Exercises for Week 3

1. Find the Continuous-Time Fourier Series (CTFS) coefficients for the following continuous-time periodic signal:

$$\tilde{x}(t) = \begin{cases} 1.5, & 0 \le t < 1 \\ -1.5 & 1 \le t < 2 \end{cases}$$

with fundamental period of $T = 2$.

2. Use L'Hopital's rule to prove

$$\frac{2\sin(\Omega T)}{\Omega} = 2T$$

When $\Omega = 0$.

3. Find the real part, imaginary part, magnitude and phase of

$$X(j\Omega) = \frac{1}{a + j\Omega}$$

4. Compute the Continuous-Time Fourier transform of

$$x(t) = e^{-\alpha|t|}, \quad \alpha > 0$$

Then find the magnitude and phase of $X(j\Omega)$.

5. Compute the Fourier transform of $x(t) = \cos(100t)$.

6. Compute the Fourier transform of $x(t) = 1$.

7. Consider the analog signal

$$x(t) = 3\cos(2000\pi t) + 5\sin(6000\pi t) + 10\cos(12000\pi t)$$

(a) What is the Nyquist rate for this signal?

(b) Assume now that we sample this signal using a sample rate $F_s = 5\ kHz$. What is the discrete-time signal $x[n]$ obtained after sampling?

(c) What is the analog signal $y(t)$ we can reconstruct from the samples if we use ideal anti-imaging lowpass reconstruction filter?

Solution

Question 1

$$\tilde{x}(t) = \begin{cases} 1.5, & 0 \le t < 1 \\ -1.5 & 1 \le t < 2 \end{cases}$$

The signal has period $T = 2$ and fundamental frequency $\Omega_0 = \pi$.

Consider the period from $t = -1$ to $t = 1$, the periodic signal $\tilde{x}(t)$ can be represented as

$$\tilde{x}(t) = \begin{cases} 1.5, & 0 \le t < 1 \\ -1.5 & -1 \le t < 0 \end{cases}$$

This representation making the Fourier coefficients easier to calculate as

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\Omega_0 t}dt = \frac{1}{2}\left[\int_{-1}^{0} -1.5e^{-jk\pi t}dt + \int_{0}^{1} 1.5e^{-jk\pi t}dt\right]$$

For $k = 0$,

$$a_0 = \frac{1}{2}\left[\int_{-1}^{0} -1.5dt + \int_{0}^{1} 1.5dt\right] = \frac{1}{2}[-1.5 + .15] = 0$$

For $k \neq 0$,

$$a_k = \frac{1}{2}\left[\int_{-1}^{0} -1.5e^{-jk\pi t}dt + \int_{0}^{1} 1.5e^{-jk\pi t}dt\right]$$

$$= \frac{3}{4}\left[\int_{-1}^{0} -e^{-jk\pi t}dt + \int_{0}^{1} e^{-jk\pi t}dt\right]$$

$$= \frac{3}{4}\left[-\frac{1}{-jk\pi}e^{-jk\pi t}\Big|_{-1}^{0} + \frac{1}{-jk\pi}e^{-jk\pi t}\Big|_{0}^{1}\right]$$

$$= \frac{3}{4jk\pi}[1 - e^{jk\pi} - e^{-jk\pi t} + 1]$$

$$= \frac{3}{4jk\pi}[2 - 2\cos(k\pi)]$$

$$= \frac{3}{j2k\pi}[1 - \cos(k\pi)]$$

Question 2

For $\Omega = 0$, prove

$$\frac{2\sin(\Omega T)}{\Omega} = 2T$$

We can use L'Hopital's rule to obtain

$$\lim_{\Omega\to 0}\frac{2\sin(\Omega T)}{\Omega} = \lim_{\Omega\to 0}\frac{\frac{d(2\sin(\Omega T))}{d\Omega}}{\frac{d\Omega}{d\Omega}} = \lim_{\Omega\to 0}\frac{2T\cos(\Omega)}{1} = 2T$$

Question 3

Find the real part, imaginary part, magnitude and phase of

$$X(j\Omega) = \frac{1}{a + j\Omega}$$

We can first simplify this expression to

$$X(j\Omega) = \frac{1}{a + j\Omega} = \frac{1}{a + j\Omega}\cdot\frac{a - j\Omega}{a - j\Omega} = \frac{a - j\Omega}{a^2 + \Omega^2} = \frac{a}{a^2 + \Omega^2} + j\frac{-\Omega}{a^2 + \Omega^2}$$

