Discrete Fourier Transform (DFT)

EE4015 Digital Signal Processing

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Message 1 : Group Project Presentation Schedule

- **The project presentation schedule is now available in the Group Project webpage. Students are recommended to check their project information. If you find any problem of the arrangement or incorrect information, please contact the instructor as soon as possible for correction or follow up action.**

- This is also just a friendly reminder to submit group project proposals. Students must submit their group project proposal in **PDF format** to the CANVAS group project proposal assignment by **September 27, 2022 at 11pm**. The details of the group project can be found in the course website:
  - http://www.ee.cityu.edu.hk/~lmpo/ee4015/pdf/projects.htmlLinks to an external site.
- This is a group submission, so **each project team needs to assign a project leader to submit the proposal to CANVAS**.
  - Filename format : Proposal_GroupNumber_ProjectName.pdf
  - Filename example: Proposal_Group01_Audio_Classification.pdf

Message 2 : Assignment 1 Submission

- This is just a friendly reminder for submitting the Assignment 1 by **October 4, 2022 at 11pm**
- Students are required to submit the answer sheets of the Assignment 1 as a pdf file to this CANVAS assignment:
  - Filename format : Assignment01_StudentName_StudentID.pdf
  - Filename example: Assignment01_Chen_Hoi_501234567.pdf
- The questions of the Assignment 1 can be downloaded from the following link:
  - http://www.ee.cityu.edu.hk/~lmpo/ee4015/pdf/2022_EE4015_Ass01.pdf

Message 3 : Quiz

- **Canvas Quiz on Week 7**
  - Canvas quiz with 30 multiple choice questions released on **October 11, 2022 at 5:00pm**.
  - Students must perform the CANVAS Quiz in the classroom of P4701.
  - Students must complete this quiz by **6:00 PM**.
  - This quiz is open-book and covers course content from Weeks 1 to 4.

A Big Picture of Transformations for Signal Processing

Continuous-Time Signals

Periodic : $\tilde{x}(t)$

- Continuous-Time Fourier Series (CTFS) : a_k
  - Commonly called Fourier Series (FS)

Non-Periodic (Aperiodic) : $x(t)$

- Continuous-Time Fourier Transform (CTFT) : $X(j\Omega)$
  - Commonly called Fourier Transform (FT)

Generalization

- Laplace Transform : $X(s) = X(\sigma + j\Omega)$
  - For system design

Discrete-Time Signals (Sequences)

Periodic : $\tilde{x}[n]$

- Discrete Fourier Series (DFS) : $\tilde{X}[k]$
  - also called Discrete-Time Fourier Series (DTFS)

Non-Periodic (Aperiodic) : $x[n]$

- Discrete-Time Fourier Transform (DTFT) : $X(e^{j\omega})$

Finite-Duration Sequences : $x[n]$

- Discrete Fourier Transform (DTF) : $X[k]$
- Fast Fourier Transform (FFT) : $X[k]$

Generalization

- The z-Transform : $X(z) = X(re^{j\omega})$

Content

Fourier Transforms For Discrete-Time Signal Analysis (Last Week)

- Discrete-Time Fourier Transform (DTFT) for Non-Periodic Sequences
- Properties of DTFT

DFS, DFT and FFT

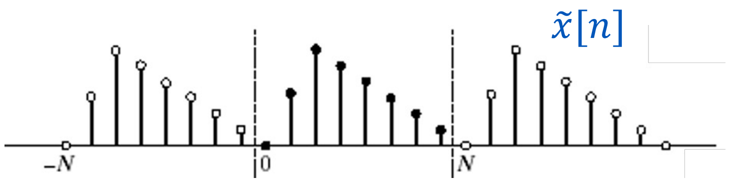
- Discrete Fourier Series (DSF) for Periodic Sequences
- Properties of DSF
- Periodic Convolution
- Discrete Fourier Transform (DFT) : Finite-Duration Sequences
- Properties of DFT
- Circular Convolution
- Zero-Padding for DFT computation of Linear Convolution
- Fast Fourier Transform (FFT) : Fast Algorithms for computing DFT
- Signal Analysis using FFT
- Signal Processing using FFT
- Spectrogram (Optional)

Why DFS, DFT and FFT?

