Exercises for Week 7

1. Consider the following discrete-time frequency responses:

$$H_1(e^{j\omega}) = \frac{1}{1 - 0.1e^{-j\omega}}$$

and

$$H_2(e^{j\omega}) = \frac{1}{1 - 10e^{-j\omega}}$$

Discuss the causality of the systems which correspond to the two spectra.

2. Consider a causal linear time-invariant system with system function

$$H(z) = \frac{1 - a^{-1}z^{-1}}{1 - az^{-1}}, \quad a \text{ is real}$$

(a) Write the difference equation that relates the input $x[n]$ and output $y[n]$ of this system.

(b) For what range of values of a is the system stable?

(c) Find the impulse response of the system.

(d) Is the system a finite impulse response (FIR) or infinite impulse response (IIR) filter?

(e) Assume $|a| < 1$. Show that the system is an all-pass system, i.e., the magnitude of the frequency response is a constant. Also, specify the value of this constant.

3. Consider a discrete-time signal $x[n]$ is passed through a system with transfer function $H(z)$ to produce an output $y[n]$.

Given $y[n]$, is it possible to get back $x[n]$?

This is referred to as an equalization or deconvolution problem which arises in many applications such as communications. For example, the transmitter sends out information $x[n]$. After passing through the transmission channel (telephone line, air, etc.), the receiver obtains $y[n]$ which is a filtered version of $x[n]$.

(Hint: consider a simple case of $H(z) = 1 - az^{-1}$)

Solution

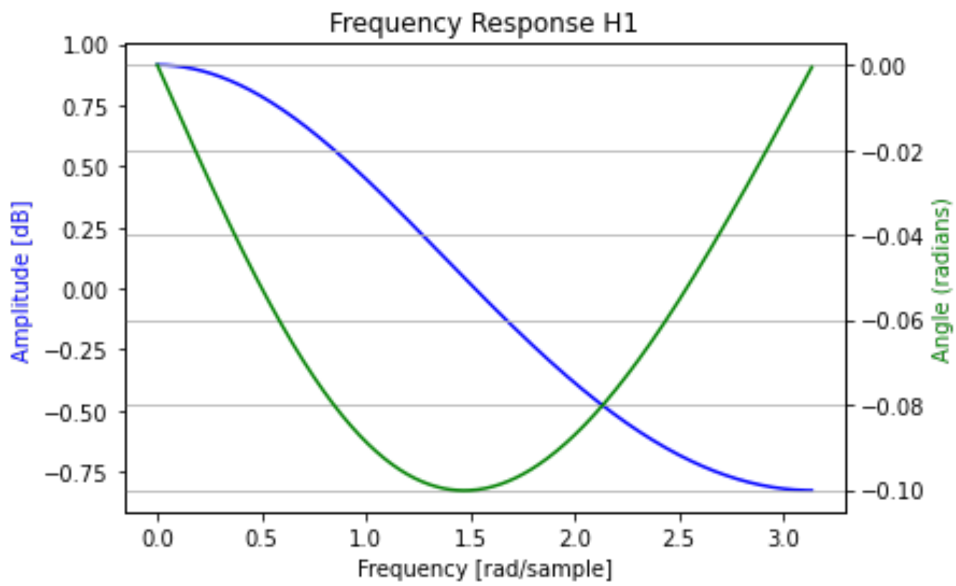
Question 1

Since $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ exist, it is clear that they are **stable** and their ROCs should include the unit circle.

That is, the ROC of $H_1(e^{j\omega})$ is $|z| > 0.1$ which corresponds to a causal system. While the ROC of $H_2(e^{j\omega})$ is $|z| < 0.1$ which corresponds to a noncausal system.

