Review of
Continuous-Time Fourier Analysis

EE4015 Digital Signal Processing

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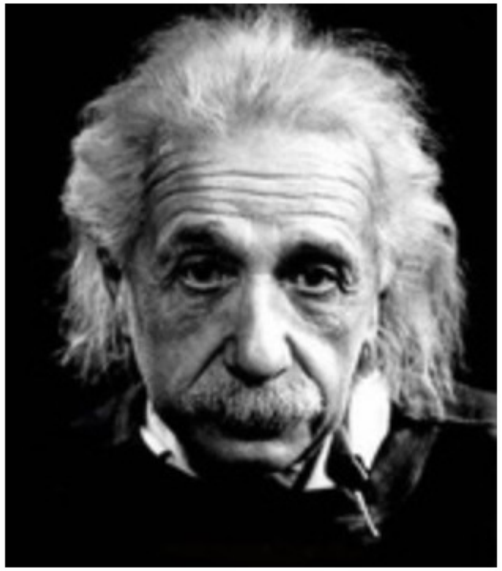
Assignment 1

- **The assignment 1 is now available in the schedule webpage for download. The deadline for the assignment 1 is Tuesday of Week 6 (Oct. 4, 2022).**
  - http://www.ee.cityu.edu.hk/~lmpo/ee4015/pdf/2022_EE4015_Ass01.pdf

- Submit the answer sheets of the Assignment 1 as a pdf file to this CANVAS assignment 1:
  - Filename format : Assignment01_StudentName_StudentID.pdf
  - Filename example: Assignment01_Chen_Hoi_501234567.pdf



“It is not that I'm so smart. But I stay with the questions much longer.”

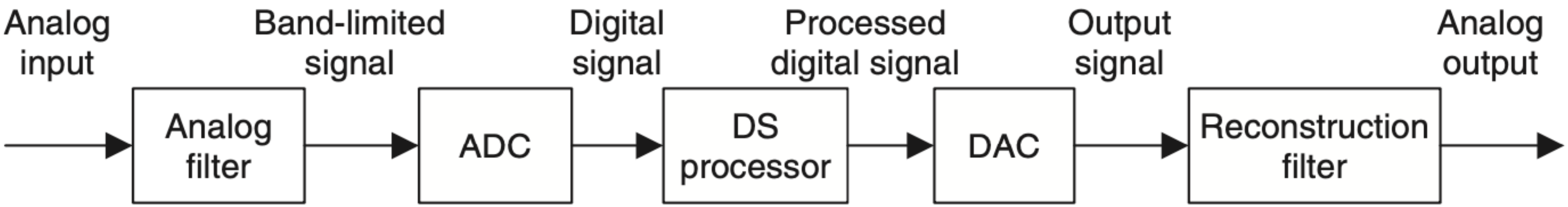
― Albert Einstein

tags: intelligence, learning, wisdom

Content

- **Review of Continuous-Time Signal Analysis in Frequency Domain**
  - Continuous-Time Fourier Series (CTFS) for Periodic Signal Analysis
  - Continuous-Time Fourier Transform (CTFT) for Non-periodic Signal Analysis
  - Laplace Transform : Generalization of CTFT and System Design of Analog Systems
- **Analog-to-Digital Conversion (ADC)**
  - Time-domain Modelling of the sampling process using modulation of the CT signal with impulse train
  - Frequency Domain Analysis of the sampling process using CTFT
  - Nyquist Sampling Theorem and Anti-aliasing Filter
- **Digital-to-Analog Conversion (DAC)**
  - Reconstruction Filter
- **Quantization**

A Digital Signal Processing System



A Big Picture of Transformations for Signal Processing

Continuous-Time Signals

Periodic : $\tilde{x}(t)$

- Continuous-Time Fourier Series (CTFS) : a_k
  - Commonly called Fourier Series (FS)

Non-Periodic (Aperiodic) : $x(t)$

- Continuous-Time Fourier Transform (CTFT) : $X(j\Omega)$
  - Commonly called Fourier Transform (FT)

Generalization

- Laplace Transform : $X(s) = X(\sigma + j\Omega)$
  - For system design

Discrete-Time Signals (Sequences)

Periodic : $\tilde{x}[n]$

- Discrete Fourier Series (DFS) : $\tilde{X}[k]$
  - also called Discrete-Time Fourier Series (DTFS)

Non-Periodic (Aperiodic) : $x[n]$

- Discrete-Time Fourier Transform (DTFT) : $X(e^{j\omega})$

Finite-Duration Sequences : $x[n]$

  - Discrete Fourier Transform (DTF) : $X[k]$
  - Fast Fourier Transform (FFT) : $X[k]$

Generalization

- The z-Transform : $X(z) = X(re^{j\omega})$

Continuous-Time Signal Analysis in Frequency Domain

Fourier series and Fourier transform are the tools for analyzing analog signals.

Basically, they are used for signal conversion between time and frequency domains

What are Fourier Series and Fourier Transform?

- Fourier Series and Fourier Transform, named after Joseph Fourier, are mathematical transformations employed to transform signals between time (or spatial) domain and frequency domain.
- They are tools that breaks a waveform (a function or signal) into alternate representations, characterized by sine and cosines.
- It shows that any waveform can be re-written as the weighted sum of sinusoidal functions.

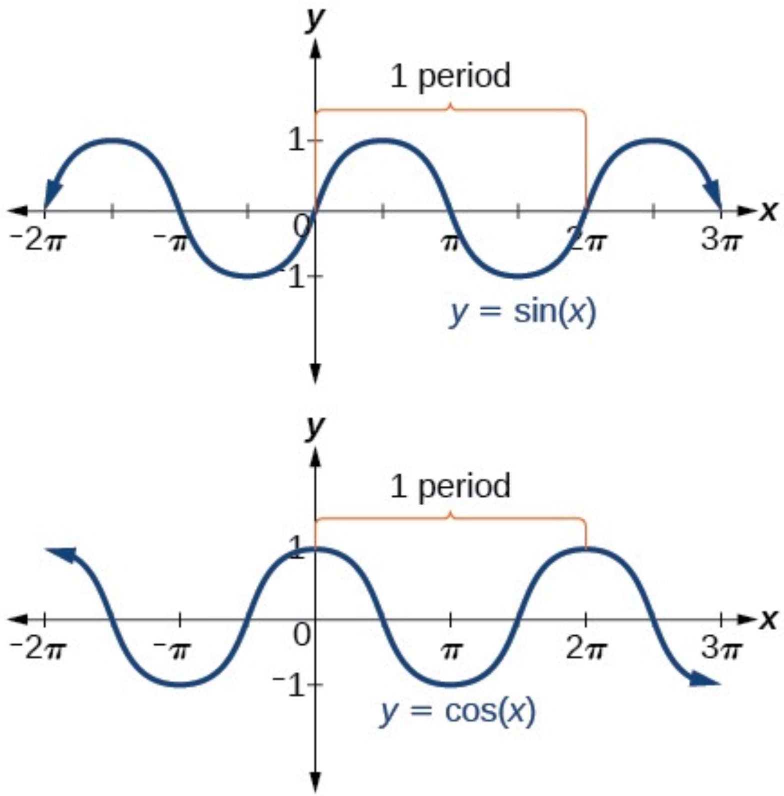
Joseph Fourier
(1768-1830)

Sine and Cosine Functions

- They are periodic function with period of 2π
  - $\sin(x + n2\pi) = \sin(x)$
  - $\cos(x + n2\pi) = \cos(x)$
- General form of sine and cosine signals:
  - $y(t) = A\sin(\Omega t + \theta)$
  - $y(t) = A\cos(\Omega t + \theta)$

where

A is Amplitude,
Ω is angular frequency in radian/sec,
θ is the phase angle in radians.

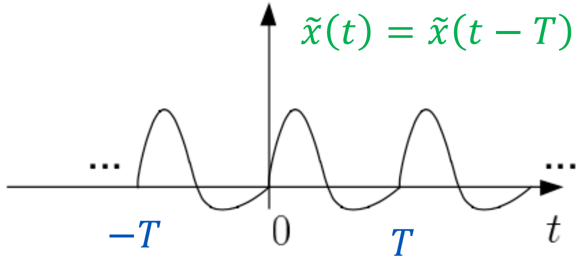


Continuous-Time Fourier Series (CTFS)

Frequency-Domain Representation of

Periodic Continuous-Time Signals $\tilde{x}(t)$

Continuous-Time Fourier Series



- **Fourier Series** is basically a way of approximating or representing a continuous-time periodic signal by a series of ***simple harmonic (sine and cosine)*** functions.

- For a periodic signal with period T, then its fundamental harmonic frequency is $\Omega_o = 2\pi/T$.