As a result, real and imaginary parts of $X(j\Omega)$ can be identifies as:

$$\frac{a}{a^2+\Omega^2} \qquad \frac{-\Omega}{a^2+\Omega^2}$$

The magnitude is

$$|X(j\Omega)| = \sqrt{\left(\frac{a}{a^2 + \Omega^2}\right)^2 + \left(\frac{-\Omega}{a^2 + \Omega^2}\right)^2} = \sqrt{\frac{a^2 + \Omega^2}{(a^2 + \Omega^2)^2}} = \frac{1}{\sqrt{a^2 + \Omega^2}}$$

And the phase is

$$\angle X(j\Omega) == \tan^{-1}\left(\frac{\frac{-\Omega}{a^2 + \Omega^2}}{\frac{a}{a^2 + \Omega^2}}\right) = \tan^{-1}(\frac{-\Omega}{a}) = -\tan^{-1}(\frac{\Omega}{a})$$

Note that the magnitude can also be found in an easier manner as follows:

$$|X(j\Omega)| = \sqrt{X(j\Omega)\cdot X^*(j\Omega)} = \sqrt{\frac{1}{a + j\Omega}\cdot\frac{1}{a - j\Omega}} = \frac{1}{\sqrt{a^2 + \Omega^2}}$$

Question 4

Compute the Continuous-Time Fourier transform of

$$x(t) = e^{-\alpha|t|}, \quad \alpha > 0$$

We can first express this signal in the following form for calculating the CTFT:

$$x(t) = \begin{cases} e^{-\alpha t}, & t \geq 0 \\ e^{\alpha t}, & t < 0 \end{cases}$$

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\,dt = \int_{-\infty}^{0} e^{\alpha t}e^{-j\Omega t}\,dt + \int_{0}^{\infty} e^{-\alpha t}e^{-j\Omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{(\alpha - j\Omega)t}\,dt + \int_{0}^{\infty} e^{-(\alpha+\Omega j)t}\,dt$$

$$= \frac{1}{(\alpha - j\Omega)}e^{(\alpha - j\Omega)t}\Big|_{-\infty}^{0} + \frac{1}{-(\alpha+\Omega j)}e^{-(\alpha+\Omega j)t}\Big|_{0}^{\infty}$$

$$= \frac{1}{\alpha - j\Omega} + \frac{1}{\alpha + \Omega j}$$

$$= \frac{1}{\alpha - j\Omega}\cdot\frac{\alpha + j\Omega}{\alpha + j\Omega} + \frac{1}{\alpha + \Omega j}\cdot\frac{\alpha - j\Omega}{\alpha - j\Omega}$$

$$= \frac{2\alpha}{\alpha^2 + \Omega^2}$$

As $X(j\Omega)$ is real and $\alpha > 0$, we have:

$$|X(j\Omega)| = X(j\Omega) = \frac{2\alpha}{\alpha^2 + \Omega^2}$$

and

$$\angle X(j\Omega) = 0$$

Question 5

Compute the Fourier transform of $x(t) = \cos(100t)$. However, we should realize that it is difficult to calculate the Fourier Transform of the sinusoidal signal using the transformation equation.

On the other hand, we know that the inverse Fourier Transform of shifted impulse in Transform domain is a complex exponential signal as shown below:

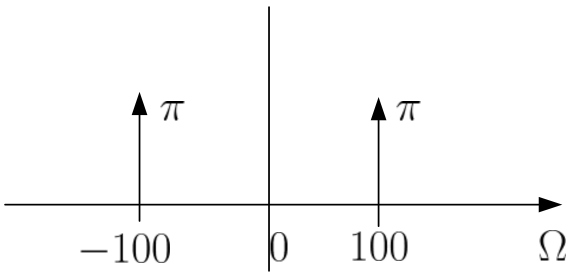
$$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega - \Omega_0)$$

According to the above expression, we know

$$\cos(\Omega_0 t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$$

As a result,

$$\cos(100t) = \frac{e^{j100t} + e^{-j100t}}{2} \leftrightarrow \pi\delta(\Omega + 100) + \pi\delta(\Omega - 100)$$