- DTFT $H(e^{j\omega})$ provides great insights in discrete-time signal processing, but it not suitable for practical digital signal analysis
  - It is because $H(e^{j\omega})$ is a function of the continuous frequency variable ω
  - It is difficult to use computers to calculate a continuum of functional values.
- **Discrete Fourier Series (DFS)** $\tilde{X}[k]$ is closely related to DTFT but allows practical computation as it is discrete in frequency for analyzing periodic sequence $\tilde{x}[n]$. DFS is also called as Discrete-Time Fourier Series (DTFS)
- **Discrete Fourier Transform (DFT)** $X[k]$ is also closely related to DTFT and discrete in frequency, but it is used for analyzing finite-length sequence $x[n]$.
- **Fast Fourier Transform (FFT)** $X[k]$ is the fast algorithms to compute DFT for efficient implementation of DFT in real applications.

Discrete Fourier Series (DFS)

- Given a periodic sequence $\tilde{x}[n]$ with period *N* so that $\tilde{x}[n] = \tilde{x}[n + rN]$
- The Fourier Series representation can be written as



$$\tilde{x}[n] = \frac{1}{N}\sum_{k} \tilde{X}[k]e^{j\frac{2\pi}{N}kn}$$

- The Fourier Series representation of continuous-time periodic signals require infinite number of complex exponentials. Note that for discrete-time periodic signals, we have

$$e^{j\frac{2\pi}{N}(k+mN)n} = e^{j\frac{2\pi}{N}kn}e^{j(2\pi mn)} = e^{j\frac{2\pi}{N}kn}$$

- Due to the periodicity of the complex exponential, we only need *N* exponentials for DFS:

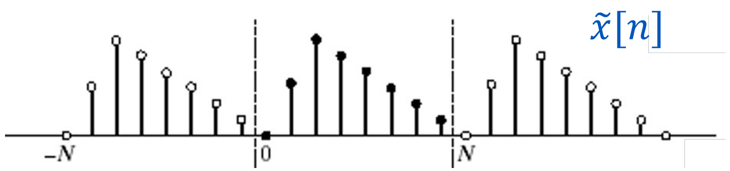
$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]e^{j\frac{2\pi}{N}kn}$$

$\omega_0 = \frac{2\pi}{N}$ is the fundamental angular frequency

DFS : Representation of Periodic Sequence

- A periodic sequence $\tilde{x}[n]$ with period N in terms of DFS coefficients as

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]e^{j\frac{2\pi}{N}kn}$$



- The DFS coefficients can be obtained via

$$\tilde{X}[n] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\frac{2\pi}{N}kn}$$

- For convenience we sometimes use $W_N = e^{-j\frac{2\pi}{N}}$

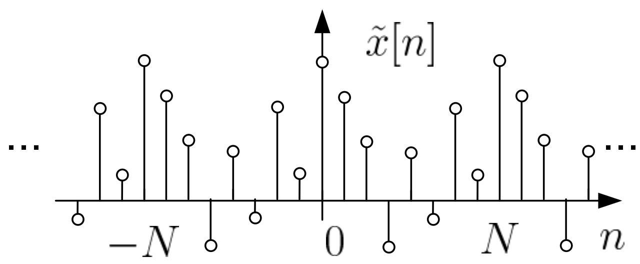
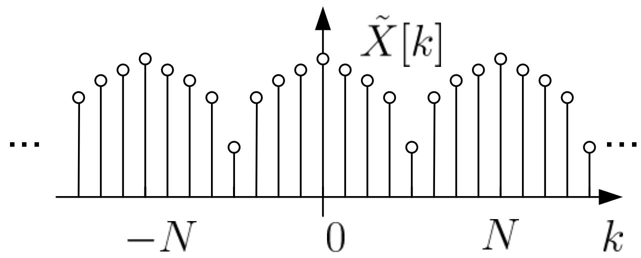
DFS Analysis Equation