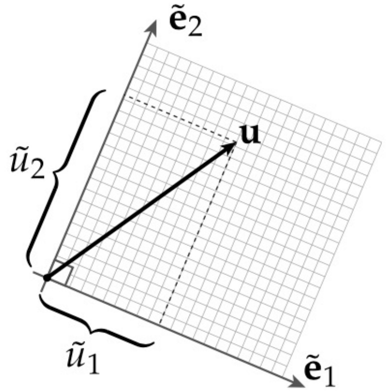
- The Fourier Series is defined as $$\tilde{x}(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\Omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\Omega_0 t)$$

- Its Fourier Series coefficients are given by

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt \qquad a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\Omega_0 t)\,dt \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\Omega_0 t)\,dt$$

Project of Function onto Sinusoids

- Projection onto bases works just like vectors in R^n



Fourier Series

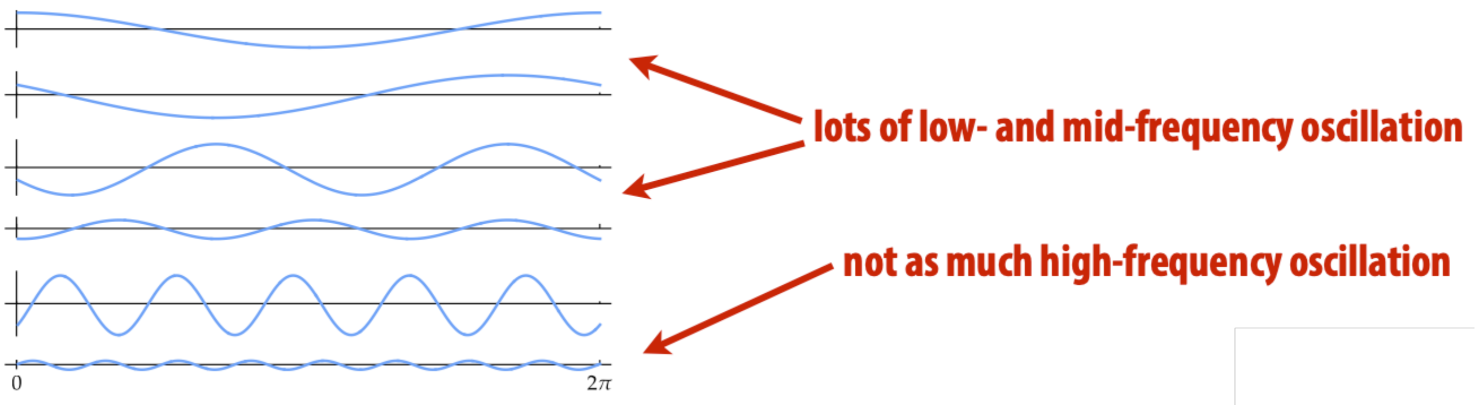
$$a_n = \langle x(t), \cos(n\Omega_0 t)\rangle$$

$$b_n = \langle x(t), \sin(n\Omega_0 t)\rangle$$

Inner Products

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\Omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\Omega_0 t)$$

- Decomposes signal into frequencies

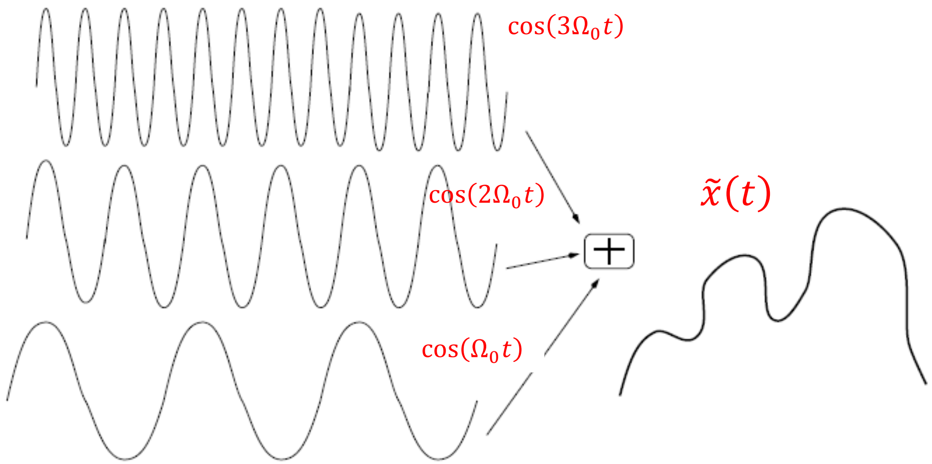


Interpretation of CT Fourier Series

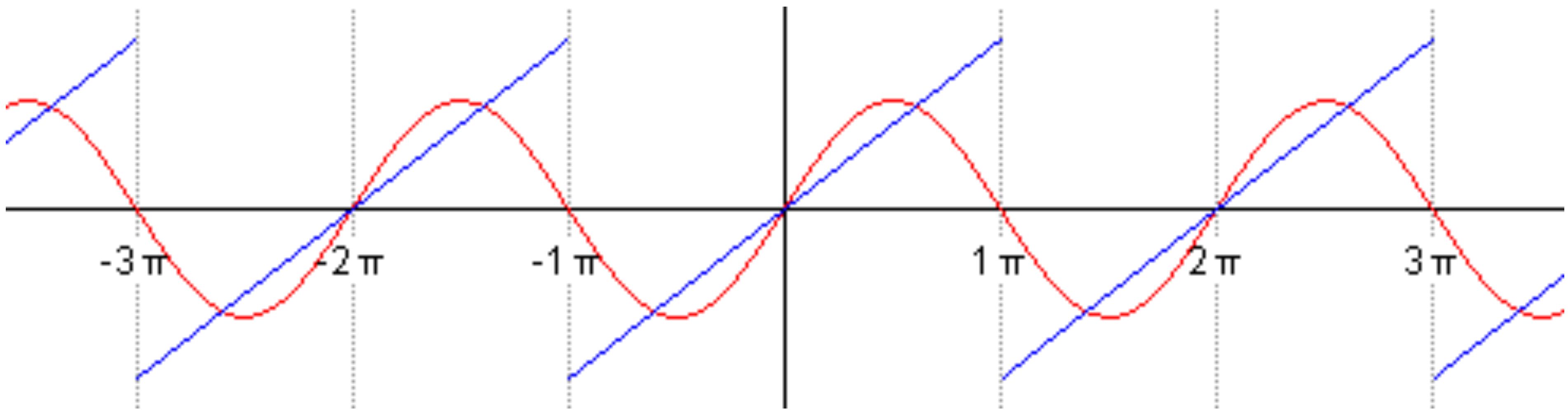
- Any periodic function $x(t)$ can be expressed as a weighted sum (infinite) of sine and cosine functions of varying frequency:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\Omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\Omega_0 t)$$

- Express periodic signals using harmonically related sinusoids with frequencies $0, \Omega_o, 2\Omega_o, \cdots$, where Ω_o is called the fundamental frequency, $2\Omega_o$ is called the first harmonic, $3\Omega_o$ is called the second harmonic, and so on



Fourier Series Example



Complex Fourier Series

- Every periodic function with period T can be expanded into a Fourier series as

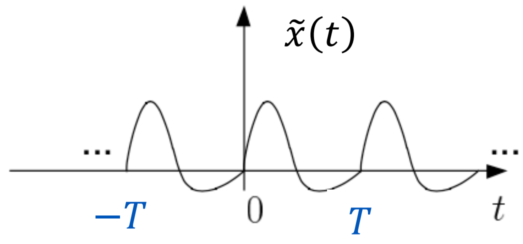
$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

where

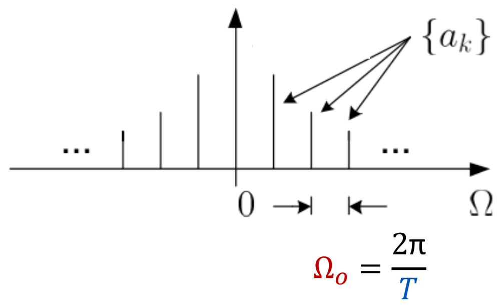
$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\Omega_0 t}\, dt, \qquad k = 0, \pm 1, \pm 2, \cdots$$

- a_k are called Fourier Series Coefficients.

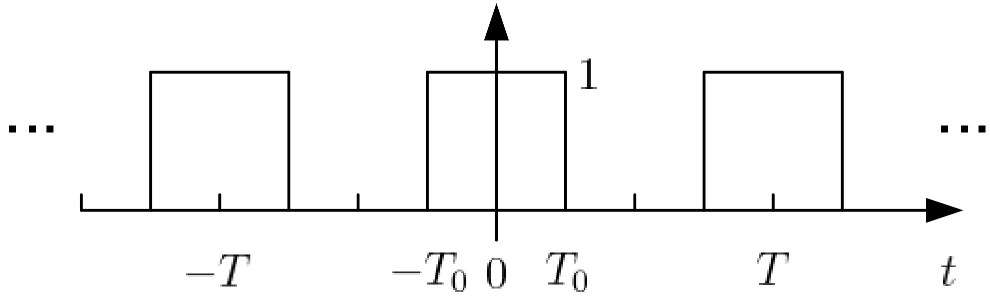
Time Domain



Frequency Domain



Fourier Series Example of Periodic Pulses



$$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & \text{otherwise} \end{cases}$$

- The fundamental frequency is $\Omega_0 = 2\pi/T$, we get:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\Omega_o t}dt = \frac{1}{T}\int_{-T_o}^{T_o} e^{-jk\Omega_o t}dt$$

