EE4015 Digital Signal Processing
Semester A 2022-2023

Solution for Assignment 3

Solution of Question 1 [25 marks]

1(a) By inverse DTFT, the required impulse response is:

$$h_d[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\,d\omega = \frac{1}{2\pi}\left[\frac{1}{jn}e^{j\omega n}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{j2\pi n} = \frac{\sin(\omega_c n)}{\pi n} = \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\ \dfrac{\sin(\omega_c n)}{\pi n}, & elsewhere \end{cases}$$

[6 marks]

1(b)
For the cut-off frequency of 800 Hz with sampling frequency 8000 Hz, the discrete-time frequency is

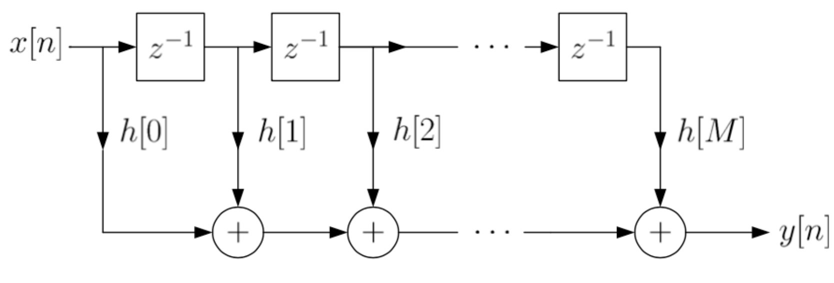
$$\omega_c = 2\pi\frac{800}{8000} = 0.2\pi$$

The impulse response of the FIR filter with 5-length is the 2-point shifted version of $h_d[n]$ and then apply 5-point rectangular window, which is given by

$$h[n] = \begin{cases} 0.2\,, & n = 2 \\ \dfrac{\sin\big(0.2\pi(n-2)\big)}{\pi(n-2)}, & n = 0,1,,3,4 \end{cases}$$

[8 marks]

1(c) The Direct Form implementation of the FIR with filter length of 5 is



where $M = 4$

[6 marks]

1(d) The major disadvantage of the rectangular window is that it can only achieve maximum -13 dB in the stopband even increasing the filter orders. To tackle this problem, we can use other window functions such as Blackman, Hanning, Hamming and Kaiser, they can achieve much higher stopband attenuation up to -90dB for Kaiser but higher order is required.

[5 marks]

<u>Solution of Question 2</u> [20 marks]

If $H_a(s)$ is a Butterworth lowpass filter, the order should be 2.

[4 marks]

For stable and causal, choose poles in the left half plan. For 2nd order, the poles are located at

$$c_k = \Omega_c e^{\frac{jk\pi}{N}} \cdot e^{\frac{j\pi}{2N}} = \Omega_c e^{\frac{jk\pi}{2}} \cdot e^{\frac{j\pi}{4}}, \qquad k = 0,1,2,3$$

But, we only select the left half plane poles with $k = 1,2$

- $c_1 = \Omega_c e^{\frac{j\pi}{2}} \cdot e^{\frac{j\pi}{4}} = \Omega_c e^{j\frac{3\pi}{4}} = -\frac{\Omega_c}{\sqrt{2}} + j\frac{\Omega_c}{\sqrt{2}}$
- $c_2 = \Omega_c e^{\frac{j2\pi}{2}} \cdot e^{\frac{j\pi}{4}} = \Omega_c e^{j\frac{5\pi}{4}} = -\frac{\Omega_c}{\sqrt{2}} - j\frac{\Omega_c}{\sqrt{2}}$

Then, the 2nd order Butterworth Transfer is given by

$$H_a(s) = \frac{\Omega_c^2}{\left(s + \frac{\Omega_c}{\sqrt{2}}\right)^2 + \frac{\Omega_c^2}{2}}$$

and we deduce that $\Omega_c = 1$.

As a result, the normalized Butterworth lowpass filter transfer function $\Omega_c = 1$ is given by:

$$H_a(s) = \frac{1}{s^2 + 2\frac{1}{\sqrt{2}}s + \frac{1}{2} + \frac{1}{2}} = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Therefore, $H_a(s) = \frac{1}{(s+a)^2 + b^2} = \frac{1}{s^2 + 2as + (a^2 + b^2)}$ can be a 2nd order Butterworth filter with two poles located at

$$-a + jb \qquad \text{and} \qquad -a - jb$$

For $a^2 + b^2 = 1$ and let $s = j\Omega$, we have

$$H_a(j\Omega) = \frac{1}{-\Omega^2 + 2aj\Omega + 1} = \frac{1}{(1-\Omega^2) + 2aj\Omega} \Rightarrow |H_a(j\Omega)| = \frac{1}{\sqrt{(1-\Omega^2)^2 + (2a\Omega)^2}}$$

