Discrete-Time Fourier Transform (DTFT) and Discrete Fourier Series (DFS)

EE4015 Digital Signal Processing

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Message 1 : Submission of Project Proposal

- This is just a friendly reminder to submit group project proposals. Students must submit their group project proposal in **PDF format** to the CANVAS group project proposal assignment by **September 27, 2022 at 11pm**, with the project title and list of members. The details of the group project can be found in the course website:
- http://www.ee.cityu.edu.hk/~lmpo/ee4015/pdf/projects.htmlLinks to an external site.
- This is a group submission, so **each project team needs to assign a project leader to submit the proposal to CANVAS**.
  - Filename format : Proposal_GroupNumber_ProjectName.pdf
  - Filename example: Proposal_Group01_Audio_Classification.pdf

Message 2 : Quiz

- **Canvas Quiz on Week 7**
  - Canvas quiz with 30 multiple choice questions released on **October 11, 2022 at 5:00pm**.
  - Students must perform the CANVAS Quiz in the classroom of P4701.
  - Students must complete this quiz by **6:00 PM**.
  - This quiz is open-book and covers course content from Weeks 1 to 4.

Message 3 : Arrangement of Leave

Hi students,

If you are unable to return to CityU due to a quarantine order or illness, please email me your medical certificate as an attachment to eelmpo@cityu.edu.hk .

With approval, you may stay at home for the lecture by Zoom.

Dr. LM Po

A Big Picture of Transformations for Signal Processing

Continuous-Time Signals

Periodic : $\tilde{x}(t)$

- Continuous-Time Fourier Series (CTFS) : a_k
  - Commonly called Fourier Series (FS)

Non-Periodic (Aperiodic) : $x(t)$

- Continuous-Time Fourier Transform (CTFT) : $X(j\Omega)$
  - Commonly called Fourier Transform (FT)

Generalization

- Laplace Transform : $X(s) = X(\sigma + j\Omega)$
  - For system design

Discrete-Time Signals (Sequences)

Periodic : $\tilde{x}[n]$

- Discrete Fourier Series (DFS) : $\tilde{X}[k]$
  - also called Discrete-Time Fourier Series (DTFS)

Non-Periodic (Aperiodic) : $x[n]$

- Discrete-Time Fourier Transform (DTFT) : $X(e^{j\omega})$

Finite-Duration Sequences : $x[n]$

- Discrete Fourier Transform (DTF) : $X[k]$
- Fast Fourier Transform (FFT) : $X[k]$

Generalization

- The z-Transform : $X(z) = X(re^{j\omega})$

Content

Fourier Transforms For Discrete-Time Signal Analysis

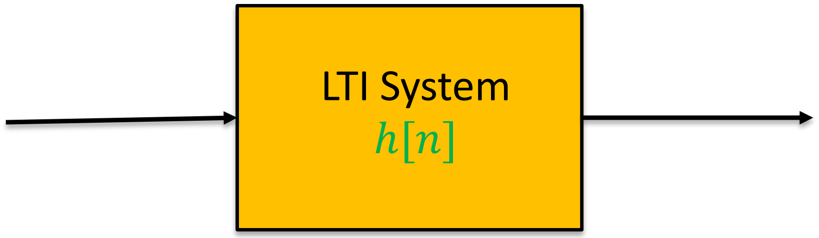
- Discrete-Time Fourier Transform (DTFT) for Non-Periodic Sequences
- Properties of DTFT
- Discrete Fourier Series (DSF) for Periodic Sequences
- Properties of DSF
- Periodic Convolution

DFT and FFT (Next Week)

- Discrete Fourier Transform (DFT) : Finite-Duration Sequences
- Properties of DFT
- Circular Convolution
- Zero-Padding for DFT computation of Linear Convolution
- Fast Fourier Transform (FFT) : Fast Algorithms for computing DFT
- Signal Analysis using FFT
- Signal Processing using FFT
- Spectrogram (Optional)

$e^{j\omega n}$ Sequences are Eigen Functions of LTI System

$e^{j\omega n}$



$y[n] = H(e^{j\omega})e^{j\omega n}$

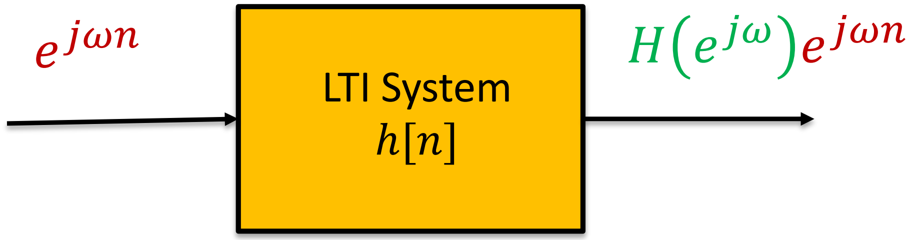
$$y[n] = e^{j\omega n} * h[n] = \sum_{k=-\infty}^{+\infty} h[k]e^{j\omega(n-k)} = e^{j\omega n} \underbrace{\sum_{k=-\infty}^{+\infty} h[k]e^{-j\omega k}}_{H(e^{j\omega})}$$

$H(e^{j\omega})$ is the Frequency Response of the Discrete-Time LTI system with impulse response of $h[n]$

Two Properties of Frequency Response $H(e^{j\omega})$

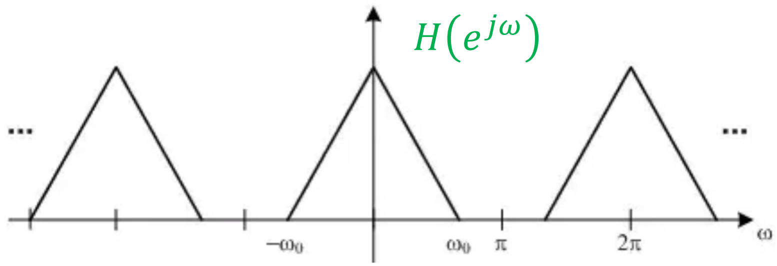
1. Frequency response is a function of continuous variable ω

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n}$$



2. Frequency response is periodic with period of 2π

$$H(e^{j\omega}) = H(e^{j(\omega+2\pi k)})$$



$H(e^{j\omega})$ is a periodic function of continuous variable ω

Fourier Series Analysis of Frequency Response

- We can consider $h[n]$ as Continuous-Time Fourier Series coefficients (a_n) of the frequency response $H(e^{j\omega})$, which is periodic with 2π

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n} d\omega \quad = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\sum_{k=-\infty}^{+\infty} h[k] e^{-j\omega k}\right] e^{j\omega n} d\omega$$

CTFS coefficients of $H(e^{j\omega})$

$$= \sum_{k=-\infty}^{+\infty} h[k] \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-k)}\, d\omega \; = h[n]$$

$n \neq k \;\rightarrow 0$

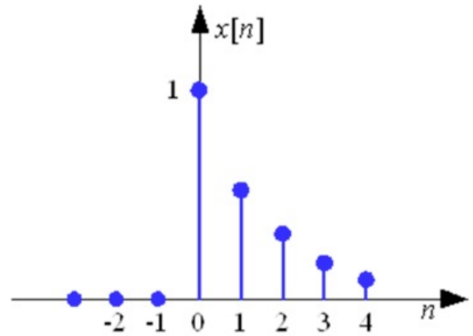
$n = k \;\rightarrow 1$

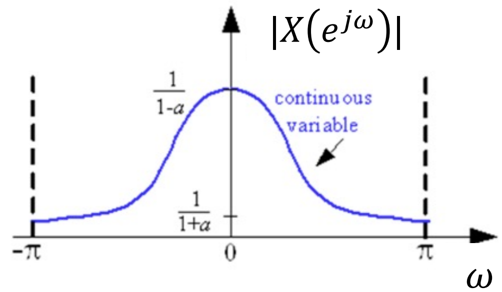
Discrete-Time Fourier Transform (DTFT)

- DTFT is defined as

Analysis Equation
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

Synthesis Equation
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n} d\omega$$