For $k \neq 0$

$$a_k = \frac{1}{T}\int_{-T_o}^{T_o} e^{-jk\Omega_o t} dt = \frac{1}{T}\left[-\frac{1}{jk\Omega_o} e^{-jk\Omega_o t}\right]_{-T_o}^{T_o} = -\frac{1}{jk\Omega_o T}\left[e^{-jk\Omega_o T_o} - e^{jk\Omega_o T_o}\right]$$

$$= \frac{1}{k\pi} \cdot \frac{1}{2j}\left[e^{jk\Omega_o T_o} - e^{-jk\Omega_o T_o}\right] = \frac{\sin(k\Omega_o T_o)}{k\pi} = \frac{\sin(2\pi k\, T_o/T)}{k\pi}$$

$$\Omega_o = 2\pi/T$$

For $k = 0$

$$a_0 = \frac{1}{T}\int_{-T_o}^{T_o} 1\, dt = \frac{2T_o}{T}$$

L'Hopital's Rule

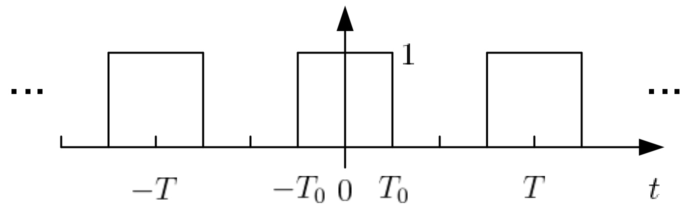
- The reason of separating the cases of $k = 0$ and $k \neq 0$ is to facilitate the computation of a_0, whose value is not straightforwardly obtained from the general expression which involves "0/0". Nevertheless, using L'Hopital's rule

$$a_0 = \lim_{k \to 0} \frac{\sin(2\pi k T_0/T)}{k\pi} = \lim_{k \to 0} \frac{\frac{d \sin(2\pi k T_0/T)}{dk}}{\frac{dk\pi}{dk}} = \lim_{k \to 0} \frac{2\pi T_0/T \cos((2\pi k T_0/T))}{\pi} = \frac{2T_0}{T}$$

Periodic Pulses Spectrum

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

Time Domain

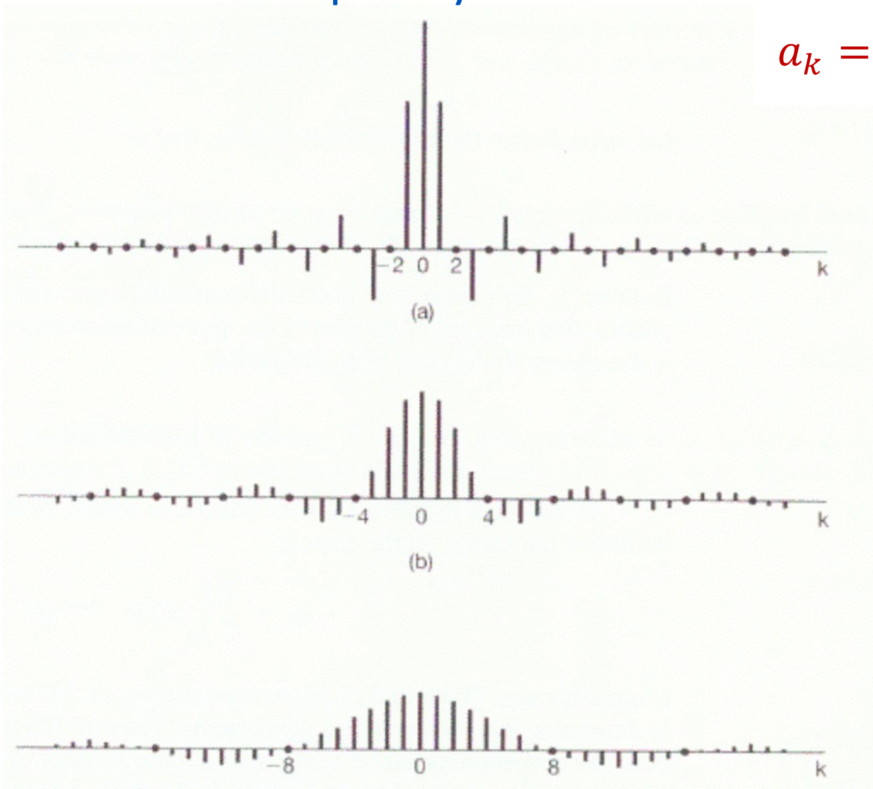


$$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & \text{otherwise} \end{cases}$$

Continuous and Periodic

Frequency Domain

$$a_k = \frac{\sin(2\pi k T_0/T)}{k\pi}$$



$T = 4T_0$

$T = 8T_0$

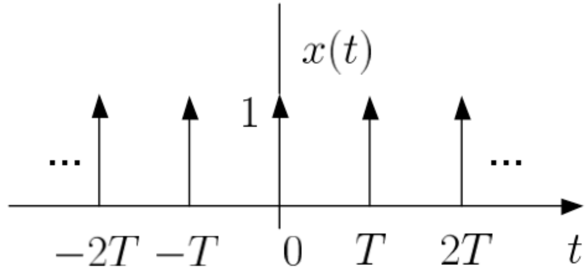
$T = 16T_0$

Discrete and Non-periodic

Fourier Series of Impulse Train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

Time Domain



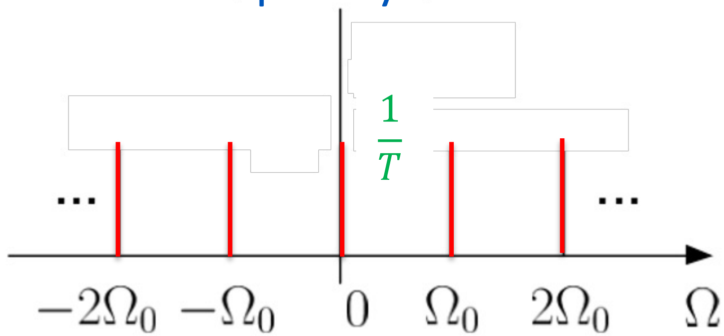
- Clearly, $x(t)$ is a periodic signal with a period of T.
- The Fourier series coefficients are

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\Omega_o t}\, dt = \frac{1}{T}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\Omega_0 t}$$

Fourier Series Representation

Frequency Domain

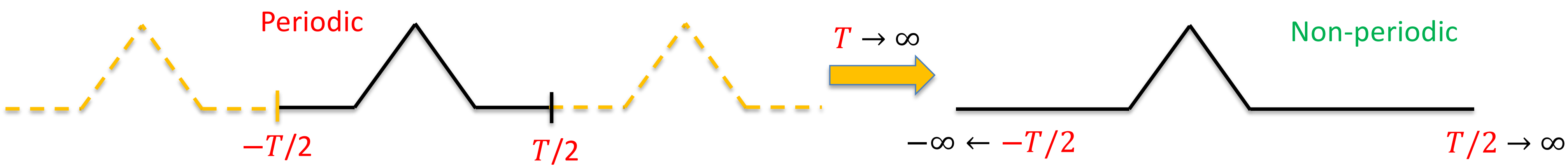


Continuous-Time Fourier Transform (CTFT)

From Fourier Series to Fourier Transform

- Fourier Series is used to represent periodic signal as weighed sum of the complex exponentials with harmonic frequencies of Ω

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\Omega t} \quad \text{with} \quad \Omega = \frac{2\pi}{T} \qquad \text{and} \qquad a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\Omega t}\,dt, \qquad k = 0, \pm 1, \pm 2, \cdots$$



- If we take the period $T \rightarrow \infty$, then $\Omega \rightarrow 0$ and the periodic signal $x(t)$ become non-periodic. Its corresponding Fourier Series can be expressed as

$$x(t) = \lim_{\Omega \rightarrow 0} \sum_{k=-\infty}^{\infty} \left[\frac{\Omega}{2\pi}\int_{-\pi/\Omega}^{\pi/\Omega} x(\tau)e^{-jk\Omega\tau}\,d\tau\right] e^{jk\Omega t} = \int_{-\infty}^{\infty} \frac{1}{2\pi}\underbrace{\left[\int_{-\infty}^{\infty} x(\tau)e^{-jk\Omega\tau}d\tau\right]}_{X(j\Omega)} e^{j\Omega t}d\Omega$$

Continuous-Time Fourier Transformation (CTFT)

- For analysis of continuous-time non-periodic signals
- Defined on a continuous range of Ω
- The **CTFT** of a continuous-time non-periodic signal $x(t)$ is:

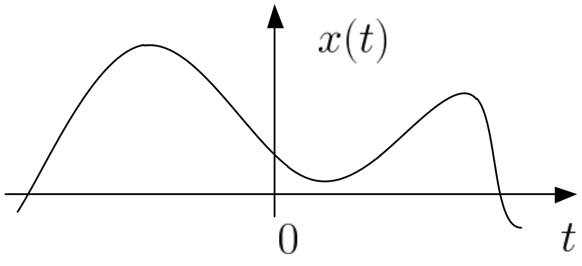
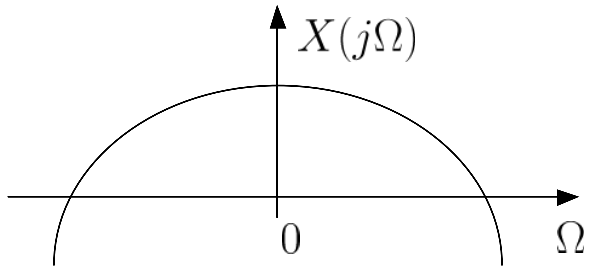
$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\, dt \qquad \textbf{Analysis Equation}$$

which is also called spectrum.

