Exercises for Week 4

1. Determine the discrete-time Fourier transforms (DTFTs) for

$$x[n] = (0.5)^n u(n) \quad \text{and} \quad y[n] = 2^n\, u(n)$$

2. Let $h_c(t)$ be the impulse response of a linear time-invariant (LTI) continuous-time system and it has the form of:

$$h_c(t) = \begin{cases} e^{-at}, & t \geq 0,\ a > 0 \\ 0, & t < 0 \end{cases}$$

(a) Determine the Fourier transform of $h_c(t)$, $H_c(j\Omega)$.
(b) The $h_c(t)$ is sampled with a sampling period of T to produce a sequence $h_c[n]$. Determine the DTFT of $h_c[n]$, $H_c(e^{j\omega})$.
(c) Find the maximum values for $|H_c(j\Omega)|$and $|H_c(e^{j\omega})|$.

3. Consider a LTI system with input $x[n]$, output $y[n]$ and impulse response $h[n]$. Let the DTFT of $h[n]$ be $H(e^{j\omega})$.

(a) If $x[n] = e^{j\omega_1 n}$, determine $y[n]$ in terms of $H(e^{j\omega})$.

(b) Extend the result of (a) when

$$x[n] = \sum_{k=1}^{K} \alpha_k e^{j\omega_k n}$$

4. Let $X(e^{j\omega})$ denote the DTFT of $x[n]$. Prove the following two properties:

(a) The DTFT of $x^*[n]$ is $X^*(e^{-j\omega})$.

(b) The DTFT of $x^*[-n]$ is $X^*(e^{j\omega})$.

Suggested Solution

Question 1

$$X(e^{j\omega}) = DTFT\{(0.5)^n u[n]\} = \sum_{n=0}^{\infty} (0.5)^n e^{-j\omega n} = \frac{1}{1 - 0.5e^{-j\omega}}$$

$$Y(e^{j\omega}) = DTFT\{(2)^n u[n]\} = \sum_{n=0}^{\infty} (2)^n e^{-j\omega n} = \infty$$

However, the summation of $\sum_{n=0}^{\infty} |(2)^n|$ is not absolutely summable, thus the DTFT of $y[n]$ is not exist.

Question 2

Given $h_c(t) = \begin{cases} e^{-at}, & t \geq 0, a > 0 \\ 0, & t < 0 \end{cases} = e^{-at}u(t)$ with $a > 0$

(a) The CTFT of $h_c(t)$ is

$$H_c(j\Omega) = \int_{-\infty}^{\infty} h_c(t) e^{-j\Omega t} dt = \int_{0}^{\infty} e^{-at} e^{-j\Omega t} dt = \int_{0}^{\infty} e^{-(a+j\Omega)t} dt$$

$$= \frac{e^{-(a+j\Omega)t}}{-(a+j\Omega)}\bigg|_0^{\infty} = \frac{e^{-(a+j\Omega)\infty}}{-(a+j\Omega)} - \frac{e^{-(a+j\Omega)0}}{-(a+j\Omega)} = \frac{1}{a+j\Omega}$$

(b) The sampled version of $h_c(t)$

$$h_c[n] = h_c(nT) = \begin{cases} e^{-aTn}, & n \geq 0, a > 0 \\ 0, & n < 0 \end{cases} = e^{-aTn}u[n]$$

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} e^{-aTn} e^{-j\omega n} = \frac{1}{1 - e^{-aT}e^{-j\omega}} = \frac{1}{1 - e^{-(aT+j\omega)}}$$

(c)

$$|H_c(j\Omega)| = \sqrt{H_c(j\Omega)H_c^*(j\Omega)} = \sqrt{\frac{1}{a+j\Omega}\cdot\frac{1}{a-j\Omega}} = \sqrt{\frac{1}{a^2+\Omega^2}}$$

When $\Omega = 0$, $|H_c(j\Omega)|$ is maximum. Thus, the maximum value of $|H_c(j\Omega)|$ is

$$|H_c(j0)| = \sqrt{\frac{1}{a^2+0^2}} = \frac{1}{a}$$

$$|H(e^{j\omega})| = \sqrt{H(e^{j\omega})H^*(e^{j\omega})} = \sqrt{\frac{1}{1-e^{-(aT+j\omega)}}\cdot\frac{1}{1-e^{-(aT-j\omega)}}}$$

$$= \sqrt{\frac{1}{1-e^{-aT}e^{j\omega}-e^{-aT}e^{-j\omega}+e^{-2aT}}} = \sqrt{\frac{1}{1-e^{-aT}[e^{j\omega}+e^{-j\omega}]+e^{-2aT}}}$$

$$= \sqrt{\frac{1}{1-2e^{-aT}\cos\omega+e^{-2aT}}}$$

When $\omega = 0$, $|H(e^{j\omega})|$ is maximum. Thus, the maximum value of $|H(e^{j\omega})|$ is

$$|H(e^{j0})| = \sqrt{\frac{1}{1-2e^{-aT}+e^{-2aT}}} = \sqrt{\frac{1}{(1-e^{-aT})^2}} = \frac{1}{1-e^{-aT}}$$

Question 3

3(a) $H(e^{j\omega})$ is the frequency response of the LTI system with unit impulse response of $h[n]$.

For the input sequence of $x[n] = e^{j\omega_1 n}$, then the output sequence

$$y[n] = x[n] * h[n] = e^{j\omega_1 n} * h[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k]e^{j\omega_1(n-k)} = e^{j\omega_1 n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_1 k} = H(e^{j\omega_1})\, e^{j\omega_1 n}$$

3(b) Use the linearity property of the LTI system, when $x[n] = \sum_{k=1}^{K} \alpha_k e^{j\omega_k n}$, the output sequence is equal to

$$y[n] = \sum_{k=1}^{K} \alpha_k H(e^{j\omega_k})e^{j\omega_k n}$$

Question 4

$X(e^{j\omega})$ denote the DTFT of $x[n]$, then

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

4(a) The DTFT of $x^*[n]$ is $X^*(e^{-j\omega})$

$$X^*(e^{j\omega}) = \left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right)^* = \sum_{n=-\infty}^{\infty} x^*[n]e^{j\omega n}$$

$$X^*(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n}$$

Then,

$$DTFT\{x^*[n]\} = \sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n} = X^*(e^{-j\omega})$$

4.(b) The DTFT of $x^*[-n]$ is $X^*(e^{j\omega})$.

$$DTFT\{x^*[-n]\} = \sum_{n=-\infty}^{\infty} x^*[-n]e^{-j\omega n}$$

Let $m = -n$

$$DTFT\{x^*[-n]\} = \sum_{m=\infty}^{-\infty} x^*[m]e^{j\omega m} = \sum_{m=-\infty}^{\infty} x^*[m]e^{j\omega m} = \left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}\right)^* = X^*(e^{j\omega})$$