) has a complexity order of $O(N^3)$. Alternatively, we can adopt a recursive algorithm [5] to compute $\boldsymbol{\Sigma}_n^{-1}$ with a complexity order of $O(N^2)$. In [5], $\boldsymbol{\Sigma}_n$ is a $N \times N$ complex number matrix while here $\boldsymbol{\Sigma}_n$ is a $2N \times 2N$ real number matrix. The complexity of inverting $\boldsymbol{\Sigma}_n$ in both cases is nearly the same. By similar reasoning, it can be also verified that the RLMMSE and CLMMSE approaches have similar complexity.

IV. Simulation Results

The performance of the CLMMSE and RLMMSE approaches for an 8-PSK modulation system are studied. A rate $R = 1/2$ convolutional code with generator $[D^2 + D + 1, D^2 + 1]$ is considered as the channel code. The frame length of the data is 2049 bits. The interleaver is generated randomly. Two time-invariant ISI channels [7], given by

$$\text{C1:} \quad \{h_m\}_{m=0}^{M-1} = \{0.227, 0.46, 0.688, 0.46, 0.227\}$$

$$\text{C2:} \quad \{h_m\}_{m=0}^{M-1} = \{0.29, 0.5, 0.58, 0.5, 0.29\}$$

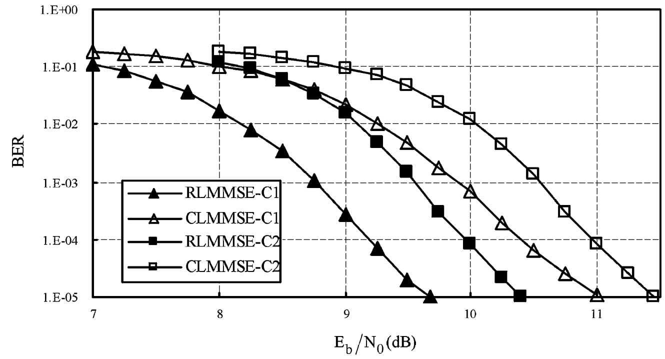

Fig. 2. Performance of the turbo equalization systems after 5 iterations.

are considered. The width of the observation window N is set to 15 $(N_1 = 5, N_2 = 9)$. The same signal mapper as that in [5] is used. Simulation results are summarized in Fig. 2. At $\text{BER} = 10^{-5}$, the RLMMSE equalizer achieves about 1 dB of performance gain over the CLMMSE equalizer in both channels.

V. Conclusions

A modified LMMSE equalizer for turbo equalization has been derived based on an augmented real matrix representation. It provides a more accurate model for the conditional distribution of the LMMSE estimates. Compared with the existing method, the proposed method achieves considerable performance improvement without increasing complexity.

References

[1] C. Berrou, A. Glavieux, and P. Thitimajshima, "Near Shannon limit error-correcting and decoding: Turbo-codes," in *Proc. IEEE Int. Conf. Commun.*, May 1993, pp. 1064–1070.

[2] C. Douillard *et al.*, "Iterative correction of intersymbol interference: Turbo-equalization," *Eur. Trans. Telecommun.*, vol. 6, pp. 507–511, Sept.–Oct. 1995.

[3] G. Bauch and V. Franz, "A comparison of soft-in/soft-out algorithms for 'Turbo detection'," in *Proc. Int. Conf. Telecommun.*, June 1998, pp. 259–263.

[4] D. Raphaeli and Y. Zarai, "Combined turbo equalization and turbo decoding," in *Proc. IEEE Global Telecommun. Conf.*, vol. 2, Nov. 1997, pp. 639–643.

[5] M. Tuchler, A. C. Singer, and R. Koetter, "Minimum mean squared error equalization using a priori information," *IEEE Trans. Signal Processing*, vol. 50, pp. 673–683, Mar. 2002.

[6] M. Tuchler, R. Koetter, and A. C. Singer, "Turbo equalization: Principles and new results," *IEEE Trans. Commun.*, vol. 50, pp. 754–767, May 2002.

[7] J. Proakis, *Digital Communications*, 3rd ed. New York: McGraw-Hill, 1995.