- The **inverse CTFT** is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}\, d\Omega \qquad \textbf{Synthesis Equation}$$

Illustration of CT Fourier Transform

<table>
<tr><th>time domain</th><th>frequency domain</th></tr>
<tr><td>
</td><td>
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$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \Longrightarrow$$

$$\Longleftarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

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<tr><td>Continuous and Non-periodic</td><td>Continuous and Non-periodic</td></tr>
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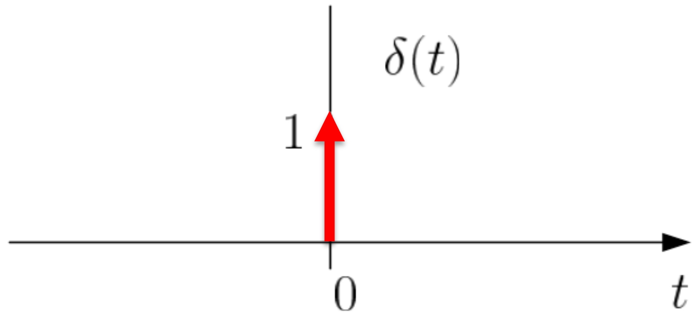
Delta Function $\delta(t)$

- The delta function $\delta(t)$ can be expressed as

$$\delta(t) = \begin{cases} \infty, & if \;\; t = 0 \\ 0, & if \;\; t \neq 0 \end{cases}$$



- It has the following characteristics

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \quad \text{and} \quad x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

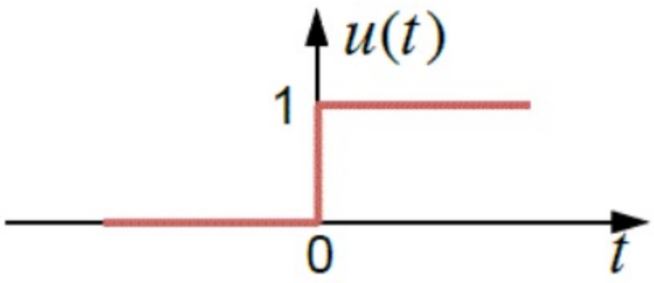
where $x(t)$ is a continuous-time signal.

- The Time shifting property $$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Unit Step Function $u(t)$

- The unit step function $u(t)$ can be expressed

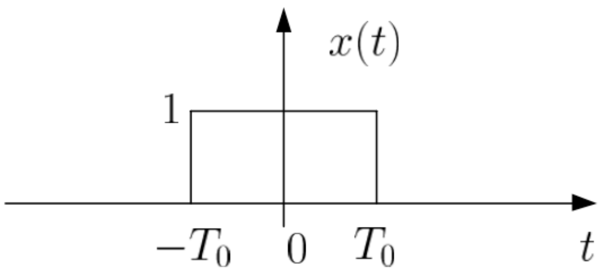
$$u(t) = \begin{cases} 1, & if \;\; t > 0 \\ 0, & if \;\; t < 0 \end{cases}$$



- As there is a sudden change from 0 to 1 at $t = 0$, $u(0)$ is not well defined.

Fourier Transform of Rectangular Pulse

$$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & otherwise \end{cases}$$



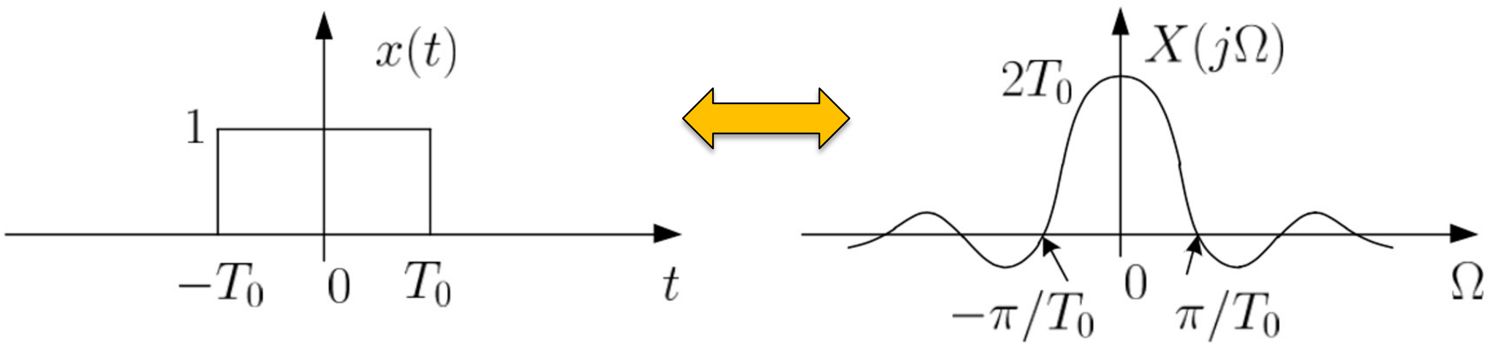
- This signal is of finite length and corresponds to one period of the periodic function. Its Fourier Transform can be expressed as

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt = \int_{-T_0}^{T_0} e^{-j\Omega t}dt = \frac{2\sin(T_0\Omega)}{\Omega}$$

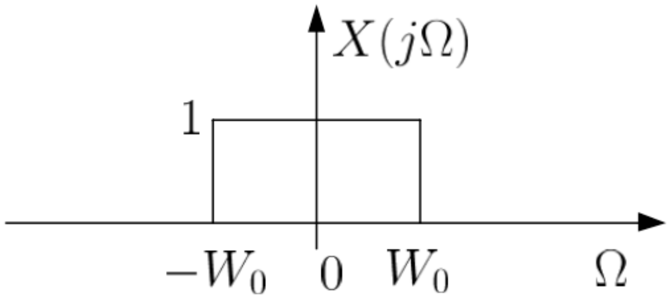
Fourier Transform Pair for Rectangular Pulse

- Define the sinc function as $$\mathrm{sinc}(\pi u) = \frac{\sin(\pi u)}{\pi u}$$
- It is seen that $X(j\Omega)$ is a scaled sinc function because

$$X(j\Omega) = \frac{2\sin(T_0\Omega)}{\Omega} = 2T_0 \,\mathrm{sinc}\left(\frac{T_0\Omega}{\pi}\right)$$



Inverse Fourier Transform of Rectangular Pulse Spectrum

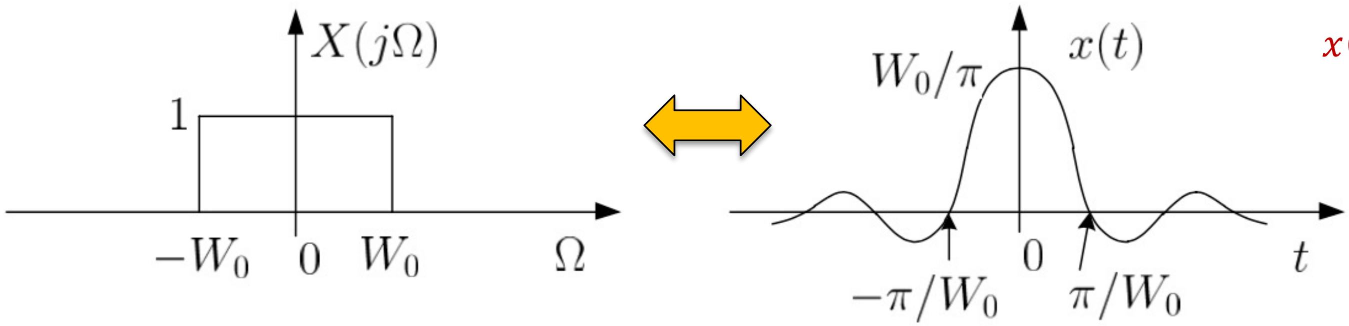


$$X(j\Omega) = \begin{cases} 1, & -W_0 < \Omega < W_0 \\ 0, & \text{otherwise} \end{cases}$$

