EE4015 Digital Signal Processing
Semester A 2022-2023

Solution for Assignment 1

Solution of Question 1

The signal has period $T = 2$ and fundamental frequency $\Omega_o = \pi$. Consider the period from $t = -1$ to $t = 1$ and use Continuous-Time Fourier Series analysis equation:

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-k\Omega_o t}dt = \frac{1}{2}\left[\int_{-1}^{0} 2e^{-jk\pi t}dt + \int_{0}^{1} e^{-jk\pi t}dt\right]$$

For $k = 0$,

$$a_0 = \frac{1}{2}\left[\int_{-1}^{0} 2dt + \int_{0}^{1} 1dt\right] = \frac{1}{2}[2+1] = 1.5$$

For $k \neq 0$,

$$\begin{aligned} a_k &= \frac{1}{2}\left[\int_{-1}^{0} 2e^{-jk\pi t}dt + \int_{0}^{1} e^{-jk\pi t}dt\right] \\ &= \frac{1}{2}\left[\frac{2}{-jk\pi}e^{-jk\pi t}\Big|_{-1}^{0} + \frac{1}{-jk\pi}e^{-jk\pi t}\Big|_{0}^{1}\right] \\ &= -\frac{1}{2jk\pi}\left[2 - 2e^{jk\pi} + e^{-jk\pi} - 1\right] \\ &= -\frac{1}{2jk\pi}\left[1 - 2e^{jk\pi} + e^{-jk\pi}\right] \end{aligned}$$

Combining the results, we have:

$$a_k = \begin{cases} 1.5, & k = 0 \\ -\frac{1}{2jk\pi}\left[1 - 2e^{jk\pi} + e^{-jk\pi}\right] & k \neq 0 \end{cases}$$

Solution of Question 2

We first re-express $x(t)$ as

$$x(t) = e^{-2|t-1|} = \begin{cases} e^{-2(t-1)}, & t > 1 \\ e^{2(t-1)}, & t < 1 \end{cases}$$

According to Continuous-Time Fourier transform, we obtain:

$$\begin{aligned} X(j\Omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \\ &= \int_{-\infty}^{1} e^{2(t-1)}e^{-j\Omega t}dt + \int_{1}^{\infty} e^{-2(t-1)}e^{-j\Omega t}dt \\ &= \frac{e^{-j\Omega}}{2-j\Omega} + \frac{e^{-j\Omega}}{2+j\Omega} \\ &= \frac{4e^{-j\Omega}}{4+\Omega^2} \end{aligned}$$

Solution of Question 3

$$y[n] = x[n]x[n-1] - 3x[n+2]$$

(a) The system is **not causal**. **(1 mark)**
It is because $y[n]$ depends on future system input, e.g., $y[n]$ depends on the future input of $x[n+2]$. **(2 marks)**

(b) The system is **not linear**. **(1 mark)**

Let $y_i[n] = T\{x_i[n]\},\ i = 1,2,3$ with $x_3[n] = ax_1[n] + bx_2[n]$. The system outputs for $x_1[n]$ and $x_2[n]$ are $y_1[n] = x_1[n]x_1[n-1] - 3x_1[n+2]$ and $y_2[n] = x_2[n]x_2[n-1] - 3x_2[n+2]$. Then

$$y_3[n] = T\{x_3[n]\} = (ax_1[n] + bx_2[n])(ax_1[n-1] + bx_2[n-1]) - 3(ax_1[n+2] + bx_2[n+2])$$

$$\neq aT\{x_1[n]\} + bT\{x_2[n]\}$$

$$= ax_1[n]x_1[n-1] - 3bx_1[n+2] + bx_2[n]x_2[n-1] - 3bx_2[n+2]$$

(2 marks)

(c)
The system is **time-invariant**. **(1 mark)**
It is because

$$y[n-n_o] = T\{x[n-n_o]\} = x[n-n_o]x[n-n_o-1] - 3x[n-n_o+2]$$

(2 marks)

(d)
The system is **stable**. **(1 mark)**

For bounded input $|x[n]| \leq S$, the output is bounded by $|y[n]| \leq S^2 + 3S$. **(2 marks)**

