EE4015 Digital Signal Processing
Semester A 2022-2023

Solution for Assignment 2

Solution of Question 1

(a) Taking the-z transform on both sides of the difference equation, we can obtain the transfer function:

$$y[n] = 0.1y[n-1] + 0.12y[n-2] + 7x[n]$$

$$Y(z)(1 - 0.1z^{-1} - 0.12z^{-2}) = 7X(z) \Rightarrow \frac{Y(z)}{X(z)} = H(z) = \frac{7}{1 - 0.1z^{-1} - 0.12z^{-2}}$$

(3marks)

(b)

$$H(z) = \frac{7z^2}{z^2 - 0.1z - 0.12} = \frac{7z^2}{(z-0.4)(z+0.3)}$$

Hence the poles are 0.4 and -0.3 and there are two zeroes at 0.

(3 marks)

(c) If the system is casual and stable, the ROC should include the unit circle, which means that the ROC is $|z| > 0.4$. Hence the impulse response is also:

$$H(z) = \frac{7}{1 - 0.1z^{-1} - 0.12z^{-2}} = \frac{4}{1 - 0.4z^{-1}} + \frac{3}{1 + 0.3z^{-1}}$$

$$h[n] = 4(0.4)^n u[n] + 3(-0.3)^n u[n]$$

(3 marks)

(d) The squared magnitude is:

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = \frac{7}{1 - 0.1e^{-j\omega} - 0.12e^{-j2\omega}} \frac{7}{1 - 0.1e^{j\omega} - 0.12e^{j2\omega}}$$

$$= \frac{49}{1.0244 - 0.176\cos(\omega) - 0.24\cos(2\omega)}$$

$$\Rightarrow |H(e^{j\omega})| = \frac{7}{\sqrt{1.0244 - 0.176\cos(\omega) - 0.24\cos(2\omega)}}$$

(3 marks)

$$H(e^{j\omega}) = \frac{7}{1 - 0.1e^{-j\omega} - 0.12e^{-j2\omega}} \left(\frac{1 - 0.1e^{j\omega} - 0.12e^{j2\omega}}{1 - 0.1e^{j\omega} - 0.12e^{j2\omega}} \right)$$

$$= 7\frac{1 - 0.1\cos(\omega) - 0.12\cos(2\omega) - j(0.1\sin(\omega) + 0.12\sin(2\omega))}{1.0244 - 0.176\cos(\omega) - 0.24\cos(2\omega)}$$

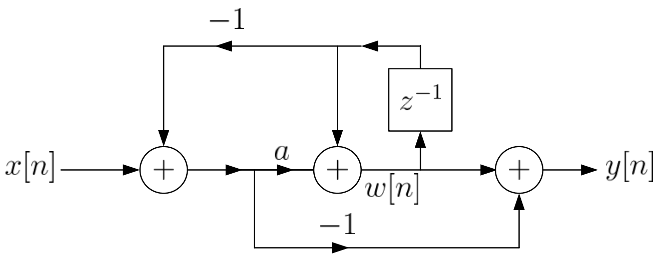
Hence the phase is:

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{0.1\sin(\omega) - 0.12\sin(2\omega)}{1 - 0.1\cos(\omega) - 0.12\cos(2\omega)}\right)$$

(3 marks)

<u>Solution of Question 2</u>

(a) First we introduce an intermediate variable $w[n]$:



$$w[n] = w[n-1] + a(x[n] - w[n-1]) => w[n] = (1-a)w[n-1] + ax[n]$$

$$y[n] = -(x[n] - w[n-1]) + w[n] => y[n] = w[n] + w[n-1] - x[n]$$

Take the z transform of the two equations, we have:

$$W(z) = (1-a)z^{-1}W(z) + aX(z) \quad => \quad W(z) = \frac{1}{1-(1-a)z^{-1}}X(z)$$

$$Y(z) = W(z)(1 + z^{-1}) - X(z)$$

Substituting the first equation into the second equation yields:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{a - 1 + z^{-1}}{1 + (a-1)z^{-1}}$$

(12 marks)