- The inverse Fourier Transform of this rectangular spectrum can be obtained by

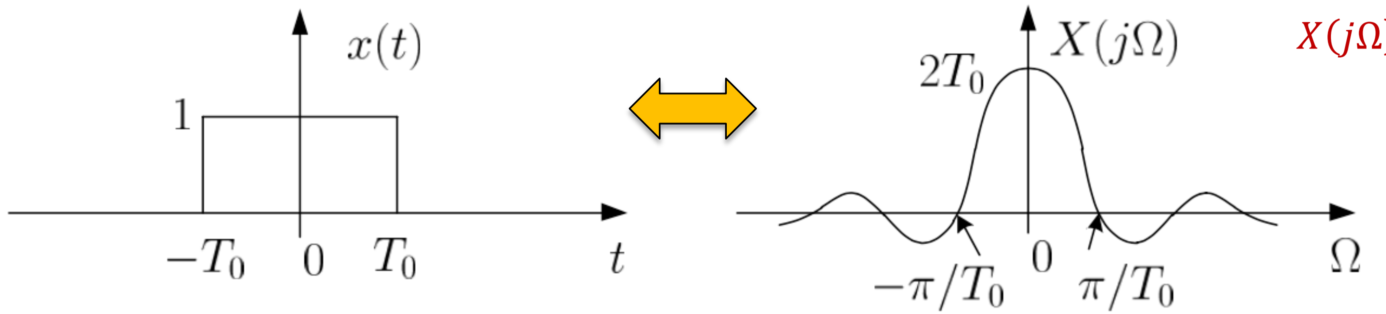
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega = \frac{1}{2\pi}\int_{-W_0}^{W_0} e^{j\Omega t}d\Omega = \frac{\sin(W_0 t)}{\pi t} = \frac{W_0}{\pi}\text{sinc}\left(\frac{W_0 t}{\pi}\right)$$

Fourier Transform Pair for Rectangular Pulse Spectrum



$$x(t) = \frac{W_0}{\pi}\,\mathrm{sinc}\left(\frac{W_0 t}{\pi}\right)$$

- We can observe **the duality property** of Fourier Transform

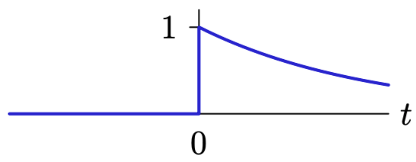


$$X(j\Omega) = 2T_0\,\mathrm{sinc}\left(\frac{T_0\Omega}{\pi}\right)$$

Fourier Transform of Exponential Function

- Right-sided Continuous-Time exponential function is defined as

$$x(t) = e^{-at}u(t) \text{ with } a > 0.$$

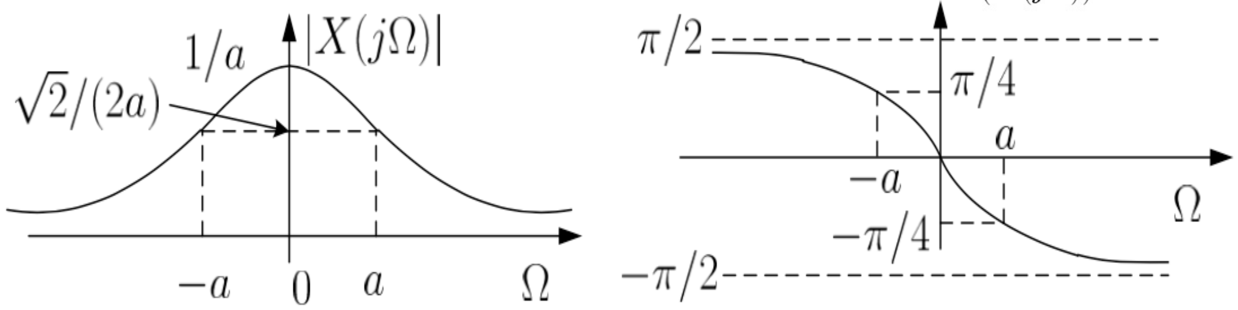


- Its Fourier Transform can be obtained by

$$X(j\Omega) = \int_0^\infty e^{-at}e^{-j\Omega t}dt = -\frac{1}{a+j\Omega}e^{-(a+j\Omega)t}\Big|_0^\infty = \frac{1}{a+j\Omega} = \frac{a-j\Omega}{a^2+\Omega^2}$$

$$|X(j\Omega)| = \frac{1}{\sqrt{a^2+\Omega^2}}$$

$$\angle(X(j\Omega)) = -\tan^{-1}\left(\frac{\Omega}{a}\right)$$



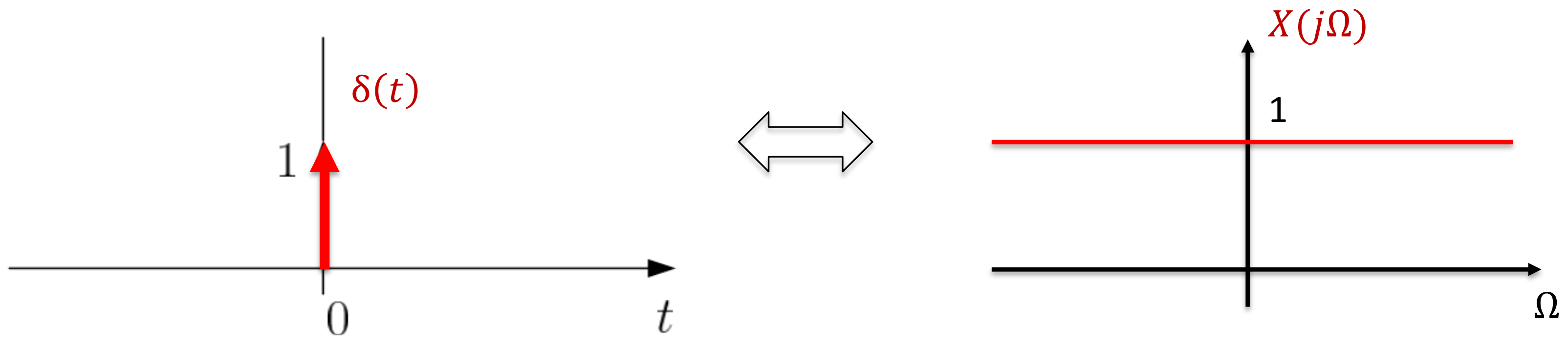
Magnitude and phase plots for $1/(a+j\Omega)$

Fourier Transform of Delta Function $\delta(t)$

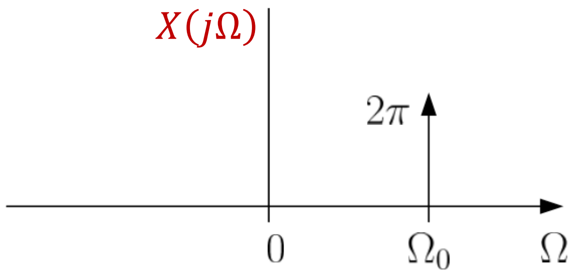
- The Fourier Transform of delta Function can be obtained as

$$X(j\Omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega \cdot 0} dt = e^{-j\Omega \cdot 0} \int_{-\infty}^{\infty} \delta(t) dt = e^{-j\Omega \cdot 0} = 1$$

- Spectrum of $\delta(t)$ has unit amplitude at all frequencies



Impulse in Frequency Domain



$$X(j\Omega) = 2\pi\delta(\Omega - \Omega_0)$$

- Based on $\delta(t)$, Fourier transform can be used to represent continuous-time periodic signals. The inverse Fourier Transform of $2\pi\delta(\Omega - \Omega_0)$ can be calculated by

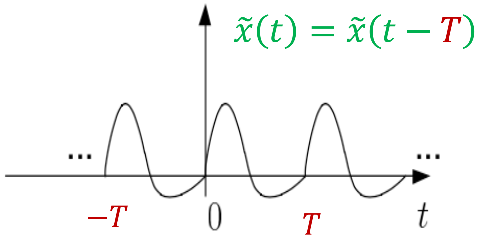
$$x(t) = \mathcal{F}^{-1}\{2\pi\delta(\Omega - \Omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0)e^{j\Omega t}d\Omega = e^{j\Omega_0 t}$$

- As a results, the Fourier Transform Pair is:

$$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega - \Omega_0)$$

Fourier Transform Pair for CT Periodic Signal

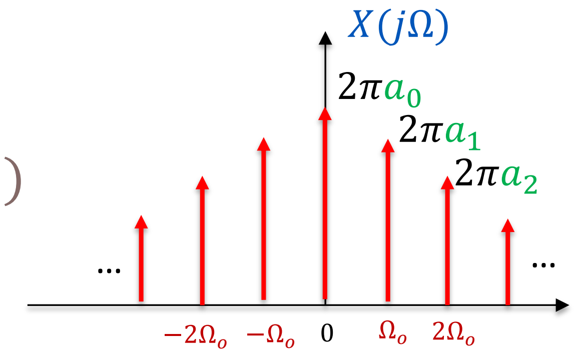
- Based on the Fourier Transform pair of Impulse in Frequency Domain, we can express the Fourier pair for any Continuous-Time Periodic Signal as



$\Omega_o = \frac{2\pi}{T}$

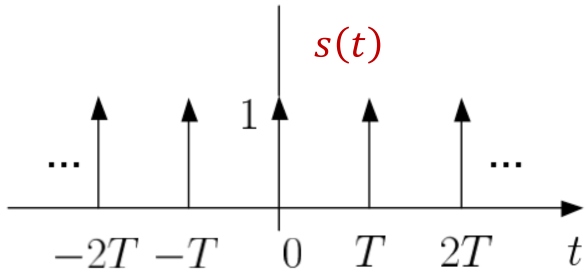
$$\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\Omega - k\Omega_0)$$

Periodic signal in
Fourier Series Expression
and a_k are Fourier Series Coefficients



Fourier Transform of Impulse Train

$$s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



- Clearly, $x(t)$ is a periodic signal with a period of T. Using the previous example, the Fourier series coefficients are