Continuous and periodic in ω

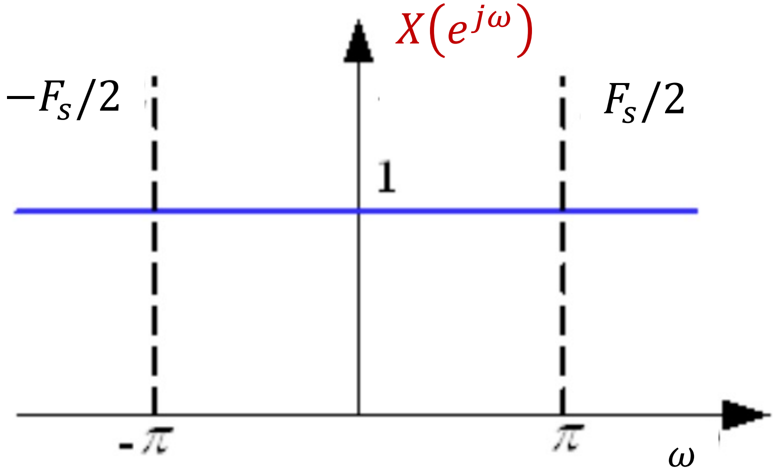
- ω is the discrete-time angular frequency $(-\pi \leq \omega \leq \pi)$
- $\omega = \Omega T$ (T is the sample period and Ω is the analog frequency)
- $X(e^{j\omega})$ is continuous and periodic in ω with period with 2π, i.e. $X(e^{j\omega}) = X(e^{j\omega+2\pi})$

DFTT Example 1

- A unit impulse signal $\delta[n]$ is transformed into its frequency domain counterpart using the DTFT as follows:

$$x[n] = \delta[n]$$

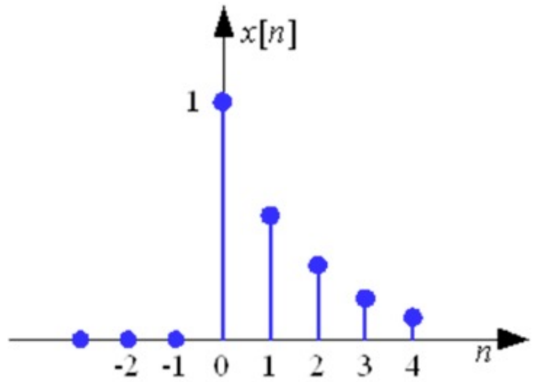
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-jn\omega} = 1$$



DTFT Example 2

- Determine the DTFT of a right-sided power sequence

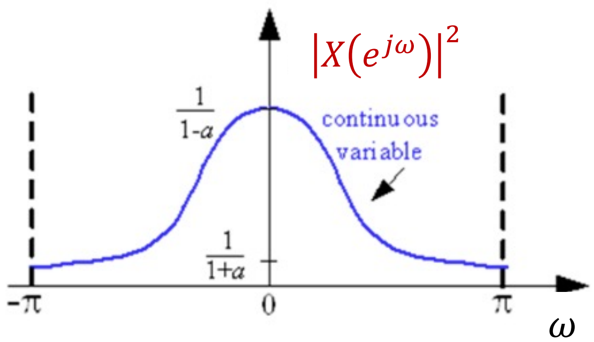
$$x[n] = \begin{cases} 0 & n < 0 \\ a^n & n \geq 0, |a| < 1 \end{cases}$$



- The DTFT is given by

$$X(e^{j\omega}) = \sum_{n=0}^{+\infty} a^n e^{-j\omega n} = \sum_{n=0}^{+\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1}{(1 - a\cos\omega) + ja\sin\omega} = \frac{(1 - a\cos\omega) - ja\sin\omega}{1 + a^2 - 2a\cos\omega}$$



$$|X(e^{j\omega})|^2 = \frac{1}{1 + a^2 - 2\,a\cos\omega}$$

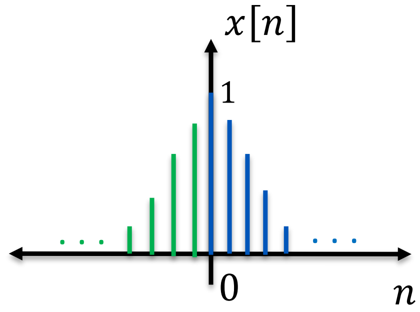
Magnitude Response

$$\angle X(e^{j\omega}) = -\tan^{-1}\left(\frac{a\sin\omega}{1 - a\cos\omega}\right)$$

Phase Response

DTFT Example 3

- Find the DTFT of both-sided sequence $x[n] = a^{-|n|}$



$$X(e^{j\omega}) = \sum_{n=-\infty}^{-1} a^{n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{-n} e^{-j\omega n}$$

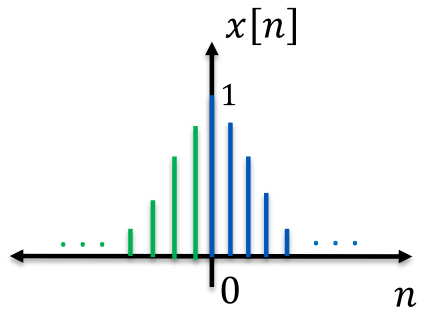
$$= \sum_{m=1}^{\infty} a^{-m} e^{j\omega m} + \sum_{n=0}^{\infty} a^{-n} e^{-j\omega n} = -1 + \sum_{m=0}^{+\infty} \left(a^{-1} e^{j\omega}\right)^{m} + \sum_{n=0}^{+\infty} \left(a^{-1} e^{-j\omega}\right)^{n}$$

$$= -1 + \frac{1}{1 - a^{-1} e^{j\omega}} + \frac{1}{1 - a^{-1} e^{-j\omega}} = \frac{a^2 - 1}{1 - 2a \cos \omega + a^2}$$

Magnitude Response $\left|X(e^{j\omega})\right| = \left|\frac{a^2 - 1}{1 - 2a \cos \omega + a^2}\right|$

Phase Response $\angle X(e^{j\omega}) = 0$

DTFT Example 3 (Detail Calculation)



- Find the DTFT of $x[n] = a^{-|n|}$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{-1} a^{n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{-n} e^{-j\omega n}$$

$$= \sum_{m=1}^{\infty} a^{-m} e^{j\omega m} + \sum_{n=0}^{\infty} a^{-n} e^{-j\omega n} = -1 + \sum_{m=0}^{+\infty} \left(a^{-1} e^{j\omega}\right)^{m} + \sum_{n=0}^{+\infty} \left(a^{-1} e^{-j\omega}\right)^{n}$$

$$= -1 + \frac{1}{1 - a^{-1} e^{j\omega}} + \frac{1}{1 - a^{-1} e^{-j\omega}} = -1 + \frac{1}{1 - a^{-1} e^{j\omega}} \left(\frac{1 - a^{-1} e^{-j\omega}}{1 - a^{-1} e^{-j\omega}}\right) + \frac{1}{1 - a^{-1} e^{-j\omega}} \left(\frac{1 - a^{-1} e^{j\omega}}{1 - a^{-1} e^{j\omega}}\right)$$

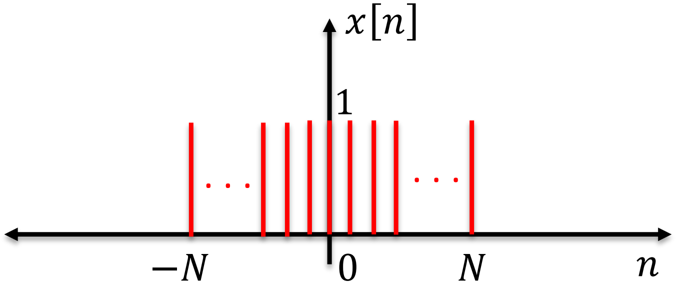
$$= -1 + \frac{2 - a^{-1}\left(e^{j\omega} + e^{-j\omega}\right)}{1 - a^{-1}\left(e^{j\omega} + e^{-j\omega}\right) + a^{-2}} = -1 + \frac{2 - 2a^{-1} \cos\omega}{1 - 2a^{-1} \cos\omega + a^{-2}}$$

$$= -1 + \frac{2a^{2} - 2a \cos \omega}{a^{2} - 2a \cos \omega + 1} = \frac{a^{2} - 1}{1 - 2a \cos \omega + a^{2}}$$

