Exercises for Week 9

1. Consider an ideal highpass filter whose frequency response in $(0, 2\pi)$ is given as:

$$H_d(e^{j\omega}) = \begin{cases} 1, & 0.3\pi \le \omega \le 1.7\pi \\ 0, & otherwise \end{cases}$$

   Use the window method with rectangular window to design a length-21 casual linear-phase finite impulse response (FIR) filter that approximates $H_d(e^{j\omega})$.

2. Use the window method with rectangular window to design a length-21 casual linear-phase finite impulse response (FIR) filter that approximates an ideal lowpass filter. It is required that the sampled version of a continuous-time signal with frequency components of 500 Hz or below can pass through it with small attenuation. The sampling frequency is 8000 Hz.

3. Use the frequency sampling method to design a length 5 linear-phase FIR filter to approximate an ideal lowpass filter whose frequency response $H_d(e^{j\omega})$ in $(-\pi, \pi)$ is

$$H_d(e^{j\omega}) = \begin{cases} 1, & |\omega| \le 0.5\pi \\ 0, & otherwise \end{cases}$$

Solution

Question 1

In the interval of $(-\pi,\pi)$, $H_d(e^{j\omega}) = 1$when $0.3\pi \le \omega \le \pi$ and $-\pi \le \omega \le -0.3\pi$ and $H_d(e^{j\omega}) = 0$ otherwise. Based on the inverse DTFT, we have:

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{-0.3\pi} e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{0.3\pi}^{\pi} e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\frac{1}{jn}e^{j\omega n}\right]_{-\pi}^{-0.3\pi} + \frac{1}{2\pi}\left[\frac{1}{jn}e^{j\omega n}\right]_{0.3\pi}^{\pi}$$

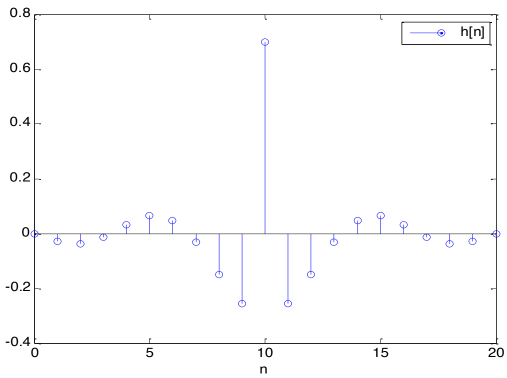
$$= \frac{e^{-j0.3\pi n} - e^{-j\pi n} + e^{j\pi n} - e^{j0.3\pi n}}{j2\pi n}$$

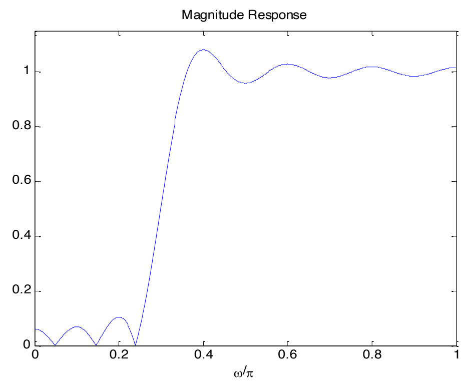
$$= \frac{\sin(\pi n)}{\pi n} - \frac{\sin(0.3\pi n)}{\pi n} = \delta[n] - 0.3\,\mathrm{sinc}(0.3n)$$

For a filter length of 21, we take $h_d[n]$ for the indices $n = -10, -9, \dots, 9, 10$. The causal FIR impulse response $h[n]$ is 10-pointed shifted version of $h_d[n]$:

$$h[n] = \begin{cases} h_d[n-10], & n = 0,1,\dots,20 \\ 0, & othererwise \end{cases}$$

$$h[n] = \begin{cases} \delta[n-10] - 0.3\,\mathrm{sinc}\big(0.3(n-10)\big), & n = 0,1,\dots,20 \\ 0, & othererwise \end{cases}$$





Question 2

The frequency response in $(-\pi, \pi)$ of the discrete-time idea lowpass filter with cut-off frequency ω_c is

$$H_d(e^{j\omega}) = \begin{cases} 1, & -\omega_c \le \omega \le \omega_c \\ 0, & otherwise \end{cases}$$

