Exercises for Week 5

1. Prove the following orthogonality identity:

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)(k-r)n} = \begin{cases} 1, & k = r \\ 0, & k = 0,1,\cdots,N-1, k \neq r \end{cases}$$

2. Let $X[k]$ be the N-point Discrete Fourier transform (DFT) of a N-point sequence $x[n]$ for $0 \leq n \leq N-1$.

   (a) Show that if $x[n]$ satisfies $x[n] = -x[N-1-n]$, then $x[0] = 0$. Consider separately the cases of odd and even N.

   (b) Show that if N is even and $x[n] = x[N-1-n]$, then $x[N/2] = 0$.

3. Determine the DFT of

$$x[n] = \begin{cases} a^n, & 0 \leq n \leq N-1 \\ 0, & otherwise \end{cases}$$

4. Let $\tilde{x}[n] \leftrightarrow \tilde{X}[k]$be a discrete Fourier Series (DFS) pair. Prove the symmetric properties:

$$\tilde{x}^*[n] \leftrightarrow \tilde{X}^*[-k]$$

and

$$\tilde{x}^*[-n] \leftrightarrow \tilde{X}^*[k]$$

Then show that if $\tilde{x}[n]$ is real, we have:

- $\mathrm{Re}\{\tilde{X}[k]\} = \mathrm{Re}\{\tilde{X}[-k]\}$ and $\mathrm{Im}\{\tilde{X}[k]\} = -\mathrm{Im}\,\{\tilde{X}[-k]\}$
- $\left|\tilde{X}[k]\right| = \left|\tilde{X}[-k]\right|$ and $\angle\tilde{X}[k] = -\angle\tilde{X}[-k]$

Solution

Question 1

For $k = r$:

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)(k-r)n} = \frac{1}{N}\sum_{n=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)\cdot 0\cdot n} = \frac{1}{N}\sum_{n=0}^{N-1} 1 = \frac{1}{N}\cdot N = 1$$

For $k \neq r$, let $l = k - r$ where $l \neq 0$. We have:

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)(k-r)n} = \frac{1}{N}\sum_{n=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)l\cdot n} = \frac{1}{N}\cdot\frac{1-\left(e^{j\left(\frac{2\pi}{N}\right)l}\right)^N}{1-\left(e^{j\left(\frac{2\pi}{N}\right)l}\right)} = \frac{1}{N}\cdot\frac{1-e^{j2\pi l}}{1-e^{j\left(\frac{2\pi}{N}\right)l}} = 0$$

Question 2

2(a)

With $W_N^{kn} = e^{-j\left(\frac{2\pi}{N}\right)kn}$

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn} \Longrightarrow X[0] = \sum_{n=0}^{N-1} x[n]$$

For even N and using $x[n] = -x[N-1-n]$, we have:

$$X[0] = \sum_{n=0}^{N-1} x[n] = (x[0] + x[N-1]) + (x[1] + x[N-2]) + \cdots + \left(x\left[\frac{N}{2}-1\right] + x\left[\frac{N}{2}\right]\right) = 0$$

For odd N and using $x[n] = -x[N-1-n]$, we get:

$$x[(N-1)/2] = -x[N-1-(N-1)/2] = -x[(N-1)/2] \Longrightarrow x[(N-1)/2] = 0$$

Hence:

$$X[0] = \sum_{n=0}^{N-1} x[n] = (x[0] + x[N-1]) + \cdots + \left(x\left[\frac{N-1}{2}-1\right] + x\left[\frac{N-1}{2}+1\right]\right) + x\left[\frac{N-1}{2}\right] = 0$$

2(b)
For even N and using $x[n] = x[N-1-n]$, we have:

$$X[N/2] = \sum_{n=0}^{N-1} x[n] W_N^{(N/2)n} = \sum_{n=0}^{N-1} x[n](-1)^n$$

$$= x[0] - x[1] + x[2] - \cdots + x[N-2] - x[N-1]$$

$$= (x[0] - x[N-1) - (x[1] - x[N-2]) + \cdots + (-1)^{\frac{N}{2}-1}(x[N/2-1] - x[N/2])$$

= 0

<u>Question 3</u>

For $0 \le k \le N-1$:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = \sum_{n=0}^{N-1} a^n e^{-j\left(\frac{2\pi}{N}\right)kn} = \sum_{n=0}^{N-1} \left(ae^{-j\left(\frac{2\pi}{N}\right)k}\right)^n$$

$$= \frac{1 - a^N e^{-j2\pi k}}{1 - ae^{-j2\pi k/N}} = \frac{1 - a^N}{1 - ae^{-j2\pi k/N}}$$