DTFT Example 4

- Determine the DTFT of a non-casual rectangular pulse sequence

$$x[n] = \begin{cases} 1, & |n| \le N \\ 0, & |n| > N \end{cases}$$



$$X(e^{j\omega}) = \sum_{n=-N}^{N} e^{-j\omega n}$$

$$= e^{j\omega N} + e^{j\omega(N-1)} + \cdots + e^{j\omega} + e^{j\omega(0)} + e^{-j\omega} + \cdots + e^{-j\omega(N-1)} + e^{-j\omega N}$$

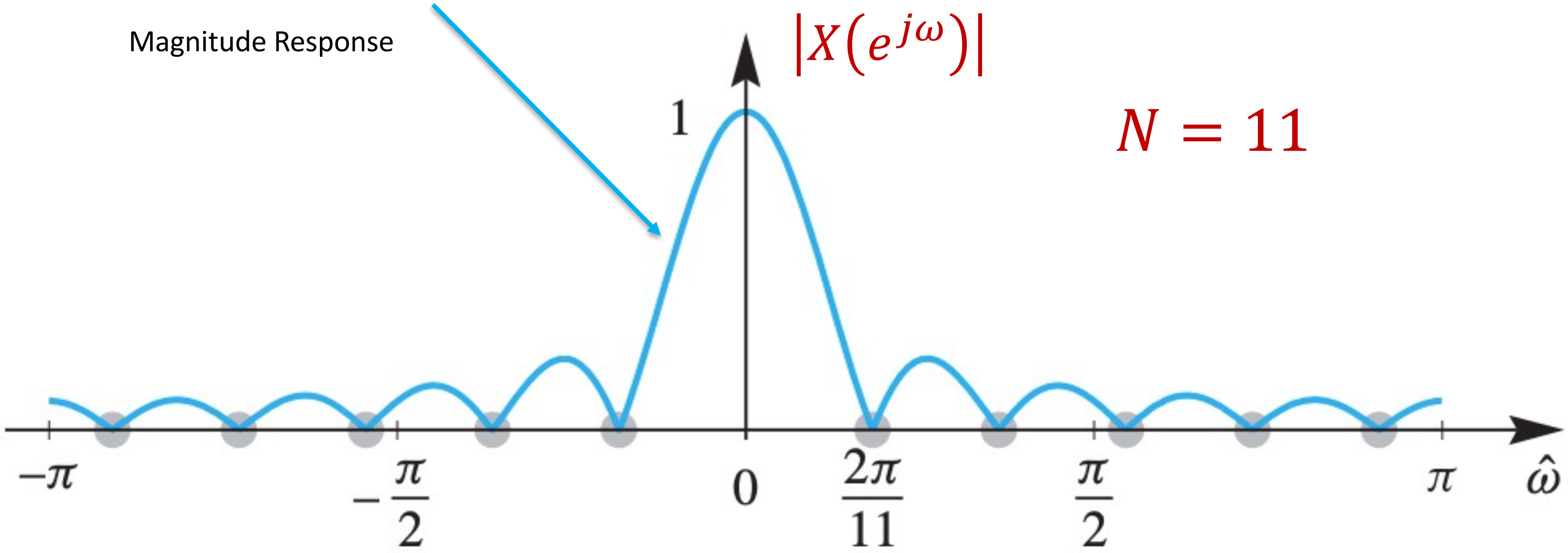
$$= e^{j\omega N}\left[1 + e^{-j\omega} + e^{-2j\omega} + \cdots + e^{-j\omega 2N}\right] = e^{j\omega N}\left[\frac{1 - e^{-j\omega(2N+1)}}{1 - e^{-j\omega}}\right]$$

$$X(e^{j\omega}) = \frac{e^{j\omega N} e^{-j\omega(N+\frac{1}{2})}}{e^{-j\omega/2}}\left[\frac{e^{j\omega(N+\frac{1}{2})} - e^{-j\omega(N+\frac{1}{2})}}{e^{j\omega/2} - e^{-j\omega/2}}\right] = \frac{\sin \omega(N + \frac{1}{2})}{\sin \omega/2}$$

$X(e^{j\omega})$ of Rectangular Pulse Sequence

$$\left|X(e^{j\omega})\right| = \left|\frac{\sin\omega(N+\frac{1}{2})}{\sin\omega/2}\right|$$

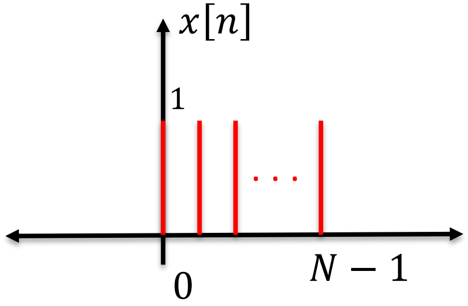
Magnitude Response



DTFT Example 5

- Determine the DTFT of a causal rectangular pulse sequence

$$x[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & Otherwise \end{cases}$$



$$X(e^{j\omega}) = \sum_{n=0}^{N-1} e^{-j\omega n} = 1 + e^{-j\omega} + e^{-2j\omega} + \cdots + e^{-j\omega(N-1)}$$

$$= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} = \frac{e^{-j\omega\frac{N}{2}}}{e^{-j\omega\frac{1}{2}}}\left[\frac{e^{j\omega\frac{N}{2}} - e^{-j\omega\frac{N}{2}}}{e^{j\omega\frac{1}{2}} - e^{-j\omega\frac{1}{2}}}\right] = e^{-j\omega\frac{N-1}{2}} \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}$$

Existence of DTFT

- For a given sequence the DTFT exist if the **infinite sum convergence**

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

- Or $\quad |X(e^{j\omega})| < \infty \quad for\ all\ \omega$

$$|X(e^{j\omega})| = \left|\sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}\right| \le \sum_{n=-\infty}^{+\infty} |x[n]||e^{-j\omega n}| = \sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$

- Therefore, the DTFT exists if a given sequence is absolute summable.
- All stable discrete-time systems are absolute summable and have DTFTs.

Properties of the DTFT (1)

$$x[n] \leftrightarrow X(e^{j\omega}) \quad and \quad y[n] \leftrightarrow Y(e^{j\omega})$$

1. **Linearity** : $ax[n] + by[n] \leftrightarrow aX(e^{j\omega}) + bY(e^{j\omega})$
2. **Time Shift** : $x[n - n_o] \leftrightarrow e^{-j\omega n_o}X(e^{j\omega})$
3. **Frequency Shift** : $x[n]e^{j\omega n_o} \leftrightarrow X(e^{j(\omega-\omega_o)})$
4. **Frequency Differentiation** : $nx[n] \leftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$

Properties of the DTFT (2)

4. **Convolution** : $x[n] * y[n] \leftrightarrow X(e^{j\omega}) \cdot Y(e^{j\omega})$

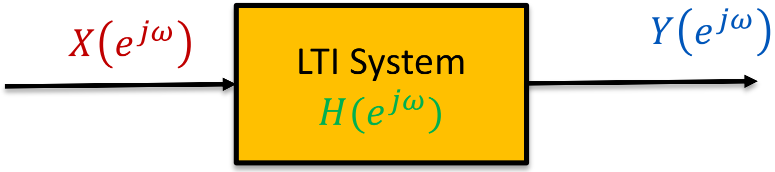
5. **Modulation** : $x[n] \cdot y[n] \leftrightarrow \frac{1}{2\pi} X(e^{j\omega}) * Y(e^{j\omega})$