By inverse DTFT, the corresponding impulse response is:

$$h_d[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left[\frac{1}{jn}e^{j\omega n}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{j2\pi n} = \frac{\sin(\omega_c n)}{\pi n} = \begin{cases} \frac{\omega_c}{\pi}, & n = 0 \\ \frac{\sin(\omega_c n)}{\pi n}, & elsewhere \end{cases}$$

For a filter length of 21, we take $h_d[n]$ for the indices $n = -10, -9, \ldots, 9, 10$. The causal FIR impulse response $h[n]$ is 10-pointed shifted version of $h_d[n]$:

$$h[n] = \begin{cases} h_d[n-10] & 0 \le n \le 20 \\ 0, & elsewhere \end{cases}$$

To determine ω_c, we need to relate the continuous-time cutoff frequency of 500Hz and sampling frequency of 8000Hz to the discrete-time cutoff frequency.

Then, we have $\omega_c = 2\pi\frac{500}{8000} = 0.125\pi$

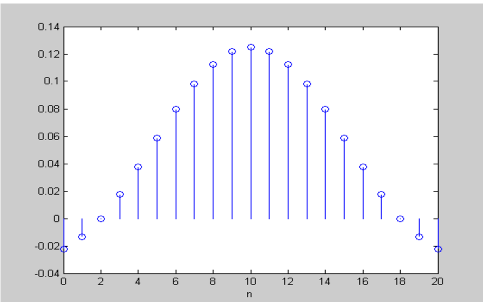
That is, by considering a continuous-time signal with frequency Ω, say, $s(t) = \sin(\Omega t)$ and its discrete-time version, say,

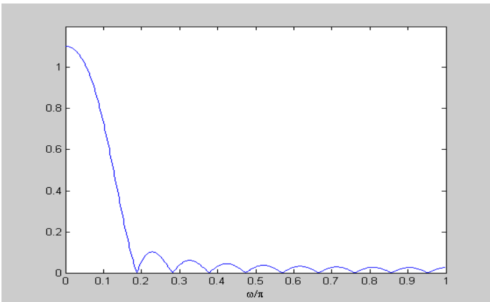
$$s[n] = s(nT) = \sin(\Omega T \cdot n) = \sin(\omega n)$$

We see that the continuous-time frequency is related to the discrete-time frequency as $\Omega T = \omega$

The impulse response of the FIR filter with 21-length is the 20-point shifted version of $h_d[n]$ and then apply 21-point rectangular window, which is given by

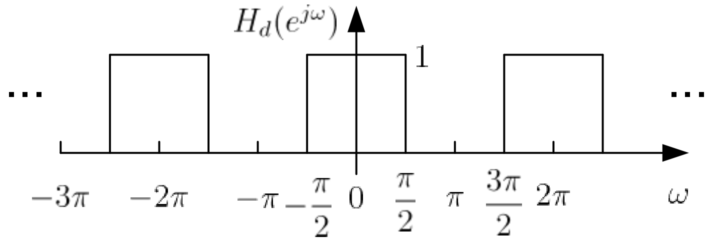
$$h[n] = \begin{cases} 0.125, & n = 10 \\ \frac{\sin(0.125\pi(n-10))}{\pi(n-10)}, & n = 0,1,\ldots,9,11,\ldots.20 \end{cases}$$





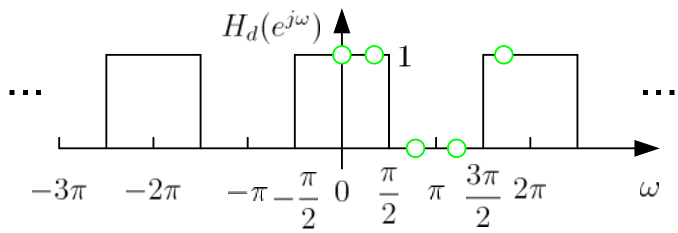
Question 3

The frequency response $H_d(e^{j\omega})$ should be periodic with a period of 2π.