The cutoff frequency Ω_c with condition of $|H_a(j\Omega_c)| = \frac{1}{\sqrt{2}}$, then can we determine the Ω_c as

$$(1 - \Omega_c^2)^2 + (2a\Omega_c)^2 = 2 \Rightarrow \Omega_c^4 + 2\Omega_c^2(2a^2 - 1) - 1 = 0$$

$$\Omega_c^2 = \frac{-2(2a^2 - 1) \pm 2\sqrt{(2a^2 - 1)^2 + 1}}{2}$$

$$\Omega_c = \sqrt{(1 - 2a^2) \pm (1 + (2a^2 - 1)^2)^{\frac{1}{2}}}$$

[16 marks]

<u>Solution of Question 3</u> [30 marks]

3(a) Based on the table of the normalized Butterworth filter transfer with $\Omega_c = 1$

$$H_{LP}(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

[4 marks]

3(b) Based on the frequency warping equation, the continuous-time cut-off frequency Ω_c

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_c}{2} = \frac{2}{0.1}\tan\left(\frac{0.5\pi}{2}\right) = 20\tan(0.25\pi) = 20\; rad\; s^{-1}$$

[4 marks]

3(c) Based on the frequency warping equation, the continuous-time cut-off frequency Ω_c

$$H_{HP}(s) = H_{LP}(s)\big|_{s=\frac{\Omega_c}{s}} = \frac{1}{\left(\frac{20}{s}\right)^2 + \sqrt{2}\left(\frac{20}{s}\right) + 1} = \frac{s^2}{s^2 + 20\sqrt{2}s + 400}$$

[4 marks]

3(d) By applying the Bilinear Transformation with T=0.1, we have

$$H_{HP}(z) = H_{HP}(s)\big|_{s=\frac{2}{0.1}\cdot\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{\left(\frac{2}{0.1}\cdot\frac{1-z^{-1}}{1+z^{-1}}\right)^2}{\left(\frac{2}{0.1}\cdot\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 20\sqrt{2}\left(\frac{2}{0.1}\cdot\frac{1-z^{-1}}{1+z^{-1}}\right) + 400}$$

$$= \frac{400\frac{(1-z^{-1})^2}{(1+z^{-1})^2}}{400\frac{(1-z^{-1})^2}{(1+z^{-1})^2} + 400\sqrt{2}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 400}$$

$$= \frac{\frac{(1-z^{-1})^2}{(1+z^{-1})^2}}{\frac{(1-z^{-1})^2}{(1+z^{-1})^2} + \sqrt{2}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1}$$

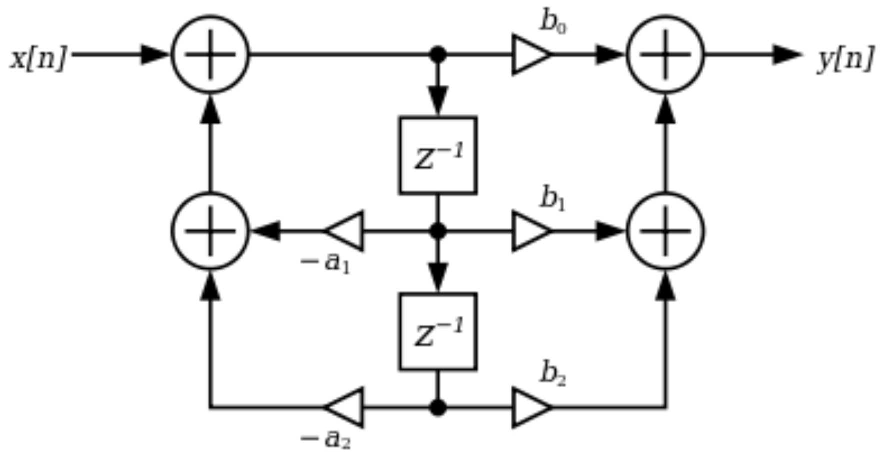
$$= \frac{1 - 2z^{-1} + z^{-2}}{(1-z^{-1})^2 + \sqrt{2}(1-z^{-1})(1+z^{-1}) + (1+z^{-1})^2}$$

$$= \frac{1 - 2z^{-1} + z^{-2}}{3.4142 + 0.5858z^{-2}}$$

$$= \frac{0.2929 - 0.5858z^{-1} + 0.2929z^{-2}}{1 + 0.1716z^{-2}}$$

[12 marks]

3(e) The canonical form implementation of the second-order IIR filter as shown below:



where $b_0 = 0.2929$, $b_1 = -0.5858$, $b_2 = 0.2929$, $a_1 = 0$, $a_2 = 0.1716$.

[6 marks]