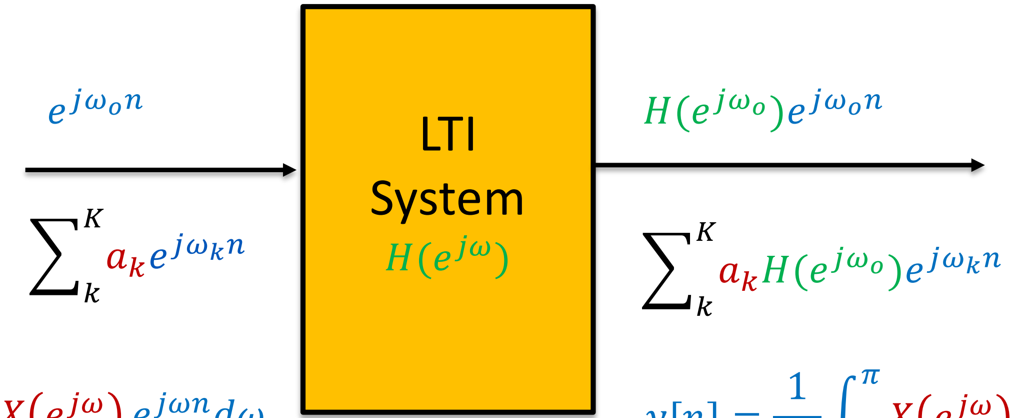
6. **Parseval's Theorem** :

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

Convolution Property of DTFT

- $y[n] = x[n] * h[n]$
- $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
- Proof





$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n} d\omega$$

$$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, H(e^{j\omega}) e^{j\omega n} d\omega$$

DTFT Symmetry Property (1)

- $X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$
- $X^*(e^{j\omega}) = X_R(e^{-j\omega}) - jX_I(e^{-j\omega})$
- $X_R(e^{j\omega}) = X_R(e^{-j\omega})$ => Even function => f(x) = f(-x)
- $X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ => Odd function => f(x) = -f(-x)
- $|X(e^{j\omega})|$ => Even function
- $\angle X(e^{j\omega})$ => Odd function

DTFT Symmetry Property (2)

- $x[n] \leftrightarrow X(e^{j\omega})$
- $x[-n] \leftrightarrow X(e^{-j\omega})$
- $x^*[n] \leftrightarrow X^*(e^{-j\omega})$
- $Re\{x[n]\} \leftrightarrow \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{-j\omega})\right]$
- $Im\{x[n]\} \leftrightarrow \frac{1}{2j}\left[X(e^{j\omega}) - X^*(e^{-j\omega})\right]$

DTFT Symmetry Property Proof Example

- $\boldsymbol{x[n]\ is\ real}$
- $\boldsymbol{X(e^{j\omega}) = X^*(e^{-j\omega})}$
- Proof

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} \Rightarrow X(e^{-j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{+j\omega n}$$

$$X^*(e^{-j\omega}) = \sum_{n=-\infty}^{+\infty} x^*[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} = X(e^{j\omega})$$

$x[n]\ is\ real$

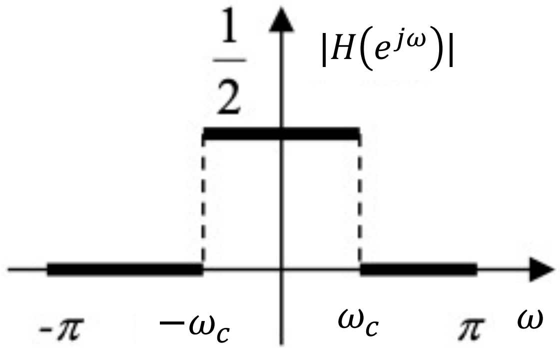
$x^*[n] = x[n]$

Inverse DTFT

Example: Impulse Response of Idea Lowpass Filter

- A discrete-time ideal lowpass filter is specificities in the fundamental interval of discrete frequency interval $-\pi \le \omega \le \pi$ as

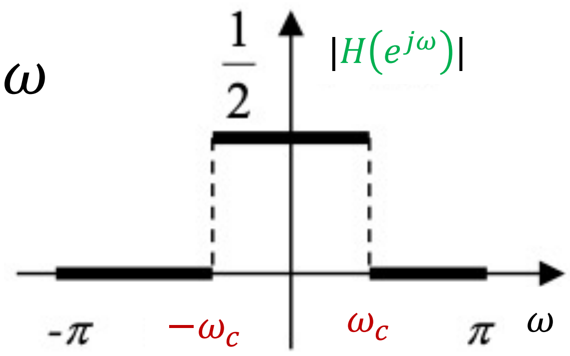
$$H(e^{j\omega}) = \begin{cases} \frac{1}{2} & 0 \le |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$



- Find the impulse response of this ideal lowpass filter.
- Sketch the impulse response for $\omega_c = \frac{\pi}{4}$

Using Inverse DTFT to find the Impulse Response

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \frac{1}{2} e^{j\omega n} d\omega$$

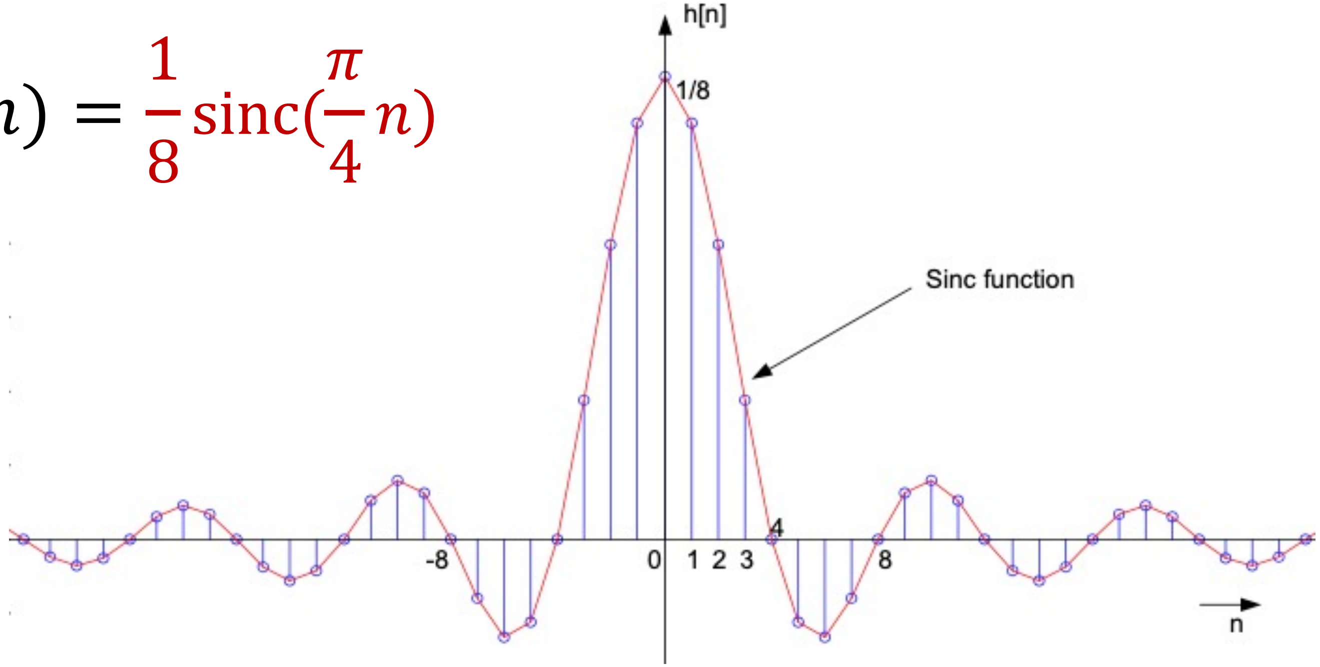


$$= \frac{1}{4\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} = \frac{1}{4\pi}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{jn}\right]$$

$$= \frac{1}{2\pi n}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right] = \frac{1}{2\pi}\frac{\sin(\omega_c n)}{n} = \frac{\omega_c}{2\pi}\frac{\sin(\omega_c n)}{\omega_c n} = \frac{\omega_c}{2\pi}\,\mathrm{sinc}(\omega_c n)$$