Question 6

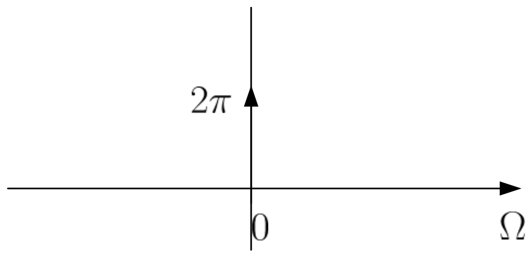
Compute the Fourier transform of $x(t) = 1$, we can apply again:

$$e^{j\Omega_0 t} \quad \leftrightarrow \quad 2\pi\delta(\Omega - \Omega_0)$$

A DC signal of $x(t) = 1$ corresponds to the frequency of $\Omega = 0$ as it can be expressed as $x(t) = 1 = e^{j\cdot 0\cdot t}$.

As a result,

$$1 = e^{j\cdot 0\cdot t} \quad \leftrightarrow \quad 2\pi\delta(\Omega)$$



Question 7

$$x(t) = 3\cos(2000\pi t) + 5\sin(6000\pi t) + 10\cos(12000\pi t)$$

(a) What is the Nyquist rate for this signal?

The frequencies existing in the analog signal are:

$$f_1 = \frac{\Omega_1}{2\pi} = \frac{2000\pi}{2\pi} = 1000\ Hz = 1\ kHz$$
$$f_2 = \frac{\Omega_2}{2\pi} = \frac{6000\pi}{2\pi} = 3000\ Hz = 3\ kHz$$
$$f_3 = \frac{\Omega_3}{2\pi} = \frac{12000\pi}{2\pi} = 6000\ Hz = 6\ kHz$$

Thus, $f_{max} = 6\ kHz$ and according to the sampling theorem,

$$F_s \geq 2\ f_{max} = 12\ kHz$$

The Nyquist rate is $12\ kHz$

(b) Assume now that we sample this signal using a sample rate $F_s = 5\ kHz$. What is the discrete-time signal $x[n]$ obtained after sampling?

$$x(t) = 3\cos(2000\pi t) + 5\sin(6000\pi t) + 10\cos(12000\pi t)$$
$$x(t) = 3\cos(2\pi \cdot 1000t) + 5\sin(2\pi \cdot 3000t) + 10\cos(2\pi \cdot 6000t)$$
$$T = \frac{1}{F_s} = \frac{1}{5000}$$

$$\begin{aligned}
x[n] = x(nT) &= 3\cos\left(2\pi\frac{1000}{5000}n\right) + 5\sin\left(2\pi\frac{3000}{5000}n\right) + 10\cos\left(2\pi\frac{6000}{5000}nt\right)\\
&= 3\cos\left(2\pi\left(\frac{1}{5}\right)n\right) + 5\sin\left(2\pi\left(\frac{3}{5}\right)n\right) + 10\cos\left(2\pi\left(\frac{6}{5}\right)n\right)\\
&= 3\cos\left(2\pi\left(\frac{1}{5}\right)n\right) + 5\sin\left(2\pi\left(1-\frac{2}{5}\right)n\right) + 10\cos\left(2\pi\left(1+\frac{1}{5}\right)n\right)\\
&= 3\cos\left(2\pi\left(\frac{1}{5}\right)n\right) + 5\sin\left(-2\pi\left(\frac{2}{5}\right)n\right) + 10\cos\left(2\pi\left(\frac{1}{5}\right)n\right)\\
&= 13\cos\left(2\pi\left(\frac{1}{5}\right)n\right) - 5\sin\left(2\pi\left(\frac{2}{5}\right)n\right)
\end{aligned}$$

- $\sin(x+y) = \sin(x)\cos(y) + \cos(y)\sin(x)$
- $\cos(x+y) = \cos(x)\cos(y) - \sin(y)\sin(x)$

(c) What is the analog signal $y(t)$ we can reconstruct from the samples if we use ideal anti-imaging lowpass reconstruction filter?

Since only frequency components at 1 kHz and 2 kHz are present in the sampled signal, the analog signal we can recover is

$$y(t) = 13\cos(2000\pi t) - 5\sin(4000\pi t)$$

Which is obviously different from the original signal $x(t)$.

The distortion of the original analog signal was caused by the aliasing effect, due to the low sampling rate of 5 kHz used, which lower than the Nyquist rate of 12 kHz.