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]W_N^{kn}$$

DFS Synthesis Equation

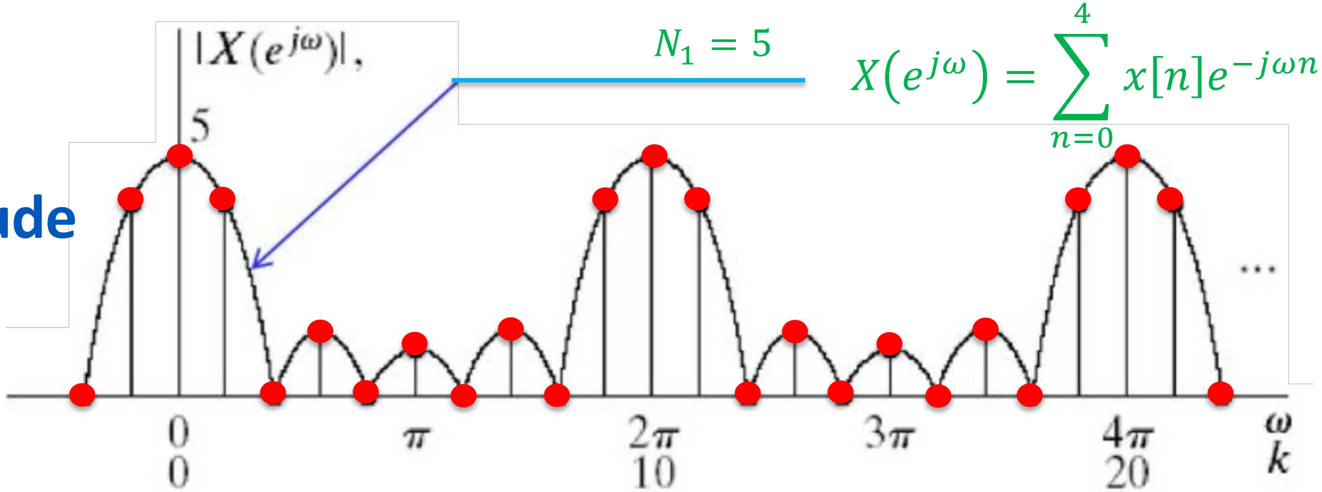
$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]W_N^{-kn}$$

Illustration of DFS

time domain	frequency domain
	
$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$	$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$
discrete and periodic	discrete and periodic

Relationship between DFS and DTFT

Magnitude



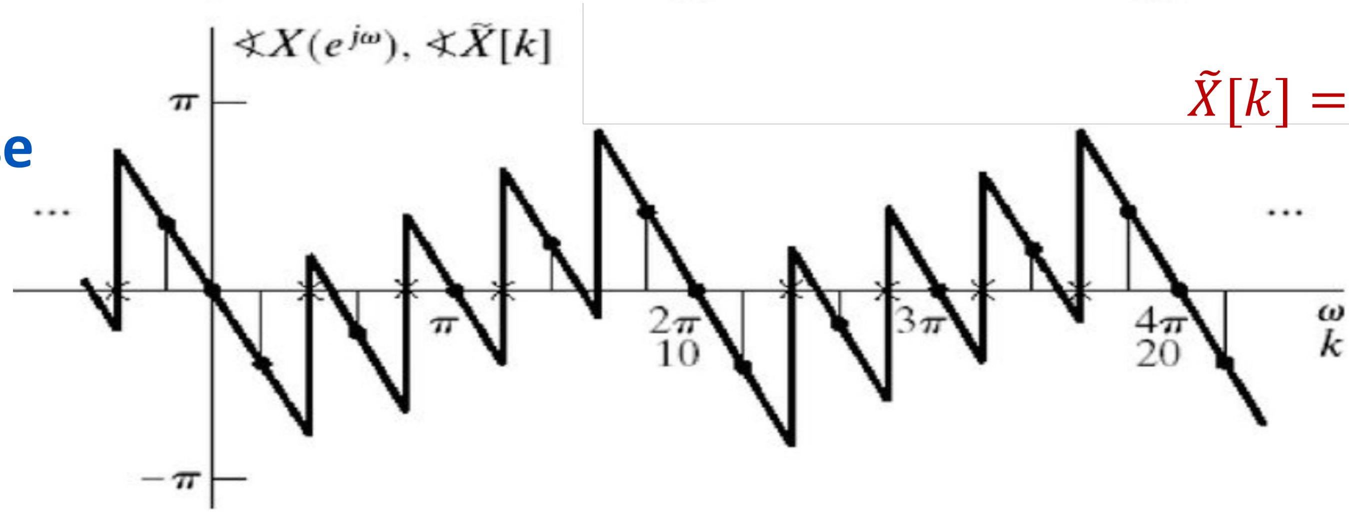
$$X(e^{j\omega}) = \sum_{n=0}^{4} x[n]e^{-j\omega n} = e^{-j2\omega}\frac{\sin(5\,\omega/2)}{\sin(\omega/2)}$$

$$\omega = \frac{2\pi k}{10}$$

Sampling

$$\tilde{X}[k] = e^{-j\frac{4\pi}{10}k}\frac{\sin(\pi k/2)}{\sin(\pi k/10)}$$

Phase



Relationship between DFS and DTFT

- Comparing the DFS $\tilde{X}[k]$ and DTFT $X(e^{j\omega})$, we have:

$$\tilde{X}[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}$$

- This is, $\tilde{X}[k]$ is equal to $X(e^{j\omega})$ sampled at *N* distinct frequencies between $\omega \in [0,2\pi]$ with a uniform frequency spacing of $2\pi/N$.
- Samples of $X(e^{j\omega})$ or DTFT of a finite-duration sequence $x[n]$ can be computed using the DFS of an infinite-duration periodic sequence $\tilde{x}[n]$, which is a periodic extension of $x[n]$.