(b)
Since the system is causal, the ROC must be $|z| > |a-1|$. Hence we have:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{a-1+z^{-1}}{1+(a-1)z^{-1}} = \frac{a-1}{1+(a-1)z^{-1}} + \frac{z^{-1}}{1+(a-1)z^{-1}}$$

$$=> \quad h[n] = (a-1)\cdot(1-a)^n u[n] + (1-a)^{n-1}u[n-1]$$
$$= -(1-a)^{n+1}u[n] + (1-a)^{n-1}u[n-1]$$

(4 marks)

(c)
If the system is stable, the ROC must include the unit circle, i.e.,

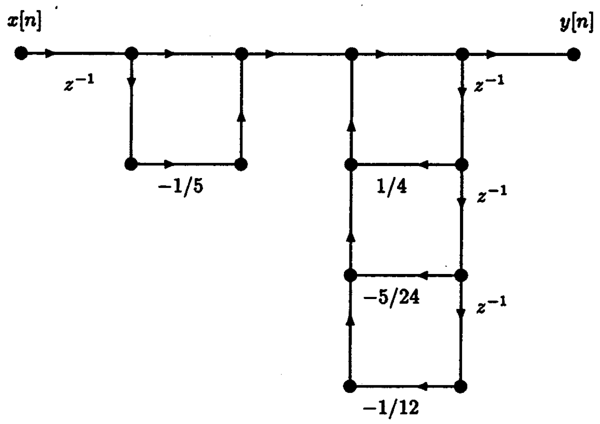
$$|a-1| < 1 \Leftrightarrow -1 < a - 1 \Leftrightarrow 0 < a < 2$$

(4 marks)

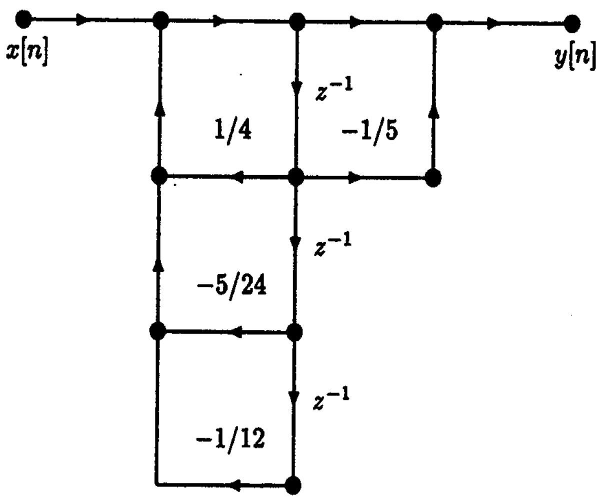
Solution of Question 3

3.(a)

$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{1 - \frac{1}{4}z^{-1} + \frac{5}{24}z^{-2} + \frac{1}{12}z^{-3}}$$

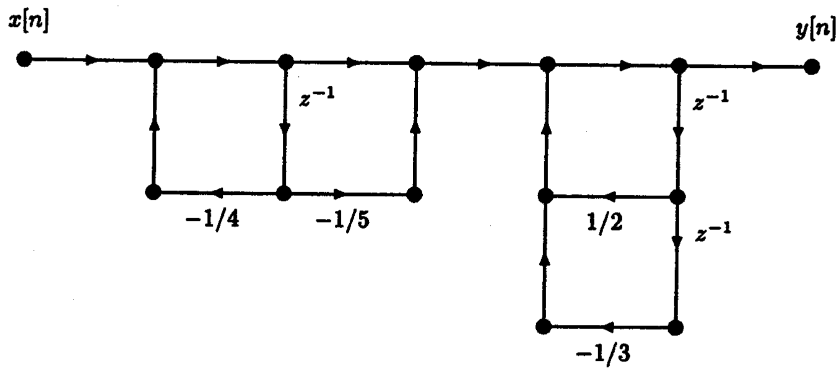


3.(b)