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\Omega_o t}\, dt = \frac{1}{T} \quad \Rightarrow \quad s(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\Omega_0 t}$$

- With $\Omega_0 = 2\pi/T$, the Fourier Transform is:

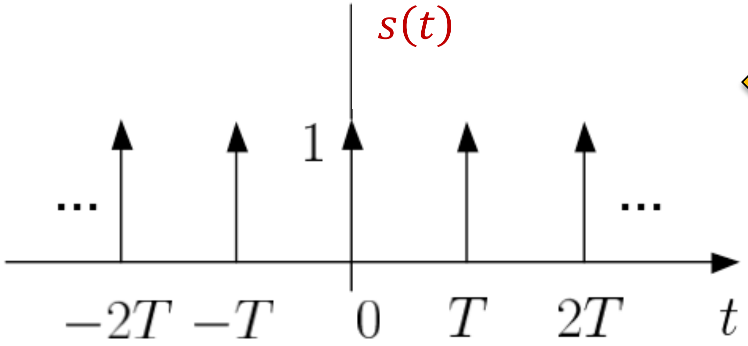
$$\Omega_o = \frac{2\pi}{T}$$

$$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega - \Omega_0)$$

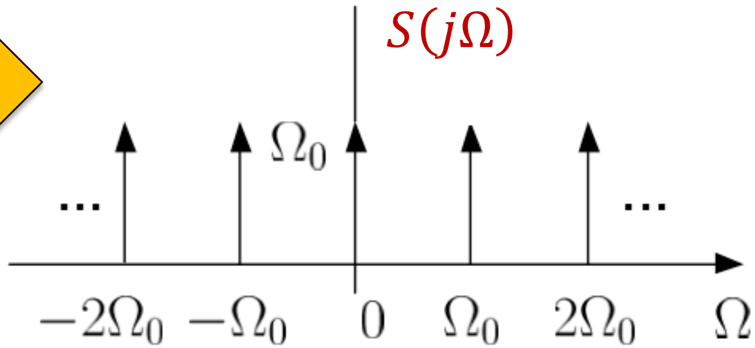
$$s(t) \leftrightarrow \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{T}\right) = \Omega_0 \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_0)$$

Fourier Transform Pair For Impulse Train

Time Domain



Frequency Domain

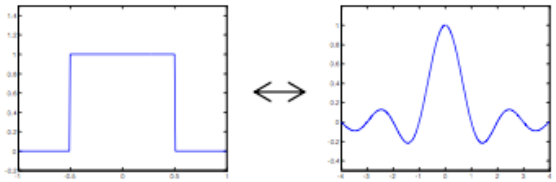


$$s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\Omega_0 t} \quad \Longleftrightarrow \quad S(j\Omega) = \Omega_0 \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_0)$$

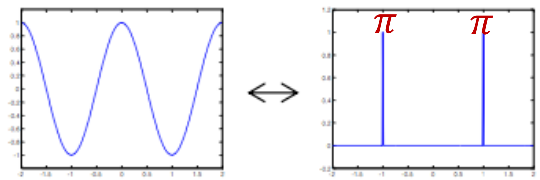
Fourier Series Expression

Important CTFT Pairs

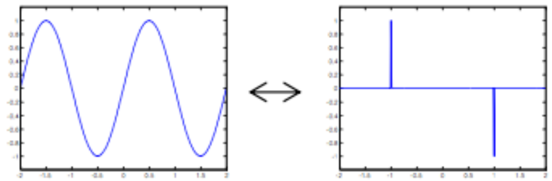
- **Rectangular Pulse**: A rectangular pulse transform to a sinc function
  - $\mathrm{rect}(t) \leftrightarrow \mathrm{sinc}(j\Omega)$



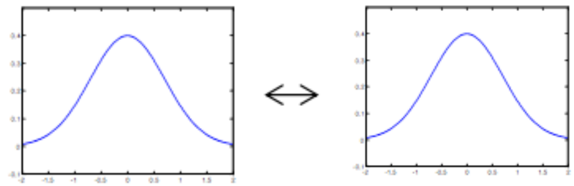
- **Cosin** : A cosine signal transforms to two impulses
  - $\cos(\Omega_0 t) \leftrightarrow \pi[\delta(\Omega+\Omega_0)+\delta(\Omega-\Omega_0)]$



- **Sine** : A sine transforms to two (imaginary) impulses
  - $\sin(\Omega_0 t) \leftrightarrow j\pi[\delta(\Omega+\Omega_0)-\delta(\Omega-\Omega_0)]$



- **Gaussian** : A Gaussian transforms to a Gaussian
  - $e^{-x^2/2\sigma^2} \leftrightarrow \sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2\Omega^2}$



Key Properties of the CTFT

1. **Linearity** : $x_1(t) \leftrightarrow X_1(j\Omega)\ and\ \ x_2(t) \leftrightarrow X_2(j\Omega)$

$$ax_1(t) + bx_1(t) \leftrightarrow aX_1(j\Omega) + bX_2(j\Omega)$$

2. **Time Shifting** : $x(t - t_o) \leftrightarrow e^{-j\Omega t_o}X(j\Omega)$
3. **Convolution** : $x(t) * h(t) \leftrightarrow X(j\Omega) \cdot H(j\Omega)$
4. **Modulation** : $x(t)\,h(t) \leftrightarrow \frac{1}{2\pi}X(j\Omega) * H(j\Omega)$
5. **Time Scaling** : $x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{j\Omega}{a}\right)$
6. **Differentiation** : $\frac{dx(t)}{dt} \leftrightarrow j\Omega X(j\Omega)$

Convergence of CTFT

Dirichlet's sufficient conditions for the convergence of Continuous-Time Fourier Transform are

1. $x(t)$ must be absolutely integrable

$$x(t) = \int_{-\infty}^{\infty} |x(t)|dt < \infty$$

2. $x(t)$ must have a finite number of maxima and minima within any finite interval.
3. $x(t)$ must have a finite number of discontinuities, all of finite size, within any finite interval.

Not all CT signals can have CTFT representations

Laplace Transform

Laplace Transform

The French Newton **Pierre-Simon Laplace**

- Developed mathematics in astronomy, physics, and statistics
- Began work in calculus which led to the Laplace Transform
- Today, Laplace Transform is widely used to solve ODE (Ordinal Differential Equation) in many application of Electrical Engineering.
- It is also widely used for Signal Processing in Analog Digital Filter Design.

Pierre-Simon Laplace
(1749-1827)

Definition of Laplace Transform

- Laplace transform maps a function $x(t)$ of time to a function of $s = \sigma + j\Omega$ in complex domain.

$$X(s) = \int x(t)e^{-st}\,dt$$

- There are two important variants:
  - Unilateral

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}\,dt$$

  - Bilateral

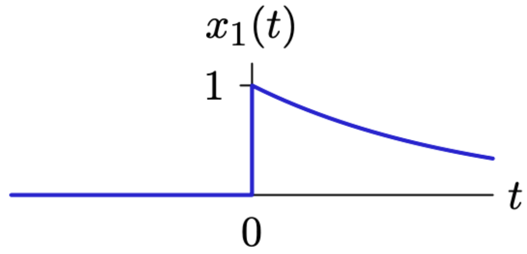
$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

- Both share important properties. We will focus on bilateral version.

Laplace Transform Example

- Find the Laplace transform of $x_1(t)$

$$x_1(t) = \begin{cases} e^{-t} & if\ t \geq 0 \\ 0 & otherwise \end{cases}$$

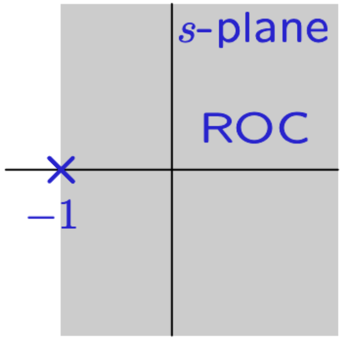


$$X_1(s) = \int_{-\infty}^{\infty} x_1(t) e^{-st}\, dt = \int_0^{\infty} e^{-t} e^{-st}\, dt = \left. \frac{e^{-(s+1)t}}{-(s+1)} \right|_0^{\infty} = \frac{1}{s+1}$$

- **Region of Convergence (ROC) :** Provided Re(s+1) > 0 which implies that Re(s) >-1

ROC

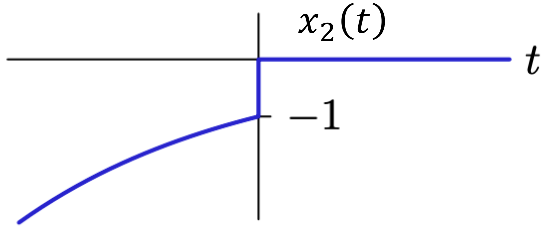
$$X_1(x) = \frac{1}{s+1}\ ; \quad \mathrm{Re(s)} > -1$$



Regions of Convergence

- Left-sided signals have left-sided Laplace transforms (bilateral only)
- Example