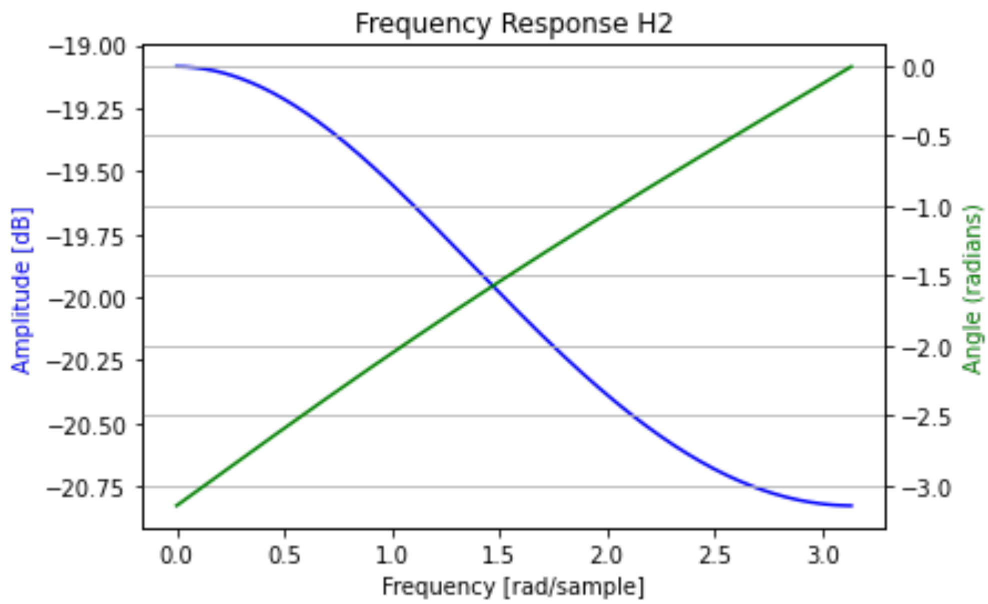
Plot the spectra of $H_1(e^{j\omega})$ using Python:

```
from scipy.signal import freqz
import numpy as np

b = np.array([1])
a = np.array([1, -0.1])
w, h = freqz(b, a)

import matplotlib.pyplot as plt
fig, ax1 = plt.subplots()
ax1.set_title('Frequency Response H1')

ax1.plot(w, 20 * np.log10(abs(h)), 'b')
ax1.set_ylabel('Amplitude [dB]', color='b')
ax1.set_xlabel('Frequency [rad/sample]')

ax2 = ax1.twinx()
angles = np.unwrap(np.angle(h))
ax2.plot(w, angles, 'g')
ax2.set_ylabel('Angle (radians)', color='g')
ax2.grid()
ax2.axis('tight')
plt.show()
```



Plot the spectra of $H_2(e^{j\omega})$ using Python:

```
from scipy.signal import freqz
import numpy as np

b = np.array([1])
a = np.array([1, -10.0])
w, h = freqz(b, a)

import matplotlib.pyplot as plt
fig, ax1 = plt.subplots()
ax1.set_title('Frequency Response H2')

ax1.plot(w, 20 * np.log10(abs(h)), 'b')
ax1.set_ylabel('Amplitude [dB]', color='b')
ax1.set_xlabel('Frequency [rad/sample]')

ax2 = ax1.twinx()
angles = np.unwrap(np.angle(h))
ax2.plot(w, angles, 'g')
ax2.set_ylabel('Angle (radians)', color='g')
ax2.grid()
ax2.axis('tight')
plt.show()
```



Since $H_1(e^{j\omega})$ is stable, the corresponding impulse response should be

$$H_1(e^{j\omega}) = \frac{1}{1 - 0.1e^{-j\omega}} \leftrightarrow h_1[n] = (0.1)^n u[n]$$

which implies that the system is causal as well.

Question 2

2(a) Cross multiplying and taking the inverse z-transform, we get:

$$y[n] - ay[n-1] = x[n] - \frac{1}{a}x[n-1]$$

2(b) Since $H(z)$ is causal and $H(z) = \frac{1-a^{-1}z^{-1}}{1-az^{-1}} = \frac{z-a^{-1}}{z-a}$ with a single pole at $z = a$, then the ROC of $H(z)$ should be $|z| > |a|$.

For stability, the ROC must include the unit circle, which implies $|a| < 1$ such that the pole $z = a$ is located inside the unit circle.