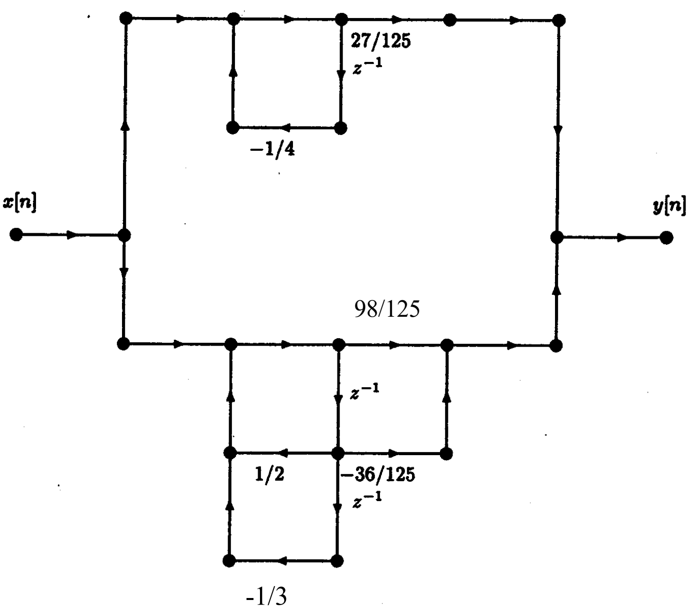
3.(c)

$$H(z)=\left(\frac{1-\frac{1}{5}z^{-1}}{1+\frac{1}{4}z^{-1}}\right)\left(\frac{1}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}}\right)$$



3.(d)

$$H(z)=\frac{\frac{27}{125}}{1+\frac{1}{4}z^{-1}}+\frac{\frac{98}{125}-\frac{36}{125}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}}$$



Solution of Question 4

4.(a)
This is a linear phase filter as the impulse response is anti-symmetric.

4.(b)
It is clear that the filter length is 2. For $0 \le k \le 1$:

$$H[k] = \sum_{n=0}^{1} h[n]W_2^{kn} = W_2^0 - W_2^k = 1 - e^{j\frac{2\pi}{2}k} = 1 - e^{-j\pi k}$$

Hence

$$H[0] = 0$$
$$H[1] = 1 - e^{-j\pi} = 1 - \cos\pi + \mathrm{j}\sin\pi = 2$$

4.(c)

$$h[n] = \delta[n] - \delta[n-1]$$

Taking the z-transform yields:

$$H(z) = 1 - z^{-1} = \frac{z-1}{z}$$

There is one pole at $z = 0$ and one zero at $z = 1$.

4.(d)

$$H(e^{j\omega}) = 1 - e^{-j\omega} = e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right) = e^{-j\frac{\omega}{2}}2j\sin\frac{\omega}{2} = e^{-j\frac{\omega}{2}}e^{j\frac{\pi}{2}}2\sin\frac{\omega}{2}$$

Hence

$$\left|H(e^{j\omega})\right| = 2\left|\sin\frac{\omega}{2}\right|$$

And

$$\angle H(e^{j\omega}) = -\frac{\omega}{2} + \frac{\pi}{2} + \angle\sin\frac{\omega}{2}$$

Alternatively,

$$H(e^{j\omega}) = 1 - e^{-j\omega} = 1 - (\cos\omega - j\sin\omega) = (1 - \cos\omega) + j\sin\omega$$

$$\left|H(e^{j\omega})\right| = \sqrt{(1-\cos\omega)^2 + (\sin\omega)^2} = 2\left|\sin\frac{\omega}{2}\right|$$

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{\sin\omega}{1-\cos\omega}\right)$$

4.(e)
At $\omega = 0$, $\left|H\left(e^{j\omega}\right)\right|$ is minimized with a value of 0.
When $\omega = \pi$, $\left|H\left(e^{j\omega}\right)\right|$ is maximized with a value of 2.
The system corresponds to a highpass filter.

4.(f)
From $h[n]$ we can obtain the difference equation and then we solve for $y[n]$:

$$y[n] = x[n] - x[n-1] = \cos\left(\frac{\pi n}{2}\right) - \cos\left(\frac{\pi(n-1)}{2}\right)$$
$$= \cos\left(\frac{\pi n}{2}\right) - \cos\left(\frac{\pi n}{2} - \frac{\pi}{2}\right) = \cos\left(\frac{\pi n}{2}\right) - \sin\left(\frac{\pi n}{2}\right)$$