Impulse Response for $\omega_c = \frac{\pi}{4}$

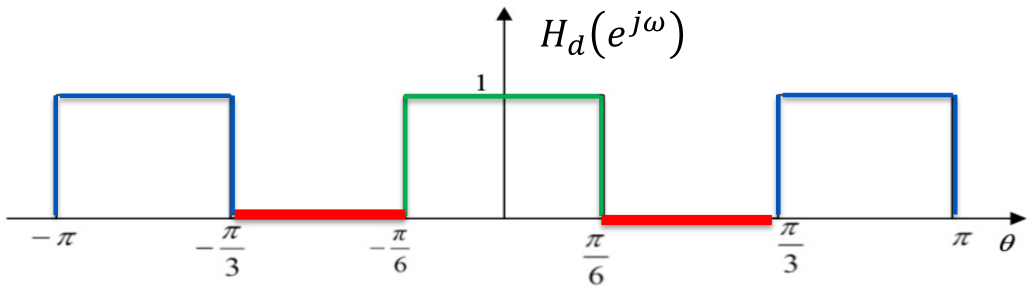
$$h[n] = \frac{\omega_c}{2\pi}\,\text{sinc}(\omega_c n) = \frac{1}{8}\text{sinc}(\frac{\pi}{4}n)$$



Example: Impulse Response of Idea Bandstop Filter

- A discrete-time ideal bandstop filter is specificities in the fundamental interval of discrete frequency interval $-\pi \leq \omega \leq \pi$ as

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \frac{\pi}{6} \\ 0 & \frac{\pi}{6} < |\omega| \leq \frac{\pi}{3} \\ 1 & \frac{\pi}{3} \leq |\omega| \leq \pi \end{cases}$$



- Find the impulse response of this ideal band stop filter.

The Impulse Response of Ideal Bandstop Filter

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n} d\omega = \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/3} e^{j\omega n} d\omega + \int_{-\pi/6}^{\pi/6} e^{j\omega n} d\omega + \int_{\pi/3}^{\pi} e^{j\omega n} d\omega\right]$$

$$= \frac{1}{2\pi}\left\{\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\pi/3} + \left[\frac{e^{j\omega n}}{jn}\right]_{-\pi/6}^{\pi/6} + \left[\frac{e^{j\omega n}}{jn}\right]_{\pi/3}^{\pi}\right\} = \frac{1}{2\pi}\left\{\frac{e^{-j\frac{\pi}{3}n} - e^{-j\pi n}}{jn} + \frac{e^{j\frac{\pi}{6}n} - e^{-j\frac{\pi}{6}n}}{jn} + \frac{e^{j\pi n} - e^{j\frac{\pi}{3}n}}{jn}\right\}$$

$$= \frac{e^{j\pi n} - e^{-j\pi n}}{j2\pi n} + \frac{e^{j\frac{\pi}{6}n} - e^{-j\frac{\pi}{6}n}}{j2\pi n} - \frac{e^{j\frac{\pi}{3}n} - e^{-j\frac{\pi}{3}n}}{j2\pi n} = \frac{\sin(\pi n)}{\pi n} + \frac{\sin(\frac{\pi}{6}n)}{\pi n} - \frac{\sin(\frac{\pi}{3}n)}{\pi n}$$

where $\frac{\sin(\pi n)}{\pi n} = 1 \text{ for } n = 0$ and zero elsewhere

$$h[n] = \delta[n] + \frac{\sin\left(\frac{\pi}{6}n\right)}{\pi n} - \frac{\sin(\frac{\pi}{3}n)}{\pi n}$$

Discrete Fourier Series (DFS)

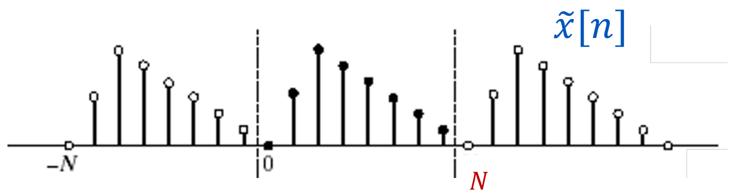
Why DFS, DFT and FFT?

- DTFT $H(e^{j\omega})$ provides great insights in discrete-time signal processing, but it not suitable for practical digital signal processing or analysis.
  - It is because $H(e^{j\omega})$ is a function of the continuous frequency variable ω
  - It is difficult to use computers to calculate a continuum of functional values.
- **Discrete Fourier Series (DFS)** $\tilde{X}[k]$ is closely related to DTFT but allows practical computation as it is discrete in frequency for analyzing **periodic sequence** $\tilde{x}[n]$. DFS is also called as Discrete-Time Fourier Series (DTFS)
- **Discrete Fourier Transform (DFT)** $X[k]$ is also closely related to DTFT and discrete in frequency, but it is used for analyzing **finite-length sequence** $x[n]$.
- **Fast Fourier Transform (FFT)** $X[k]$ is the **fast algorithms** to compute DFT for efficient implementation of DFT in real applications.

Discrete Fourier Series (DFS)

- Given a periodic sequence $\tilde{x}[n]$ with period N so that $\tilde{x}[n] = \tilde{x}[n + rN]$
- The Fourier Series representation can be written as

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=-\infty}^{\infty} \tilde{X}[k] e^{j\frac{2\pi}{N}kn}$$



- The Fourier Series representation of continuous-time periodic signals require infinite number of complex exponentials. Note that for discrete-time periodic signals, we have

$$e^{j\frac{2\pi}{N}(k+mN)n} = e^{j\frac{2\pi}{N}kn} e^{j(2\pi mn)} = e^{j\frac{2\pi}{N}kn}$$

- Due to the periodicity of the complex exponential, we only need *N* exponentials for DFS:

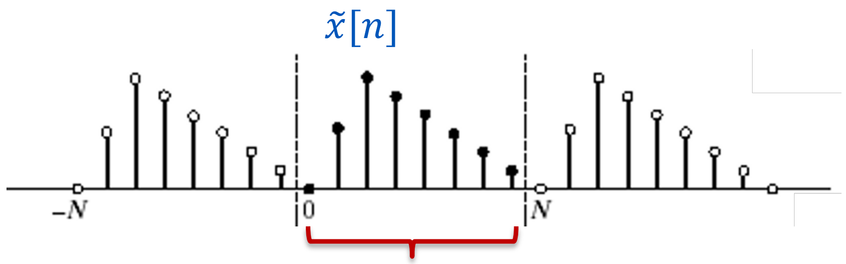
$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}kn}$$

$\omega_0 = \frac{2\pi}{N}$ is the fundamental angular frequency

DFS : Representation of Periodic Sequence

- A periodic sequence $\tilde{x}[n]$ with period N in terms of DFS coefficients as

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}kn}$$



- The DFS coefficients can be obtained via

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$$

- For convenience we sometimes use $W_N = e^{-j\frac{2\pi}{N}}$

DFS Analysis Equation

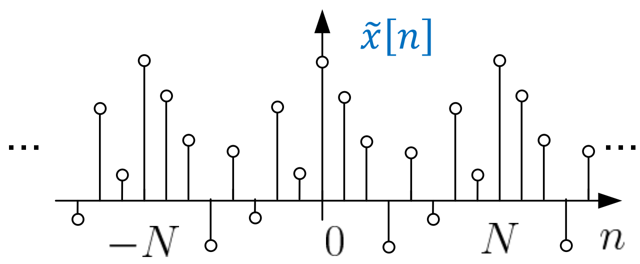
$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

DFS Synthesis Equation

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

Illustration of DFS

<table>
<tr><th>time domain</th><th>frequency domain</th></tr>
<tr><td>



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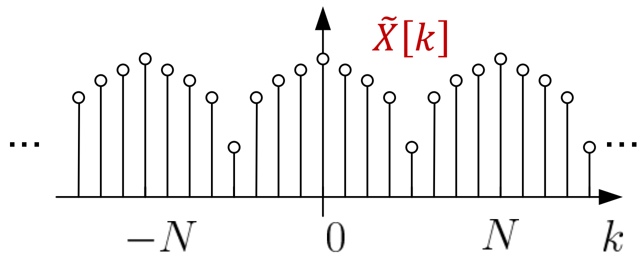


</td></tr>
<tr><td>

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

</td><td>

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

</td></tr>
<tr><td>discrete and periodic</td><td>discrete and periodic</td></tr>
</table>

DFS Example 1 : Sinusoidal Sequence

Find the DFS of the sequence $\tilde{x}[n] = \cos\left(\frac{2\pi}{8}n\right)$ for N=8.