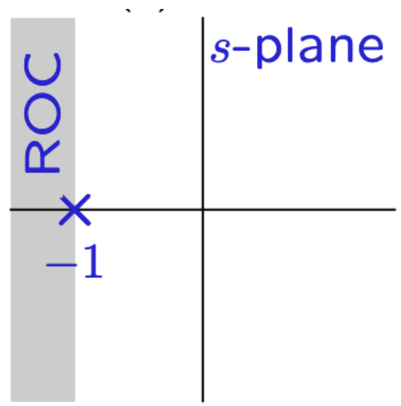
$$x_2(t) = \begin{cases} -e^{-t} & if\ t \le 0 \\ 0 & otherwise \end{cases}$$



$$X_2(s) = \int_{-\infty}^{\infty} x_2(t) e^{-st}\, dt = \int_{-\infty}^{0} e^{-t} e^{-st}\, dt = \left. \frac{-e^{-(s+1)t}}{-(s+1)} \right|_{-\infty}^{0} = \frac{1}{s+1}$$

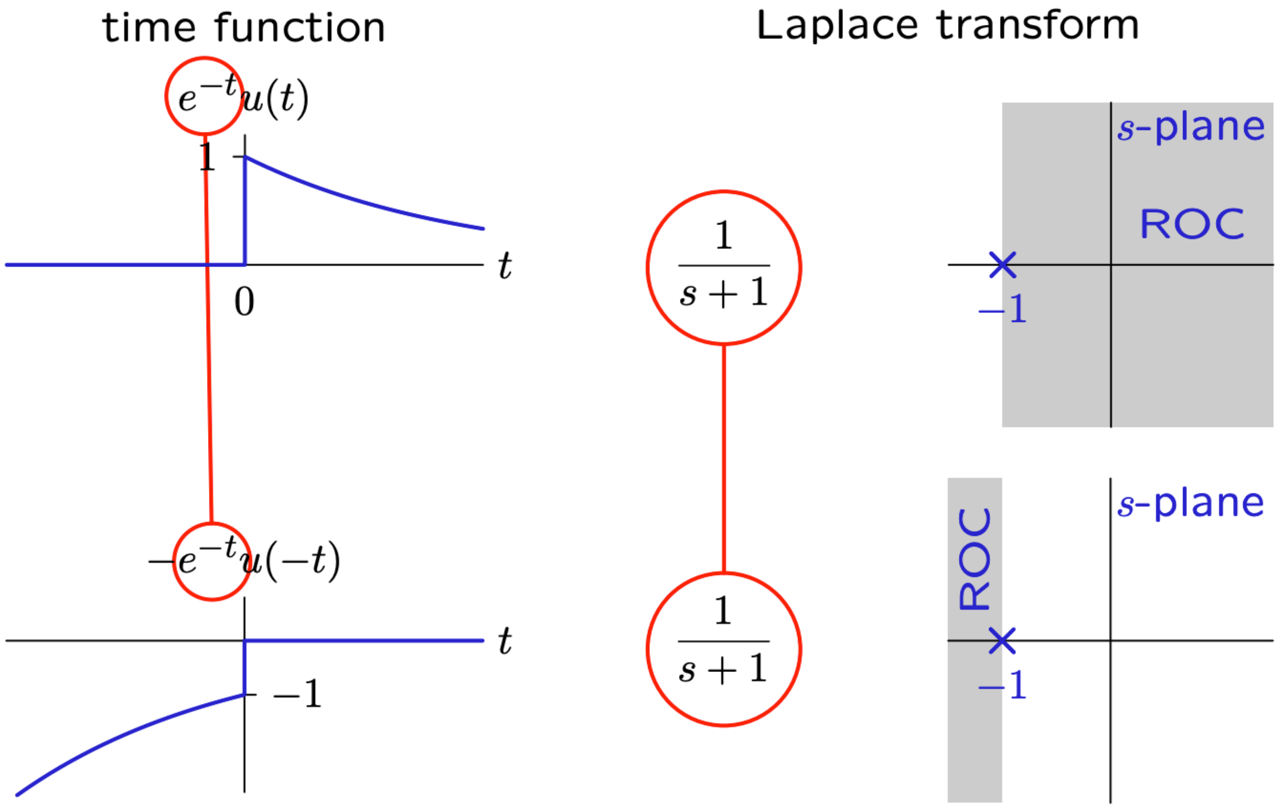
- Provided Re(s+1) < 0 which implies that Re(s) < -1

$$\frac{1}{s+1}; \quad \mathrm{Re(s)} < -1$$



Left-Sided and Right-Sided ROCs

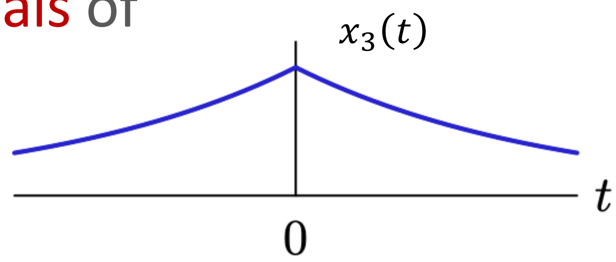
Laplace transforms of left- and right-sided exponentials have the same form (except −); with left- and right-sided ROCs, respectively.



Laplace Transform of Both-Sided Signals (1)

- Find the Laplace transform of a both-sided signals of

$$x_3(t) = e^{-|t|}$$



$$X_3(s) = \int_{-\infty}^{\infty} e^{-|t|} e^{-st}\, dt = \int_{-\infty}^{0} e^{(1-s)t}\, dt + \int_{0}^{\infty} e^{-(1+s)t}\, dt$$

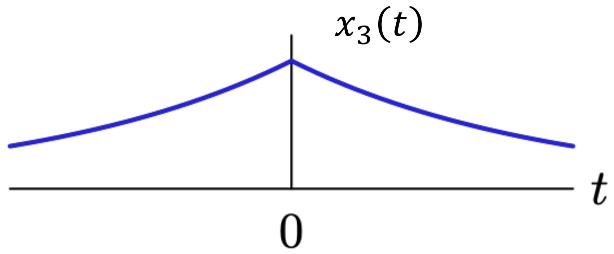
$$= \left.\frac{e^{(1-s)t}}{(1-s)}\right|_{-\infty}^{0} + \left.\frac{e^{-(s+1)t}}{-(1+s)}\right|_{0}^{\infty} = \underbrace{\frac{1}{1-s}}_{\text{Re(s)<1}} + \underbrace{\frac{1}{1+s}}_{\text{Re(s)>1}} = \frac{1+s+1-s}{(1-2)(1+s)} = \frac{2}{1-s^2}$$

- The ROC is the intersection of Re(s) < 1 and Re(s) > -1

Laplace Transform of Both-Sided Signals (1)

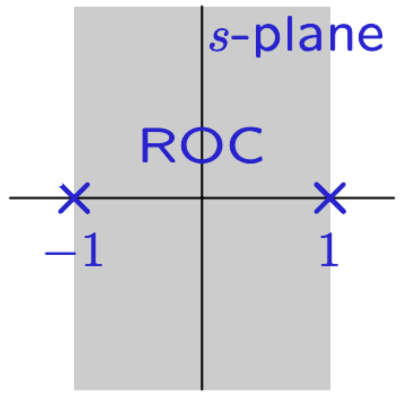
- the Laplace transform of a signal is both-sided is a vertical strip.