(e)
The input is now:

$$x[n] = 2u[n-2] + u[n-1]$$

As a result,

$$\begin{aligned} y[n] &= (2u[n-2] + u[n-1])(2u[n-3] + u[n-2]) - 3(2u[n] + u[n+1]) \\ &= 4u[n-3] + 2u[n-3] + 2u[n-2] + u[n-2] - 6[u[n]] - 3u[n+1] \\ &= 6u[n-3] + 3u[n-2] - 6[u] - 3u[n+1] \end{aligned}$$

(4 marks)

Hence, $y[0] = -9$, $y[1] = -9$, $y[2] = -6$, $y[3] = 0$

(4 marks)

Solution of Question 4

Consider two discrete-time signals $x[n] = u[-1-n]$ and $h[n] = (0.5)^n u[n]$, and compute $y[n] = x[n] * h[n]$ using the convolution formula.

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \sum_{m=-\infty}^{\infty} u[-m-1] \cdot (0.5)^{n-m} u[n-m] \\ &= \sum_{m=-\infty}^{-1} (0.5)^{n-m} u[n-m] \\ &= \sum_{l=1}^{\infty} (0.5)^{n+l} u[n+l] \end{aligned}$$

For $n \geq -1$, all $\{u[n+l]\}$ correspond to 1 and we have:

$$y[n] = \sum_{l=1}^{\infty} (0.5)^{n+l} = (0.5)^n \sum_{l=1}^{\infty} (0.5)^l = (0.5)^n \cdot \frac{0.5}{1-0.5} = (0.5)^n$$

For $n < -1$, $u[n+l] = 1$ when $n + l \geq 0$ or $l \geq -n$

$$y[n] = \sum_{l=-n}^{\infty} (0.5)^{n+l} = (0.5)^n \sum_{l=-n}^{\infty} (0.5)^l = (0.5)^n \cdot \frac{(0.5)^{-n}}{1-0.5} = 2$$

Combining the results, we have:

$$y[n] = \begin{cases} (0.5)^n, & n \geq -1 \\ 2, & n < -1 \end{cases}$$

Solution of Question 5

Let $x[n] = \{1,4,0,2\}$ and $h[n] = \{1,2,1\}$. Find their convolution with both of the sequence start at $n = 0$.

$$x[n] * h[n] = \{1,4,0,2\} * \{1,2,1\} = \{1, 6, 9, 6, 4, 2\}$$

Solution of Question 6

$x[0] = 0$, $x[1] = 1$, $x[2] = 0$, $x[3] = -1$ and $x[4] = 0$.

Yes. $x[n]$ is a periodic signal.

Solution of Question 7

$$h[n] = h_1[n] * h_3[n] * h_4[n] + h_1[n] * h_2[n]$$

$$h[n] = h_1[n] * (h_3[n] * h_4[n] + h_2[n])$$

Solution of Question 8

For an analog signal of $x(t) = 3\cos(70\pi t)$, find the Nyquist sampling rate in Hz and also determine the discrete-time angular frequency and the discrete-time signal $x[n]$ mathematical expression of $x(t)$ sampled at the Nyquist rate.

The maximum frequency component of $x(t)$ is

$$F_{max} = \frac{70\pi}{2\pi} = 35\ Hz$$

Therefore, according to sampling theorem, we the Nyquist sampling rate is

$$F_s = 2F_{max} = 70\ Hz$$

The discrete-time angular frequency of the sampled signal is

$$\omega = 2\pi\frac{35}{70} = \pi$$

The discrete-time signal $x[n]$ can be expressed as

$$x[n] = 3\cos(n\pi)$$

Solution of Question 9

A difference equation for a particular discrete-time system is given by

$$y[n] = 0.1x[n] - 0.1x[n-1] + 0.8x[n-3] + 0.1x[n-4] + 0.6x[n-6]$$

This is a difference equation of an FIR discrete-time system, then the impulse response is {0.1, -0.1, 0, 0.8, 0.1, 0, 0.6}, the first sample is the index n=0. For n < 0 and n >6 are zero.