- Using the Euler's Formula, this sequence can be expressed as

$$\tilde{x}[n] = \cos\left(\frac{2\pi}{8}n\right) = \frac{1}{2}\left[e^{j\frac{2\pi}{8}n} + e^{-j\frac{2\pi}{8}n}\right] = \frac{1}{2}e^{j\frac{2\pi}{8}n} + \frac{1}{2}e^{-j\frac{2\pi}{8}n}$$

- Compared with the DFS synthesis equation with N=8,

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi}{N}kn} = \frac{1}{2}e^{j\frac{2\pi}{8}n} + \frac{1}{2}e^{-j\frac{2\pi}{8}n} = \frac{1}{8}\left[4e^{j\frac{2\pi}{8}(1)n} + 4e^{j\frac{2\pi}{8}(8-1)n}\right] = \frac{1}{8}\left[4e^{j\frac{2\pi}{8}(1)n} + 4e^{j\frac{2\pi}{8}(7)n}\right]$$

$$= \frac{1}{8}\left[\tilde{X}[0]e^{j\frac{2\pi}{8}(0)n} + \tilde{X}[1]e^{j\frac{2\pi}{8}(1)n} + \tilde{X}[2]e^{j\frac{2\pi}{8}(2)n} + \tilde{X}[3]e^{j\frac{2\pi}{8}(3)n} + \tilde{X}[4]e^{j\frac{2\pi}{8}(4)n} + \tilde{X}[5]e^{j\frac{2\pi}{8}(5)n} + \tilde{X}[6]e^{j\frac{2\pi}{8}(6)n} + \tilde{X}[7]e^{j\frac{2\pi}{8}(7)n}\right]$$

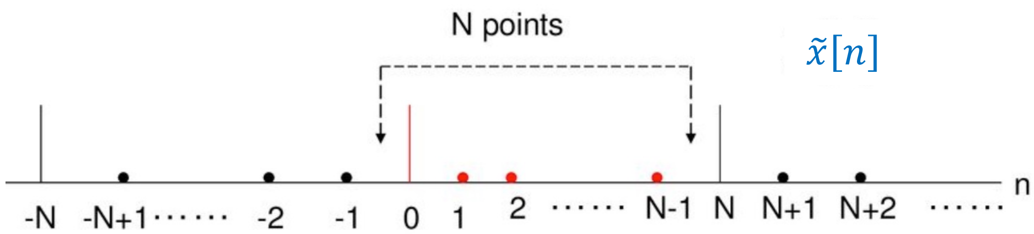
- We can find that only k=1 and 7 are non-zero of the DFS $\tilde{X}[k]$

$$\tilde{X}[1] = \tilde{X}[7] = 4 \qquad \tilde{X}[0] = \tilde{X}[2] = \tilde{X}[3] = \tilde{X}[4] = \tilde{X}[5] = \tilde{X}[6] = 0$$

DFS Example 2 : A Periodic Impulse Train

- DFS of a periodic impulse train

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \begin{cases} 1, & n = rN \\ 0, & otherwise \end{cases}$$

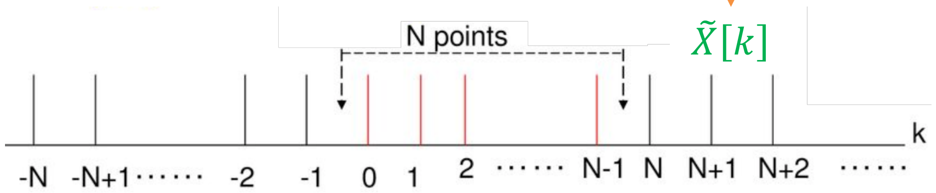


- Since the period of the signal is *N*

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} \delta[n] e^{-j\frac{2\pi}{N}kn} = e^{-j\frac{2\pi}{N}k0} = 1$$

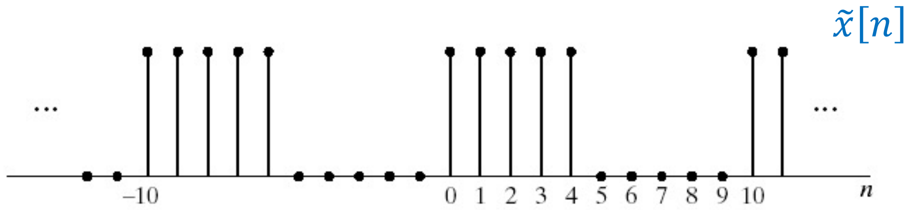
- We can represent the signal with the DFS coefficients as

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}kn}$$



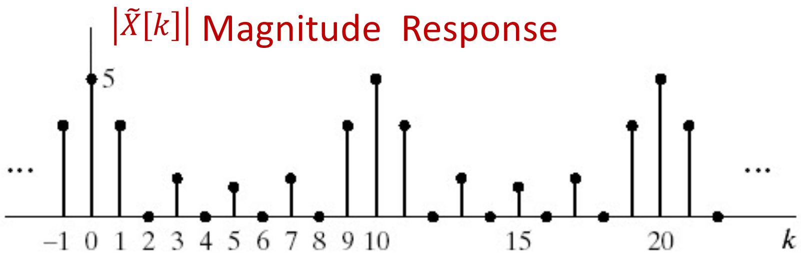
DFS Example 3 : A Periodic Rectangular Pulse Train

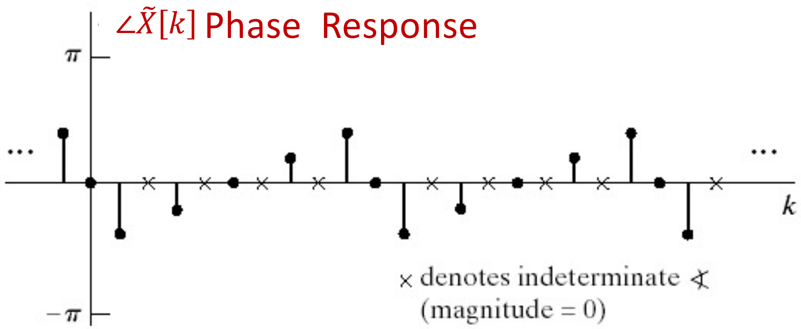
- DFS of a periodic rectangular pulse train with period N=10



- The DFS coefficients

$$\tilde{X}[k] = \sum_{n=0}^{9} \tilde{x}[n] e^{-j\frac{2\pi}{10}kn} = \sum_{n=0}^{4} e^{-j\frac{2\pi}{10}kn} = \frac{1 - e^{-j\frac{2\pi}{10}k5}}{1 - e^{-j\frac{2\pi}{10}k}} = e^{-j\frac{4\pi}{10}k} \frac{\sin(\pi k/2)}{\sin(\pi/10)}$$

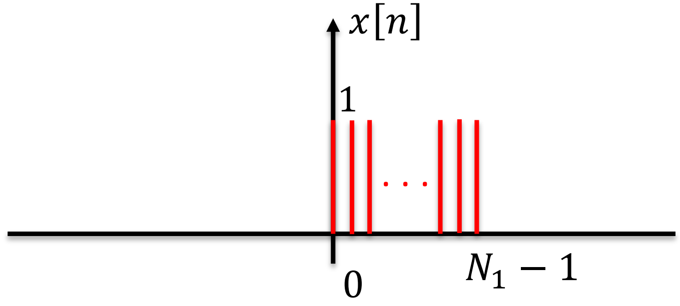




DTFT of Causal Rectangular Pulse Sequence

- Determine the DTFT of a causal rectangular pulse sequence

$$x[n] = \begin{cases} 1, & 0 \le n \le N_1 - 1 \\ 0, & Otherwise \end{cases}$$



$$X(e^{j\omega}) = \sum_{n=0}^{N_1-1} e^{-j\omega n} = 1 + e^{-j\omega} + e^{-2j\omega} + \cdots + e^{-j\omega(N_1-1)}$$

