<u>Exercises for Week 10</u>

1. Show the steps in designing and implementing an infinite impulse response (IIR) lowpass filter with the following magnitude-only specifications for $\left|H\left(e^{j\omega}\right)\right|^2$:

Pass-band ripple	$R_p = 10\ dB$
Stop-band attenuation	$A_s = 50\ dB$
Pass-band frequency	$\omega_p = 0.1\pi$
Stop-band frequency	$\omega_s = 0.3\pi$

Use Butterworth lowpass filter as the analog filter design. Assume that **impulse invariance transformation** is employed with sampling $T = 1$.

Solution

Step 1: In using the impulse invariance method with sampling period $T = 1$, the first step is to determine the analog passband and stopband frequencies. They are given by

$$\Omega_p = \frac{\omega_p}{T} = \frac{0.1\pi}{1} = 0.1\pi$$

$$\Omega_s = \frac{\omega_s}{T} = \frac{0.3\pi}{1} = 0.3\pi$$

Step 2: With the Ω_p and Ω_s frequencies, we can determine the Butterworth filter order by

$$N = \left\lceil \frac{\log_{10}\left[\left(10^{R_p/10} - 1\right)/\left(10^{A_s/10} - 1\right)\right]}{2\log_{10}\left(\Omega_p/\Omega_s\right)} \right\rceil$$

$$N = \left\lceil \frac{\log_{10}\left[\left(10^{10/10} - 1\right)/\left(10^{50/10} - 1\right)\right]}{2\log_{10}(0.1\pi/0.3\pi)} \right\rceil = \lceil 4.2397 \rceil = 5$$

Therefore, the required order is 5.

Step 3: With the Ω_p, Ω_s and N, we can determine the cutoff frequency Ω_c of the analogy Butterworth by

$$\Omega_c \in \left[\frac{\Omega_p}{\left(10^{\frac{R_p}{10}} - 1\right)^{1/(2N)}}, \frac{\Omega_s}{\left(10^{\frac{A_s}{10}} - 1\right)^{1/(2N)}} \right]$$

$$\Omega_c \in \left[\frac{0.1\pi}{\left(10^{\frac{10}{10}} - 1\right)^{\frac{1}{2\cdot 5}}}, \frac{0.3\pi}{\left(10^{\frac{50}{10}} - 1\right)^{\frac{1}{2\cdot 5}}} \right] = [0.2522, 0.2980]$$

We can choose Ω_c between 0.2522 and 0.2980. Let us choose Ω_c as 0.29.

Step 4: Based on the 5th Order Butterworth filter transfer faction with cut-off frequency $\Omega_c = 0.29$, The magnitude squared response $|H_a(j\Omega)|^2$ is then:

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{0.29}\right)^{10}}$$

There are 10 poles in $|H_a(j\Omega)|^2$ at $p_k = 0.29e^{\frac{jk\pi}{10}}$, *k=0,…, 9.*

Find Laplace transform of the Butterworth low-pass filter. We should choose the 5 poles on the left half plane to form $H_a(s)$. Let these 5 poles be p_a, p_b, p_c, p_d and p_e. Hence we have:

$$H_a(s) = \frac{0.29^5}{(s-p_a)(s-p_b)(s-p_c)(s-p_d)(s-p_e)}$$

Step 5: Express $H_c(s)$ in a partial fraction form:

$$H_a(s) = \frac{A}{s-p_a} + \frac{B}{s-p_b} + \frac{C}{s-p_c} + \frac{D}{s-p_d} + \frac{E}{s-p_e}$$

Step 6: To transform $H_a(s)$ into a digital filter $H(z)$ using impulse invariance method, the main idea is to convert a pole in s-plane into a pole in z-plane via the use of $z = e^{sT}$, where $T = 1$ is the sampling interval. As a result,

$$H(z) = \frac{A}{1-e^{p_aT}z^{-1}} + \frac{B}{1-e^{p_bT}z^{-1}} + \frac{C}{1-e^{p_cT}z^{-1}} + \frac{D}{1-e^{p_dT}z^{-1}} + \frac{E}{1-e^{p_eT}z^{-1}}$$

For direct or canonic form:

$$H(z) = \frac{b_0 + b_1z^{-1} + b_2z^{-2} + b_3z^{-3} + b_4z^{-4}}{1 + a_1z^{-1} + a_2z^{-2} + a_3z^{-3} + a_4z^{-4} + a_5z^{-5}}$$

For cascade form:

$$H(z) = \frac{b_{0,1} + b_{1,1}z^{-1} + b_{2,1}z^{-2}}{(1-e^{p_aT}z^{-1})(1-e^{p_bT}z^{-1})} \cdot \frac{b_{0,2} + b_{1,2}z^{-1} + b_{2,2}z^{-2}}{(1-e^{p_cT}z^{-1})(1-e^{p_dT}z^{-1})} \cdot \frac{b_{0,3}}{1-e^{p_eT}z^{-1}}$$

For parallel form:

$$H(z) = \frac{\gamma_{0,1} + \gamma_{1,1}z^{-1}}{(1-e^{p_aT}z^{-1})(1-e^{p_bT}z^{-1})} + \frac{\gamma_{0,2} + \gamma_{1,2}z^{-1}}{(1-e^{p_cT}z^{-1})(1-e^{p_dT}z^{-1})} + \frac{\gamma_{0,3}}{1-e^{p_eT}z^{-1}}$$