We can notice that DTFT considers the interval $[-\pi,\pi]$ while DFT considers the interval $[0,2\pi]$.

The filter length is 5. So we should use the frequency samples at ω=2πk/5, $k = 0,1,2,3,4$, i.e., ω=0, 2π/5, 4π/5, 6π/5 and $8\pi/5$.



In order to fulfil linear-phase property, we should modify the phases of the DFT coefficients are follows:

For $0 < \omega < \pi$, $\angle H(e^{j\omega}) = -\omega\tau$

For $0 < \omega < 2\pi$, $\angle H(e^{j\omega}) = -(\omega - 2\pi)\tau$

Note that $\alpha = \frac{M-1}{2} = \frac{5-1}{2} = 2$.

$$H[0] = H\left(e^{j\frac{2\pi\cdot 0}{5}}\right) = 1$$
$$H[1] = H\left(e^{j\frac{2\pi\cdot 1}{5}}\right) = 1 \cdot e^{-j2\frac{2\pi\cdot 1}{5}} = e^{-j\frac{4\pi}{5}}$$

$$H[2] = H\left(e^{j\frac{2\pi\cdot 2}{5}}\right) = 0 \cdot e^{-j2\frac{2\pi\cdot 2}{5}} = 0$$

$$H[3] = H\left(e^{j\frac{2\pi\cdot 3}{5}}\right) = 0 \cdot e^{-j2\left(\frac{2\pi\cdot 3}{5}-2\pi\right)} = 0$$

$$H[4] = H\left(e^{j\frac{2\pi\cdot 4}{5}}\right) = 1 \cdot e^{-j2\left(\frac{2\pi\cdot 4}{5}-2\pi\right)} = e^{j\frac{4\pi}{5}}$$

Moreover, making use of $H[k] = H^*[N-k]$ for real coefficients, $H[3]$ and $H[4]$ can be easily obtained as

$$H[3] = H^*[5-3] = H^*[2]$$

and

$$H[4] = H^*[1]$$

Using Python,

```
H=[1,np.exp(-1j*4*np.pi/5),0,0, np.exp(1j*4*np.pi/5)];
h=signal.fft.ifft(H)
```

-0.1236, 0.3236, 0.6000, 0.3236, -0.1236
Apart from using computer, we can produce the filter coefficients analytically for this simple case via inverse DFT:

$$h[n] = \frac{1}{N}\sum_{k=0}^{N-1} H[k]W_N^{-kn}, \qquad n = 0,1,2,3,4$$

$$= \frac{1}{5}\sum_{k=0}^{4} H[k]e^{j\frac{2\pi kn}{5}}$$

$$= \frac{1}{5}H[0] + \frac{1}{5}H[1]e^{j\frac{2\pi n}{5}} + \frac{1}{5}H[4]e^{j\frac{2\pi n\cdot 4}{5}}$$

$$= \frac{1}{5} + \frac{1}{5}e^{-j\frac{4\pi}{5}}\cdot e^{j\frac{2\pi n}{5}} + \frac{1}{5}e^{j\frac{4\pi}{5}}\cdot e^{j\frac{2\pi n\cdot 4}{5}}$$