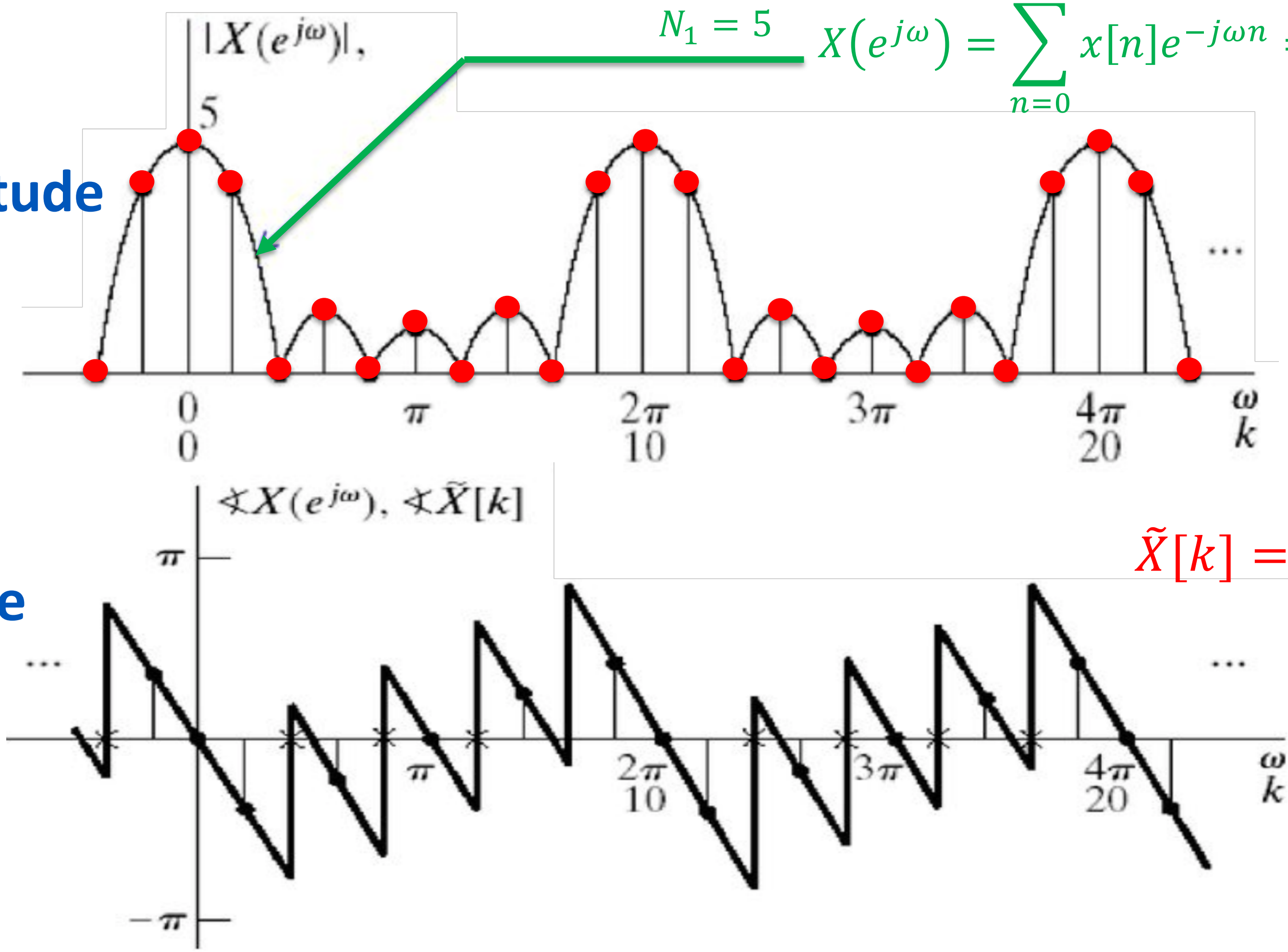
$$= \frac{1 - e^{-j\omega N_1}}{1 - e^{-j\omega}} = \frac{e^{-j\omega\frac{N_1}{2}}}{e^{-j\omega\frac{1}{2}}}\left[\frac{e^{j\omega\frac{N_1}{2}} - e^{-j\omega\frac{N_1}{2}}}{e^{j\omega\frac{1}{2}} - e^{-j\omega\frac{1}{2}}}\right] = e^{-j\omega\frac{N_1-1}{2}} \frac{\sin\frac{\omega N_1}{2}}{\sin\frac{\omega}{2}}$$

Relationship between DFS and DTFT

$N_1 = 5$ $\quad X(e^{j\omega}) = \sum_{n=0}^{4} x[n]e^{-j\omega n} = e^{-j2\omega}\frac{\sin(5\,\omega/2)}{\sin(\omega/2)}$

Magnitude

Phase



$\omega = \frac{2\pi}{10}k$

Sampling

$$\tilde{X}[k] = e^{-j\frac{4\pi}{10}k}\frac{\sin(\pi k/2)}{\sin(\pi k/10)}$$

Relationship between DFS and DTFT

- Comparing the DFS $\tilde{X}[k]$ and DTFT $X(e^{j\omega})$, we have:

$$\tilde{X}[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k}$$

- This is, $\tilde{X}[k]$ is equal to $X(e^{j\omega})$ sampled at *N* distinct frequencies between $\omega \in [0,2\pi]$ with a uniform frequency spacing of $2\pi/N$.
- Samples of $X(e^{j\omega})$ or DTFT of a finite-duration sequence $x[n]$ can be computed using the DFS of an infinite-duration periodic sequence $\tilde{x}[n]$, which is a periodic extension of $x[n]$.

Properties of the DFS

$$\tilde{x}[n] \leftrightarrow \tilde{X}[k] \quad and \quad \tilde{y}[n] \leftrightarrow \tilde{Y}[k]$$

1. **Linear Property** : $a\tilde{x}[n] + b\tilde{y}[n] \leftrightarrow a\tilde{X}[k] + b\tilde{Y}[k]$
2. **Time Shift Property** : $\tilde{x}[n - n_o] \leftrightarrow e^{-j\frac{2\pi}{N}n_o}\tilde{X}[k]$
3. **Duality** : $\tilde{x}[n] \leftrightarrow \tilde{X}[k]$, then $\tilde{X}[n] \leftrightarrow N\tilde{x}[-k]$
4. **Symmetry** : $\tilde{x}[n] \leftrightarrow \tilde{X}[k]$, then $\tilde{x}^*[n] \leftrightarrow \tilde{X}^*[-k]$ and $\tilde{x}^*[-n] \leftrightarrow \tilde{X}^*[k]$

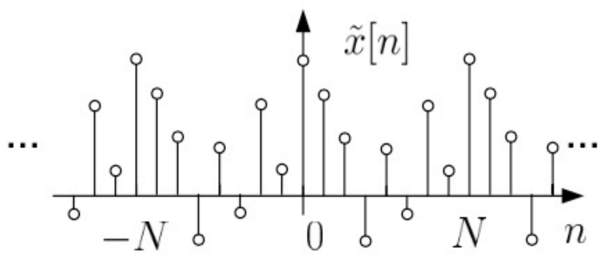
5. Periodicity Property of DFS

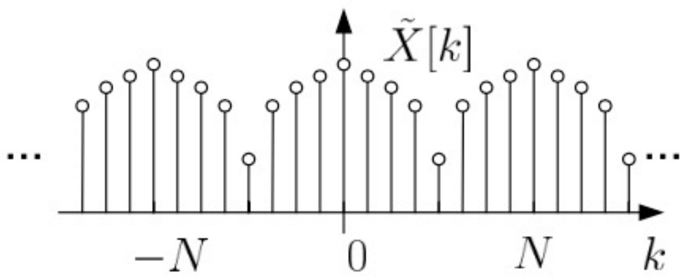
Periodicity : $\tilde{x}[n] = \tilde{x}[n + rN] \leftrightarrow \tilde{X}[k] = \tilde{X}[k + rN]$ r is integer.