As a result, we have:

$$X[k] = \begin{cases} \dfrac{1 - a^N}{1 - ae^{-j2\pi k/N}}, & 0 \le k \le N-1 \\ 0, & otherwise \end{cases}$$

Question 4

Let $\tilde{x}_1[n] = \tilde{x}^*[n]$. Then

$$\widetilde{X}_1[k] = \sum_{n=0}^{N-1} \tilde{x}^*[n] W_N^{kn} = \left(\sum_{n=0}^{N-1} \tilde{x}[n] W_N^{(-k)n} \right)^* = \widetilde{X}^*[-k]$$

Let $\tilde{x}_2[n] = \tilde{x}^*[-n]$. Then

$$\begin{aligned}\widetilde{X}_2[k] &= \sum_{n=0}^{N-1} \tilde{x}^*[-n] W_N^{kn} = \sum_{n=0}^{N-1} \tilde{x}^*[N-n] W_N^{kn} = \sum_{m=N}^{1} \tilde{x}^*[m] W_N^{k(N-m)} \\ &= \sum_{m=1}^{N} \tilde{x}^*[m] W_N^{-km} = \sum_{m=0}^{N-1} \tilde{x}^*[m] W_N^{-km}, \quad \because \tilde{x}^*[0] W_N^{-k\cdot 0} = \tilde{x}^*[N] W_N^{-k\cdot N} \\ &= \left(\sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} \right)^* = \widetilde{X}^*[k]\end{aligned}$$

Note that $\tilde{x}^*[0] W_N^{-k\cdot 0} = \tilde{x}^*[0] = \tilde{x}^*[N] = \tilde{x}^*[N] W_N^{-k\cdot N}$.

If $\tilde{x}[n]$ is real, then $\tilde{x}[n] = \tilde{x}^*[n]$. So we have:

$$\tilde{X}[k] = \tilde{X}^*[-k]$$

=>

$$\text{Re}\{\tilde{X}[k]\} + j\,\text{Im}\{\tilde{X}[k]\} = \text{Re}\{\tilde{X}^*[-k]\} + j\,\text{Im}\{\tilde{X}^*[-k]\} = \text{Re}\{\tilde{X}[-k]\} - j\,\text{Im}\{\tilde{X}[-k]\}$$

Equating the real and imaginary parts on both sides, we get:

$$\text{Re}\{\tilde{X}[k]\} = \text{Re}\{\tilde{X}[-k]\}$$
$$\text{Im}\{\tilde{X}[k]\} = -\text{Im}\{\tilde{X}[-k]\}$$

Moreover,

$$|\tilde{X}[k]| = \sqrt{\left(\text{Re}\{\tilde{X}[k]\}\right)^2 + \left(\text{Im}\{\tilde{X}[k]\}\right)^2} = \sqrt{\left(\text{Re}\{\tilde{X}[-k]\}\right)^2 + \left(-\text{Im}\{\tilde{X}[-k]\}\right)^2}$$

$$= \sqrt{\left(\text{Re}\{\tilde{X}[-k]\}\right)^2 + \left(\text{Im}\{\tilde{X}[-k]\}\right)^2} = |\tilde{X}[-k]|$$

$$\angle\tilde{X}[k] = \tan^{-1}\left(\frac{\text{Im}\{\tilde{X}[k]\}}{\text{Re}\{\tilde{X}[k]\}}\right) = \tan^{-1}\left(\frac{-\text{Im}\{\tilde{X}[-k]\}}{\text{Re}\{\tilde{X}[-k]\}}\right) = -\tan^{-1}\left(\frac{\text{Im}\{\tilde{X}[-k]\}}{\text{Re}\{\tilde{X}[-k]\}}\right) = -\angle\tilde{X}[-k]$$

With circular shift, DFT has the same properties as DFS, the following results can also be deduced accordingly:

$$\mathrm{Re}\{X[k]\} = \mathrm{Re}\{X[N-k]\}$$

$$\mathrm{Im}\{X[k]\} = -\mathrm{Im}\{X[N-k]\}$$

$$|X[k]| = |X[N-k]|$$

$$\angle X[k] = -\angle X[N-k], \quad k = 0,1,\ldots,N-1$$

=> For real signals, we only need to compute around half of the DFT coefficients ($k = 0,1,\ldots,N/2$ for even N)