2(c)

$$H(z) = \frac{1}{1-az^{-1}} - \frac{a^{-1}z^{-1}}{1-az^{-1}}, \qquad |z| > |a|$$

$$\Rightarrow h[n] = (a)^n u[n] - \frac{1}{a}(a)^{n-1}u[n-1]$$

2(d) The system is an IIR filter

2(e)
If $|a| < 1$, the DTFT converges, we have

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \frac{1-a^{-1}e^{-j\omega}}{1-ae^{-j\omega}}$$

$$|H(e^{j\omega})|^2 = H(e^{j\omega}) \cdot H^*(e^{j\omega}) = \frac{1-a^{-1}e^{-j\omega}}{1-ae^{-j\omega}} \cdot \frac{1-a^{-1}e^{j\omega}}{1-ae^{j\omega}}$$

$$|H(e^{j\omega})| = \left(\frac{1+\frac{1}{a^2}-\frac{2}{a}\cos\omega}{1+a^2-2a\cos\omega}\right)^{\frac{1}{2}} = \frac{1}{|a|}\left(\frac{a^2+1-2a\cos\omega}{1+a^2-2a\cos\omega}\right)^{\frac{1}{2}} = \frac{1}{|a|}$$

As a result, the magnitude frequency response is a constant of $\frac{1}{|a|}$.

Question 3

Consider $H(z) = 1 - az^{-1}$ which is causal and stable:

$$Y(z) = H(z) \cdot X(z) = X(z) - az^{-1}X(z) \Longrightarrow y[n] = x[n] - ax[n-1]$$

To get back $x[n]$ from $y[n]$, we need to pass $y[n]$ through the following system:

$$G(z) = \frac{1}{1 - az^{-1}}$$

$$R(z) = G(z) \cdot Y(z) = \frac{1}{1 - az^{-1}} Y(z)$$

$$\Longrightarrow R(z)(1 - az^{-1}) = Y(z) \Longrightarrow r[n] = ar[n-1] + y[n]$$

In practical implementation, the inverse system should be causal and stable. That is, the impulse response is $g[n] = a^n u[n]$ which requires $|a| < 1$.

This also means that the zero of (i.e., the pole of $1/H(z)$) should be inside the unit circle.

For the general case:

$$H(z) = \frac{B(z)}{A(z)}$$

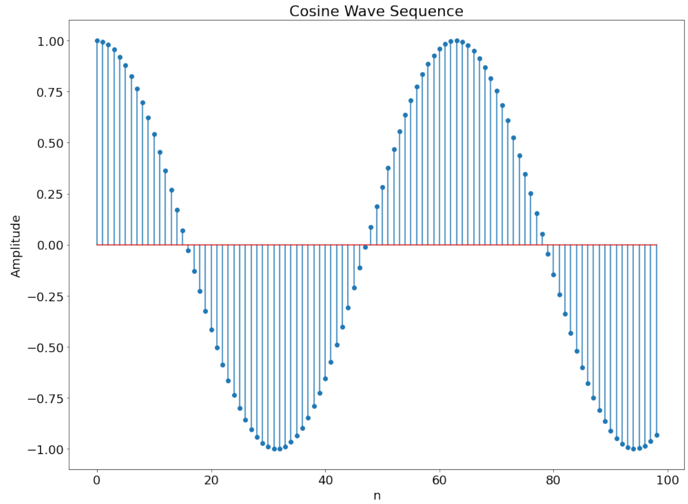
where $A(z)$ and $B(z)$ are polynomials in z^{-1}.

For a feasible $H(z)$, it should be stable and thus all its poles should be inside the unit circle.

For a feasible inverse system, i.e., $1/H(z)$, it should be stable and thus all its poles should be inside the unit circle, which means all the zeros of $H(z)$ should be inside the unit circle. The system is referred to as minimum phase system.

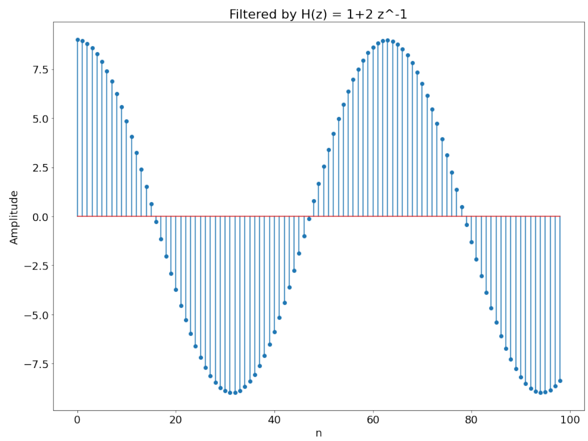
Demonstration via Python example:

```
# Generate Cosin wave
import matplotlib.pyplot as plt
import numpy as np

n = np.arange(0,99,1)
x = np.cos(0.1*n)

plt.rcParams['figure.figsize']=[16,8]
plt.rcParams.update({'font.size':18})

plt.stem(n,x);
plt.title('Cosine Wave Sequence')
plt.xlabel('n')
plt.ylabel('Amplitude')
plt.show()
```



