<u>Exercises for Week 2</u>

1. Determine whether the following discrete-time system, with input signal $x[n]$ and output signal $y[n]$ is stable, causal, linear, and/or time-invariant:

$$y[n] = ax[n+1] + b, \qquad 0 < |a| < \infty,\ 0 < |b| < \infty$$

2. Consider a linear time-invariant (LTI) discrete-time system with input $x[n]$, output $y[n]$ and impulse response $h[n]$. With the use of the convolution formula, show that the following two statements regarding causality are identical:

   (i) Output at time n depends on input up to time n

   (ii) $h[n] = 0$ for $n < 0$

3. Describe the operation of a LTI discrete-time system whose impulse response is $h[n] = 0.5\delta[n] + 0.5\delta[n-1]$ by relating the input and output. Is the system stable? Why? Is it causal? Why?

4. Compute $y[n]$ for all values of n:

$$y[n] = a^n u[n] * a^n u[n]$$

5. Determine the convolution of the following two discrete-time signals:

$$x[n] = \begin{cases} n^2, & -2 \le n \le -1 \\ 0, & otherwise \end{cases}$$

and

$$h[n] = \begin{cases} n-1, & 2 \le n \le 3 \\ 0, & otherwise \end{cases}$$

Solutions for Exercise 2

Question 1

$$y[n] = ax[n+1] + b, \qquad 0 < |a| < \infty,\ 0 < |b| < \infty$$

Stability

If $x[n]$ is bounded, then $y[n] = ax[n+1] + b$ is bounded for bounded a and b. As a result, $y[n]$ is bounded and the system is stable. In a more rigorous manner, we have:

$$|x[n]| < B_x < \infty \qquad \text{(Bounded Input)}$$

$$|y[n]| = |ax[n+1] + b| \le |a|B_x + |b| = B_y < \infty \quad \text{(Bounded Output)}$$

where we see $|x[n+1]| < B_x < \infty$ or $|x[n]| < B_x < \infty$ must give $|y[n]| < B_y < \infty$.

Causality

It is not causal because $y[n]$ depends on future input, namely, $x[n+1]$.

Linearity

The system outputs for $x_1[n]$ and $x_2[n]$ are:

$$y_1[n] = ax_1[n+1] + b \quad \text{and} \quad y_2[n] = ax_2[n+1] + b$$

Consider $x_3[n] = \alpha x_1[n] + \beta x_2[n]$, its system output is then:

$$\begin{aligned} y_3[n] &= ax_3[n+1] + b = a(\alpha x_1[n+1] + \beta x_2[n+1]) + b \\ &= a\alpha x_1[n+1] + a\beta x_2[n+1] + b \\ &\neq \alpha y_1[n] + \beta y_2[n] \\ &= \alpha a x_1[n+1] + \alpha b + \beta a x_2[n+1] + \beta b \end{aligned}$$

As a result, this system is not linear.

Time-invariance

First, we have:

$$y[n] = ax[n+1] + b$$
$$y[n-n_0] = ax[n-n_0+1] + b$$

Consider $x_1[n] = x[n-n_0]$, its system output is

$$y_1[n] = ax[n-n_0+1] + b = y[n-n_0]$$

Hence the system is time-invariant.

Question 2

Expanding the convolution formula as:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k][x[n-k]$$

$$= \cdots + h[-2] \cdot x[n+2] + h[-1] \cdot x[n+1] + h[0] \cdot x[n] + h[1] \cdot x[n-1] + h[2] \cdot x[n-2] + \cdots$$

If $y[n]$ does not depend on future inputs $x[n+1]$, $x[n+2], \ldots,$ we must have $h[-1] = h[-2] = \cdots = 0$ or $h[n] = 0$ for $n < 0$.

As a result, the two statements regarding causality are identical.

Question 3

Let $x[n]$ and $y[n]$ be the discrete-time input and output of the system with impulse response $h[n]$. Using the relationship of convolution, we have:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[k] \cdot 0.5 \cdot (\delta[n-k] + \delta[n-1-k])$$

$$= 0.5(x[n] + x[n-1])$$

The system operates as an averager which gives an average value of $x[n]$ and $x[n-1]$ at time n. It is stable because

$$\sum_{n=-\infty}^{\infty} |h[n]| = |0.5| + |0.5| = 1 < \infty$$

It is causal because $h[n] = 0$ when $n < 0$. Note that the relationship between system input and output can be obtained from impulse response or vice versa.

Question 4

$$y[n] = a^n u[n] * a^n u[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} a^k u[k] \cdot a^{n-k} u[n-k]$$

$$= \sum_{k=0}^{\infty} a^n u[n-k] = a^n \sum_{k=0}^{\infty} u[n-k]$$

$$= a^n \sum_{m=-\infty}^{n} u[k], \qquad m = n - k$$

Since $u[m] = 0$ for $m < 0$, $y[n] = 0$ for $n < 0$. For example,

$$y[-1] = a^n \sum_{m=-\infty}^{-1} u[m] = a^n[\cdots + 0 + 0] = 0$$

For $n \geq 0$, $y[n]$ is:

$$y[n] = a^n \sum_{m=0}^{n} 1$$

Evaluating $y[n]$ for some $n \geq 0$:

$$y[0] = a^n \sum_{m=0}^{0} 1 = a^n$$

$$y[1] = a^n \sum_{m=0}^{1} 1 = a^n(1+1) = 2a^n$$

$$y[2] = a^n \sum_{m=0}^{2} 1 = a^n(1+1+1) = 3a^n$$

$\Rightarrow$

$$y[n] = a^n \sum_{m=0}^{n} 1 = (n+1)a^n$$

As a result, we have

$$y[n] = (n+1)a^n u[n]$$

Question 5

The convolution is computed as:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

From the signal description, we see that $x[n]$ and $h[n]$ are non-zero only for

$$x[-2] = 4 \text{ and } x[-1] = 1$$

$$h[2] = 1 \text{ and } h[3] = 2$$

Expanding the convolution sum, we have:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k][h[n-k]$$

$$= \cdots + x[-2] \cdot h[n+2] + x[-1] \cdot h[n+1] + x[0] \cdot h[n] + x[1] \cdot h[n-2] + \cdots$$

=> $$y[n] = x[-2] \cdot h[n+2] + x[-1] \cdot h[n+1]$$

via considering the non-zero values of $x[n]$.

By considering the non-zero values of $h[n]$ ($h[2] = 1$ and $h[3] = 2$) as well:

$y[0] = x[-2] \cdot h[0+2] + x[-1] \cdot h[0+1] = x[-2] \cdot h[2] + x[-1] \cdot h[1] = 4 \cdot 1 + 1 \cdot 0 = 4$

$y[1] = x[-2] \cdot h[1+2] + x[-1] \cdot h[1+1] = x[-2] \cdot h[3] + x[-1] \cdot h[2] = 4 \cdot 2 + 1 \cdot 1 = 9$

$y[2] = x[-2] \cdot h[2+2] + x[-1] \cdot h[2+1] = x[-2] \cdot h[4] + x[-1] \cdot h[3] = 4 \cdot 0 + 1 \cdot 2 = 2$

As a result, we have:

$$y[n] = \begin{cases} 4, & n = 0 \\ 9, & n = 1 \\ 2, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Or

$$y[n] = 4\delta[n] + 9\delta[n-1] + 2\delta[n-2]$$