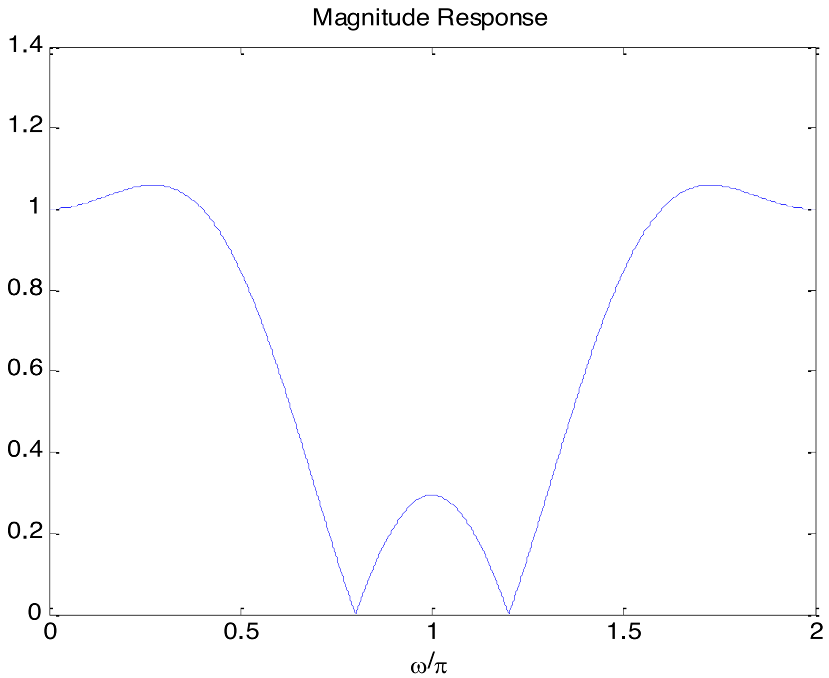
Therefore,

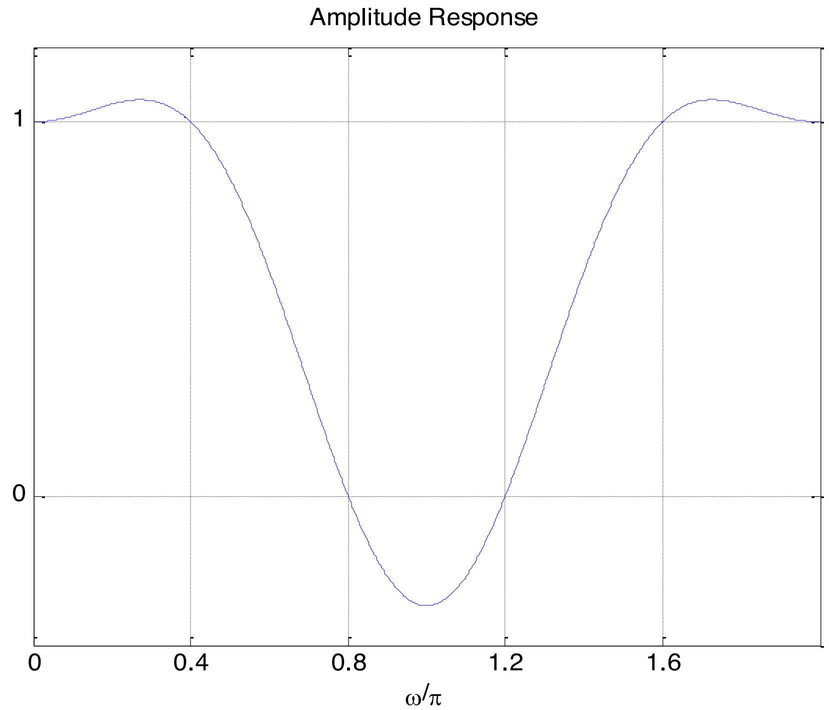
$$h[0] = \frac{1}{5} + \frac{1}{5}e^{-j\frac{4\pi}{5}} + \frac{1}{5}e^{j\frac{4\pi}{5}} = -0.1236$$

$$h[1] = \frac{1}{5} + \frac{1}{5}e^{-j\frac{4\pi}{5}}\cdot e^{j\frac{2\pi}{5}} + \frac{1}{5}e^{j\frac{4\pi}{5}}\cdot e^{j\frac{2\pi\cdot 4}{5}} = \frac{1}{5} + \frac{1}{5}e^{-j\frac{2\pi}{5}} + \frac{1}{5}e^{j\frac{2\pi}{5}} = 0.3236$$

$$h[2] = \frac{1}{5} + \frac{1}{5}e^{-j\frac{4\pi}{5}}\cdot e^{j\frac{4\pi}{5}} + \frac{1}{5}e^{j\frac{4\pi}{5}}\cdot e^{j\frac{2\pi\cdot 2\cdot 4}{5}} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = 0.6$$

Similarly, $h[3] = 0.3236$ and $h[4] = -0.1236$


Magnitude Response
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
ω/π


Amplitude Response
1
0
0
0.4
0.8
1.2
1.6
ω/π