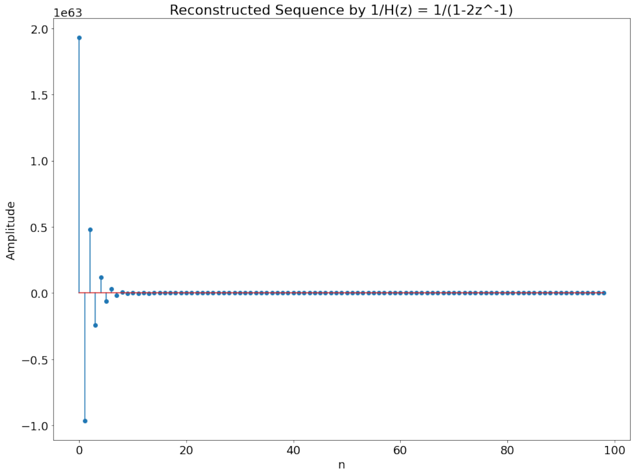
Filter this sequence by casual $H(z) = 1 + 2z^{-1}$ and stable output is observed

```
from scipy import signal
import matplotlib.pyplot as plt

y = signal.filtfilt(np.array([1, 2]),
np.array([1]), x)

plt.stem(n,y);
plt.title('Filtered by H(z) = 1+2 z^-
1')
plt.xlabel('n')
plt.ylabel('Amplitude')
plt.show()
```



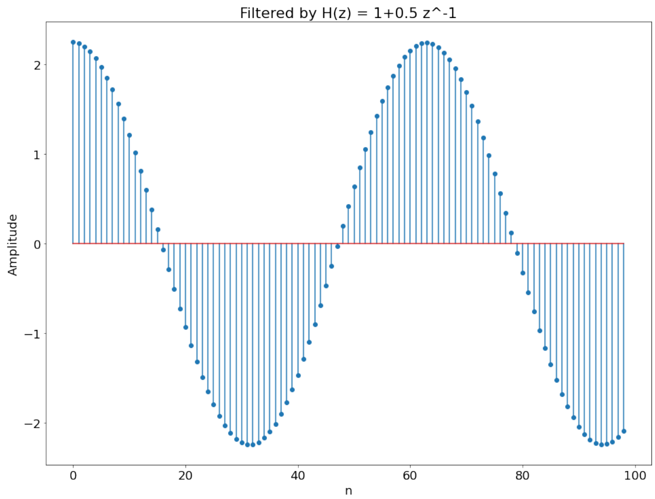
Reconstruct the sequence by filtering y[n] with $\frac{1}{H(z)} = \frac{1}{1+2z^{-1}}$ and unstable inverse is observed.

```
r = signal.filtfilt(np.array([1]),
np.array([1,2]), y)

plt.stem(n,r);
plt.title('Reconstructed Sequence by
1/H(z) = 1/(1-2z^-1)')
plt.xlabel('n')
plt.ylabel('Amplitude')
plt.show()
```



Filter this sequence by casual $H(z) = 1 + 0.5z^{-1}$ and stable output is obtained

```
from scipy import signal
import matplotlib.pyplot as plt

y = signal.filtfilt(np.array([1,
0.5]), np.array([1]), x)

plt.stem(n,y);
plt.title('Filtered by H(z) = 1+0.5
z^-1')
plt.xlabel('n')
plt.ylabel('Amplitude')
plt.show()
```



Reconstruct the sequence by filtering y[n] with $\frac{1}{H(z)} = \frac{1}{1+0.5z^{-1}}$ and stable inverse x[n] is observed.

```
r = signal.filtfilt(np.array([1]),
np.array([1, 0.5]), y)

plt.stem(n,r);
plt.title('Reconstructed Sequence
by 1/H(z) = 1/(1-0.5z^-1)')
plt.xlabel('n')
plt.ylabel('Amplitude')
plt.show()
```