Proof

$$\tilde{X}[k+rN] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{(k+rN)n} = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{nk} W_N^{n(rN)}$$

$$= \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{nk} = \tilde{X}[k]$$





6. Periodic Convolution Property of the DFS

- Let $\tilde{x}_1[n] \leftrightarrow \tilde{X}_1[k]\ and\ \ \tilde{x}_2[n]\ \leftrightarrow \tilde{X}_2[k]$ be two DFS pairs with same period of *N*, We have

$$\tilde{x}_1[n] \ \widetilde{\otimes}\ \tilde{x}_2[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\,\tilde{x}_2[n-m] \ \leftrightarrow \tilde{X}_1[k]\,\tilde{X}_2[k]$$

- Analogous to conventional convolution, $\widetilde{\otimes}$ denotes discrete-time convolution within one period of the periodic sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$

Proof of Periodic Convolution

$$\sum_{n=0}^{N-1}\left[\sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m]\right]W_N^{nk} = \sum_{m=0}^{N-1}\tilde{x}_1[m]\left[\sum_{n=0}^{N-1}\tilde{x}_2[n-m]W_N^{nk}\right]$$

$$= \sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{X}_2[k]W_N^{mk}$$

$$= \tilde{X}_2[k]\left[\sum_{m=0}^{N-1}\tilde{x}_1[m]W_N^{mk}\right]$$

$$= \tilde{X}_1[k]\tilde{X}_2[k]$$

- To compute $\tilde{x}[n] \widetilde{\otimes} \tilde{y}[n]$ where both $\tilde{x}[n]$ and $\tilde{y}[n]$ are of period N, we indeed only need the samples with n = 0,1,2,…,N-1

Calculation of Periodic Convolution (1)

- Let $\tilde{z}[n] = \tilde{x}[n] \,\widetilde{\otimes}\, \tilde{y}[n]$, which can be expressed as

$$\tilde{z}[n] = \tilde{x}[0]\tilde{y}[n] + \cdots + \tilde{x}[N-2]\tilde{y}[n-(N-2)] + \tilde{x}[N-1]\tilde{y}[n-(N-1)]$$

For $n = 0$:

$$\begin{aligned}
\tilde{z}[0] &= \tilde{x}[0]\tilde{y}[0] + \cdots + \tilde{x}[N-2]\tilde{y}[0-(N-2)] + \tilde{x}[N-1]\tilde{y}[0-(N-1)] \\
&= \tilde{x}[0]\tilde{y}[0] + \cdots + \tilde{x}[N-2]\tilde{y}[0-(N-2)+N] + \tilde{x}[N-1]\tilde{y}[0-(N-1)+N] \\
&= \tilde{x}[0]\tilde{y}[0] + \cdots + \tilde{x}[N-2]\tilde{y}[2] + \tilde{x}[N-1]\tilde{y}[1]
\end{aligned}$$

For $n = 1$:

$$\begin{aligned}
\tilde{z}[1] &= \tilde{x}[0]\tilde{y}[1] + \cdots + \tilde{x}[N-2]\tilde{y}[1-(N-2)] + \tilde{x}[N-1]\tilde{y}[1-(N-1)] \\
&= \tilde{x}[0]\tilde{y}[1] + \cdots + \tilde{x}[N-2]\tilde{y}[1-(N-2)+N] + \tilde{x}[N-1]\tilde{y}[1-(N-1)+N] \\
&= \tilde{x}[0]\tilde{y}[1] + \cdots + \tilde{x}[N-2]\tilde{y}[3] + \tilde{x}[N-1]\tilde{y}[2]
\end{aligned}$$

Calculation of Periodic Convolution (2)

- A period $\tilde{z}[n]$ of can be computed in matrix form as

$$\begin{bmatrix} \tilde{z}[0] \\ \tilde{z}[1] \\ \vdots \\ \tilde{z}[N-2] \\ \tilde{z}[N-1] \end{bmatrix} = \begin{bmatrix} \tilde{y}[0] & \tilde{y}[N-1] & \cdots & \tilde{y}[2] & \tilde{y}[1] \\ \tilde{y}[1] & \tilde{y}[0] & \cdots & \tilde{y}[3] & \tilde{y}[2] \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \tilde{y}[N-2] & \tilde{y}[N-3] & \cdots & \tilde{y}[0] & \tilde{y}[N-1] \\ \tilde{y}[N-1] & \tilde{y}[N-2] & \cdots & \tilde{y}[1] & \tilde{y}[0] \end{bmatrix} \begin{bmatrix} \tilde{x}[0] \\ \tilde{x}[1] \\ \vdots \\ \tilde{x}[N-2] \\ \tilde{x}[N-1] \end{bmatrix}$$

Periodic Convolution Example

- Given two periodic sequences $\tilde{x}[n]$ and $\tilde{y}[n]$, with period 4 :
  - $[\ \tilde{x}[0], \tilde{x}[1], \tilde{x}[2], \tilde{x}[3]\ ] = [4,\ -3,\ 2,\ -1]$
  - $[\ \tilde{y}[0], \tilde{y}[1], \tilde{y}[2], \tilde{y}[3]\ ] = [1,\ \ 2,\ \ 3,\ \ 4]$

- Compute $\tilde{z}[n] = \tilde{x}[n] \ \widetilde{\otimes}\ \tilde{y}[n]$, which can be computed as

$$\begin{bmatrix} \tilde{z}[0] \\ \tilde{z}[1] \\ \tilde{z}[2] \\ \tilde{z}[3] \end{bmatrix} = \begin{bmatrix} \tilde{y}[0] & \tilde{y}[3] & \tilde{y}[2] & \tilde{y}[1] \\ \tilde{y}[1] & \tilde{y}[0] & \tilde{y}[3] & \tilde{y}[2] \\ \tilde{y}[2] & \tilde{y}[1] & \tilde{y}[0] & \tilde{y}[3] \\ \tilde{y}[3] & \tilde{y}[2] & \tilde{y}[1] & \tilde{y}[0] \end{bmatrix} \begin{bmatrix} \tilde{x}[0] \\ \tilde{x}[1] \\ \tilde{x}[2] \\ \tilde{x}[3] \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 10 \\ 4 \\ 10 \end{bmatrix}$$

Convolution of Finite-Duration Sequences

Periodic convolution can be utilized to compute convolution of finite-duration sequences as follows.

- Let $x[n]$ and $y[n]$ be finite-duration sequences with lengths M and N, respectively, and $z[n] = x[n] \otimes y[n]$ which has a length of (M+N-1)
- We append (N-1) and (M-1) zeros at the ends of $x[n]$ and $y[n]$ for constructing periodic $\tilde{x}[n]$ and $\tilde{y}[n]$ where both are of period (M+N-1) $z[n]$ is then obtained from one period of $\tilde{x}[n] \widetilde{\otimes} \tilde{y}[n]$.

Example

- Compute the convolution of $x[n]$ and $y[n]$ with the use of periodic convolution.
- The lengths of $x[n]$ and $y[n]$ are 2 and 3 as
  - $[x[0], x[1]] = [2, \ 3]$
  - $[y[0], y[1], y[2]] = [1, \ -4, \ 5]$

- The length of $x[n] \otimes y[n]$ is (2+3-1)=4. As a result, we append two zeros and one zero in of $x[n]$ and $y[n]$, respectively. Then,
- $x[n] \otimes y[n] = [2, \ 3, 0, 0] \ \widetilde{\otimes} \ [1, -4, \ 5, 0] = [2, -5, -2, \ 15]$

Python : scipy.signal.convolve

```
from scipy import signal
x = np.array([2, 3])
y = np.array([1, -4, 5])
z = signal.convolve(x, y)
print("z = ", z)
```

z = [2 -5 -2 15]

https://docs.scipy.org/doc/scipy/reference/generated/scipy.signal.convolve.html