$$x_3(t) = e^{-|t|}$$



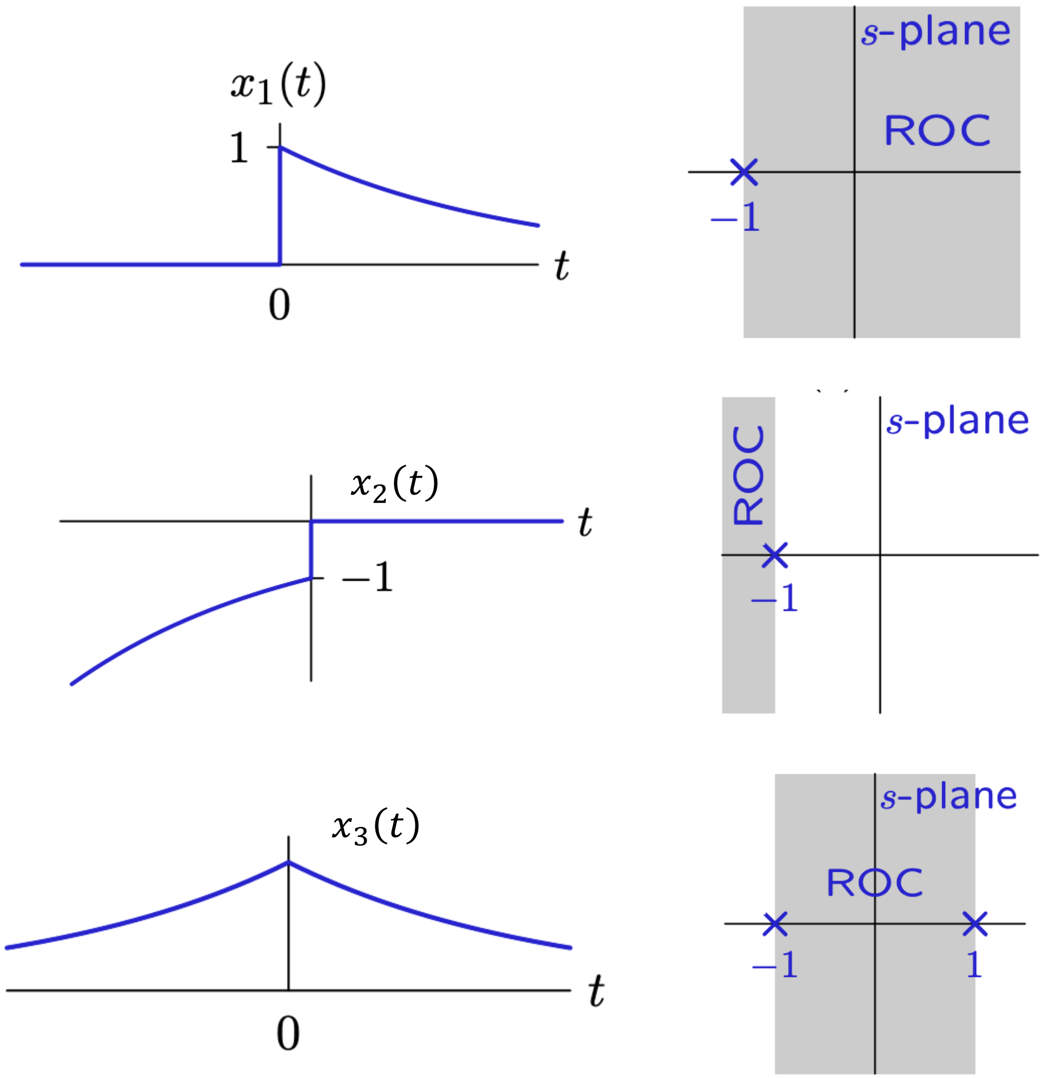
$$X_3(s) = \frac{2}{1 - s^2}$$

-1 < Re(s) < 1



Time-Domain Interpretation of ROC

$$X(s) = \int_{-\infty}^{\infty} e^{-st}\, dt$$



Fourier Transform Interpretation of Laplace Transform

- In Laplace Transform, $s = \sigma + j\Omega$ is a complex number, then we can express the transform as

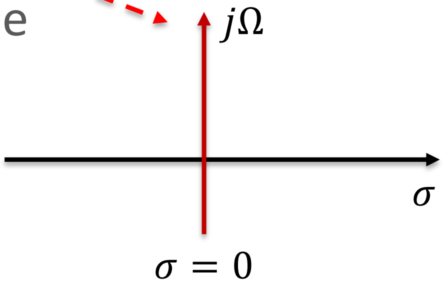
$$X(s) = X(\sigma + j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\Omega)t}\,dt = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\Omega t}\,dt$$

- Thus, the Laplace Transform can be interpreted as CTFT of the signal $x(t)$ that weighted by $e^{-\sigma t}$. This is equivalent to taking CTFT of the signal $x(t)e^{-\sigma t}$ as

$$\mathfrak{F}\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\Omega t}\,dt$$

- If we set $\sigma = 0$, then $s = j\Omega$. The Laplace Transform of $X(0 + j\Omega)$ is corresponding to the CTFT for $\sigma = 0$ (Imaginal axis) is within the ROC in s-plane

$$X(s) = X(j\Omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\Omega t}\,dt$$



Laplace Transform is Generalization of Fourier Transform

$$X(s) = X(\sigma + j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t} e^{-j\Omega t}\, dt = \mathfrak{F}\{x(t)e^{-\sigma t}\}$$

- For some signals, they cannot converge for CFTF, but we still can transform them to Laplace transform in the s-plane for analysis and system design.
- The inverse Laplace Transform can be considered as inverse CTFT of the signal $x(t)e^{-\sigma t}$

$$x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathfrak{F}\{x(t)e^{-\sigma t}\} e^{j\Omega t}\, d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)\, e^{j\Omega t}\, d\Omega$$

$$x(t) = x(t)e^{-\sigma t} e^{+\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s) e^{(\sigma + j\Omega)t}\, d\Omega = \frac{1}{2\pi j}\int_{\sigma-\infty}^{\sigma+\infty} X(s)\, e^{st}\, ds$$

Important Laplace Transform Pairs

Function	Laplace Transform
a	$\frac{a}{s}$
e^{at}	$\frac{1}{s-a}$
te^{at}	$\frac{1}{(s-a)^2}$
$\sin \Omega t$	$\frac{\Omega}{s^2+\Omega^2}$
$\cos \Omega t$	$\frac{s}{s^2+\Omega^2}$

Key Properties of the Laplace Transform

1. **Linearity** : $x_1(t) \leftrightarrow X_1(\text{s})\ and\ \ x_2(t) \leftrightarrow X_2(s)$

$$ax_1(t) + bx_1(t) \leftrightarrow aX_1(s) + bX_2(s)$$

2. **Time Shifting** : $x(t - t_o) \leftrightarrow e^{-st_o}X(s)$
3. **Convolution** : $x(t) * h(t) \leftrightarrow X(s) \cdot H(s)$
4. **Scaling Property** : $x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{s}{a}\right)$
5. **Time Differentiation** : $\frac{dx(t)}{dt} \leftrightarrow sX(s) - x(0)$

Continuous-Time Differential Equations

- CT systems whose input-output response can be described by **linear constant-coefficient ordinary differential equations** with a forced response

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- If the equation involves derivative operators on $y(t)$ ($N>0$) or $x(t)$, it has memory.
- The system stability depends on the coefficients a_k. For example, a 1st order LTI differential equation with $a_0 = 1$:

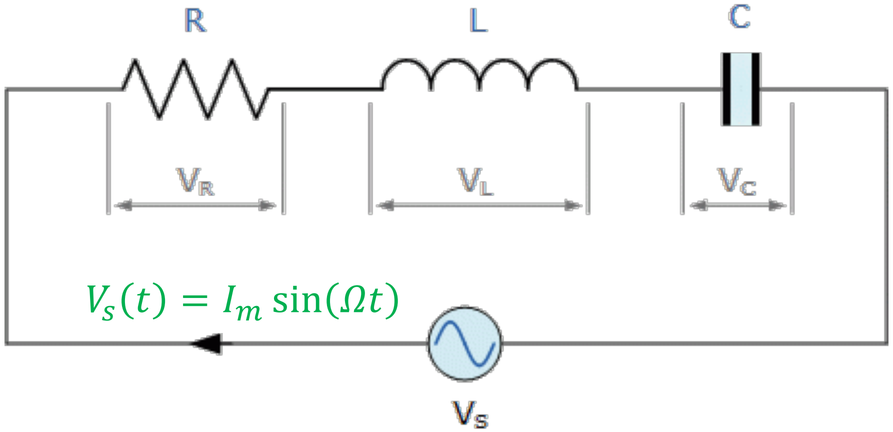
$$\frac{dy(t)}{dt} - a_1 y(t) = 0 \quad \Rightarrow \quad y(t) = Ae^{a_1 t}$$

- If $a_1>0$, the system is unstable as its impulse response represents a growing exponential function of time
- If $a_1<0$ the system is stable as its impulse response corresponds to a decaying exponential function of time

Differential Equations

- Analog systems can be represented by differential equations

$$I_m \sin(\Omega t) = L\frac{d\,i(t)}{dt} + R\,i(t) + \frac{1}{C}\int i(t)dt$$



$$I_m\Omega\cos(\Omega t) = L\frac{d^2 i(t)}{d^2 t} + R\frac{d\,i(t)}{dt} + \frac{1}{C}i(t)$$

This is a second order Ordinal Differential Equation (ODE).

Solving ODE by Laplace Transform

Ordinal Differential Equations (ODEs) can be easily solved by Laplace Transform **using differential property**. It can transform an ODE to Algebraic expression.

- $\frac{dx(t)}{dt} \leftrightarrow sX(s) - x(0)$
- $\frac{d^2x(t)}{dt^2} \leftrightarrow s^2X(s) - sx(0) - x'(0)$

- For example, $\frac{d^2x(t)}{dt^2} + 5\frac{dx(t)}{dt} + 4x(t) = 0$ can be expressed in Laplace transform as
  - $s^2X(s) - sx(0) - x'(0) + 5(sX(s) - x(0)) + 4X(s) = 0$
  - For $x(0) = 2$ and $x'(0) = -5$, then
    - $(s^2 + 5s + 4)X(s) = 2s - 5 + 10 \Rightarrow (s^2 + 5s + 4)X(s) = 2s + 5$
  - $X(s) = \frac{2s+5}{s^2+5s+4} = \frac{2s+5}{(s+4)(s+1)} = \frac{1}{s+4} + \frac{1}{s+1}$
- Inverse Laplace Transform of $X(s)$, we have the solution $x(t) = e^{-4t}u(t) + e^{-t}u(t)$

Summary

- **Continuous-Time Fourier Series (CTFS)** is used for Continuous-Time Periodic Signals analysis in frequency domain
- **Continuous-Time Fourier Transform (CTFT**) is used for both Continuous-Time periodic and non-periodic signals analysis in frequency domain
- **Laplace Transform** is a generalization transformation of CTFT.
- In signal processing, we always use Laplace Transform for LTI system design such as analog filter design and system stability analysis.