EE4015 Exercises for Week 6

1. Determine the z-transform of

$$x[n] = (0.5)^n (u[n+5] - u[n-5])$$

Specify its region of convergence (ROC).

2. A causal linear time-invariant (LTI) system has impulse response $h[n]$ and its z-transform is

$$H(z) = \frac{1 + z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{4}z^{-1}\right)}$$

(a) What is the ROC of $H(z)$?
(b) Is the system stable? Why?
(c) Find the impulse response $h[n]$ of the system.

3. If the input $x[n]$ to a LTI system is $x[n] = u[n]$, the system output $y[n]$ is

$$y[n] = \left(\frac{1}{2}\right)^{n-1} u[n+1]$$

(a) Find $H(z)$, the z-transform of the system impulse response, and plot its pole-zero diagram.
(b) Find the system impulse response $h[n]$.
(c) Is the system stable?
(d) Is the system causal?

Hint: Consider the z-transform convolution property:

$$x_1[n] * x_2[n] \overset{Z}{\leftrightarrow} X_1(z) \cdot X_2(z), \qquad ROC \ \ includes\ R_{x1} \cap \ R_{x2}$$

<u>**Solution**</u>

1.

$$X(z) = \sum_{n=-5}^{4} (0.5z^{-1})^n$$

$$= \left(\frac{0.5}{z}\right)^{-5} + \left(\frac{0.5}{z}\right)^{-4} + \cdots + \left(\frac{0.5}{z}\right)^{4}$$

$$= (0.5)^{-5}z^5 + (0.5)^{-4}z^4 + \cdots + (0.5)^4 z^{-4}$$

$$= (0.5z^{-1})^{-5}\frac{1-(0.5z^{-1})^{10}}{1-0.5z^{-1}}$$

$$= \frac{(0.5z^{-1})^{-5} - (0.5z^{-1})^5}{1-0.5z^{-1}}$$

Basically, this is a finite duration sequence with 10 non-zero samples. Thus, the ROC is $0 < |z| < \infty$.

In the last expression, the pole at $z = 0.5$ is cancelled by the zero at $z = 0.5$.

2.(a)
There are two poles, namely, 0.5 and -0.25. Since the system is causal, $h[n]$ should be a right-sided sequence, hence the ROC should be $|z| > 0.5$.

2.(b)
The system is stable since the ROC includes the unit circle.

2.(c)

$$H(z) = \frac{1+z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)} = \frac{2}{1-\frac{1}{2}z^{-1}} - \frac{1}{1+\frac{1}{4}z^{-1}}, \quad |z| > 0.5$$

$$=> \quad h[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{4}\right)^n u[n]$$

3.(a)
The z-transform of the input signal is

$$x[n] \overset{Z}{\leftrightarrow} X(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

The output signal can be written as:

$$y[n] = \left(\frac{1}{2}\right)^{n-1} u[n+1] = 4\left(\frac{1}{2}\right)^{n+1} u[n+1]$$

Using $a^n u[n] \overset{Z}{\leftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| > |a|$and the time shifting property of $x[n+1] \overset{Z}{\leftrightarrow} zX(z)$, we have

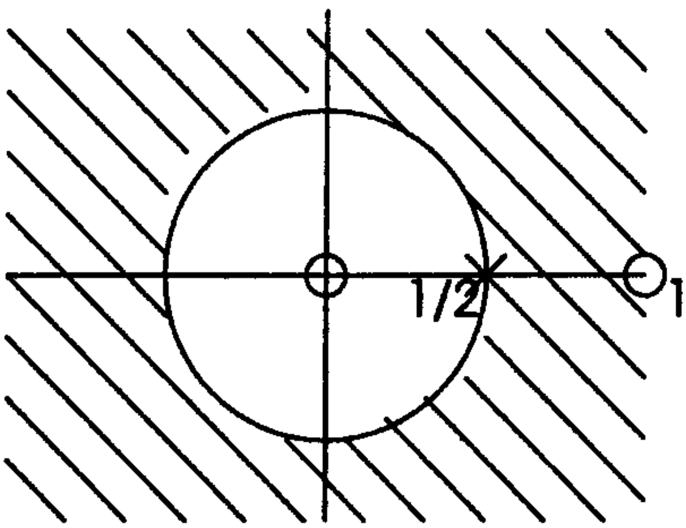
$$y[n] \overset{Z}{\leftrightarrow} Y(z) = \frac{4z}{1 - 0.5z^{-1}}, \quad |z| > 0.5$$

As a result, $H(z)$ is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{4z(1 - z^{-1})}{1 - 0.5z^{-1}} = \frac{4z(z-1)}{z - 0.5}, \quad |z| > 0.5$$

Note that:

- One pole in $H(z)$ means that there are only two ROC possibilities: $|z| > 0.5$ or $|z| < 0.5$
- $|z| > 0.5$ includes $|z| > 1 \cap |z| > 0.5$



3.(b)

$$H(z) = \frac{4z(z-1)}{z-0.5} = \frac{4z}{1-0.5z^{-1}} - \frac{4}{1-0.5z^{-1}}, \quad |z| > 0.5$$

$$=> \quad h[n] = 4(0.5)^{n+1}u[n+1] - 4(0.5)^n u[n] = 4\delta[n+1] - 2(0.5)^n u[n]$$

3.(c)

It is stable because $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$. (Or the ROC contains the unit circle)

3.(d)
It is not causal as $h[-1] = 4 \neq 0$.

It is noteworthy that although $|z| > 0.5$ implies the discrete-time signal is a right-sided sequence, it can be non-causal. That is, right-sided sequence is not necessarily a causal signal.