Discrete Fourier Transform (DFT)

- **DFT** is closely related to both DTFT and DFS, but it is used for analyzing finite-duration sequence $x[n]$.
- Let $x[n]$ be a finite-duration sequence of length N such that $x[n] = 0$ outside $0 \leq n \leq N-1$. The DFT pair of $x[n]$ is:

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} & 0 \leq k \leq N-1 \\ 0 & otherwise \end{cases}$$

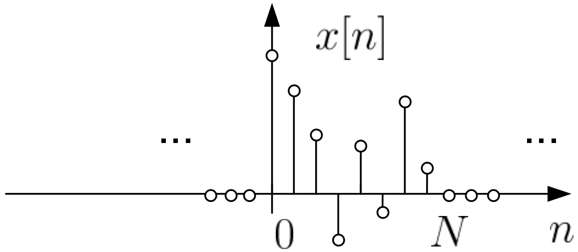
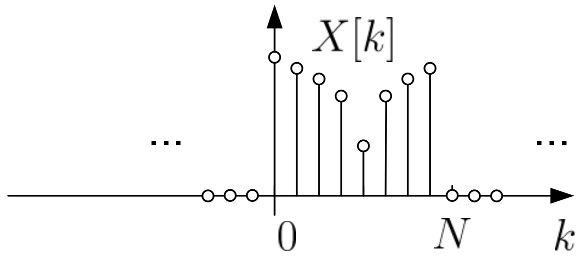
Analysis Equation

$$x[n] = \begin{cases} \frac{1}{N} \sum_{n=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} & 0 \leq k \leq N-1 \\ 0 & otherwise \end{cases}$$

Synthesis Equation

DFT and DFS are equivalent within the interval of $[0, N-1]$

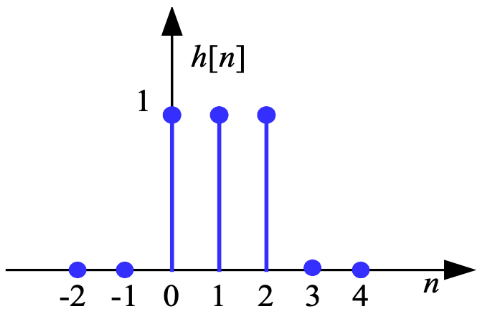
Illustration of DFT

time domain	frequency domain
	
$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$	$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$
discrete and finite	discrete and finite

DFT Example 1

- Find the DFT coefficients of a three-sample average system with finite impulse response of $h[n]$

$$h[n] = \begin{cases} \frac{1}{3} & 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$



- Using the analysis equation of DTFT with $N = 3$, we have

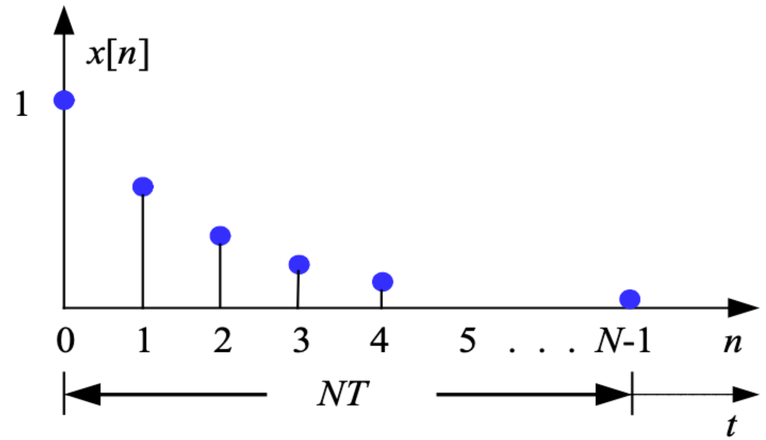
$$H[k] = \sum_{n=0}^{N-1} h[n] e^{-jn(\frac{2\pi}{N}k)} = \sum_{n=0}^{2} \frac{1}{3} e^{-jn(\frac{2\pi}{N}k)}$$

$$H[k] = e^{-j(\frac{2\pi}{N}k)} \frac{(1 + 2\cos(\frac{2\pi k}{N}))}{3}$$

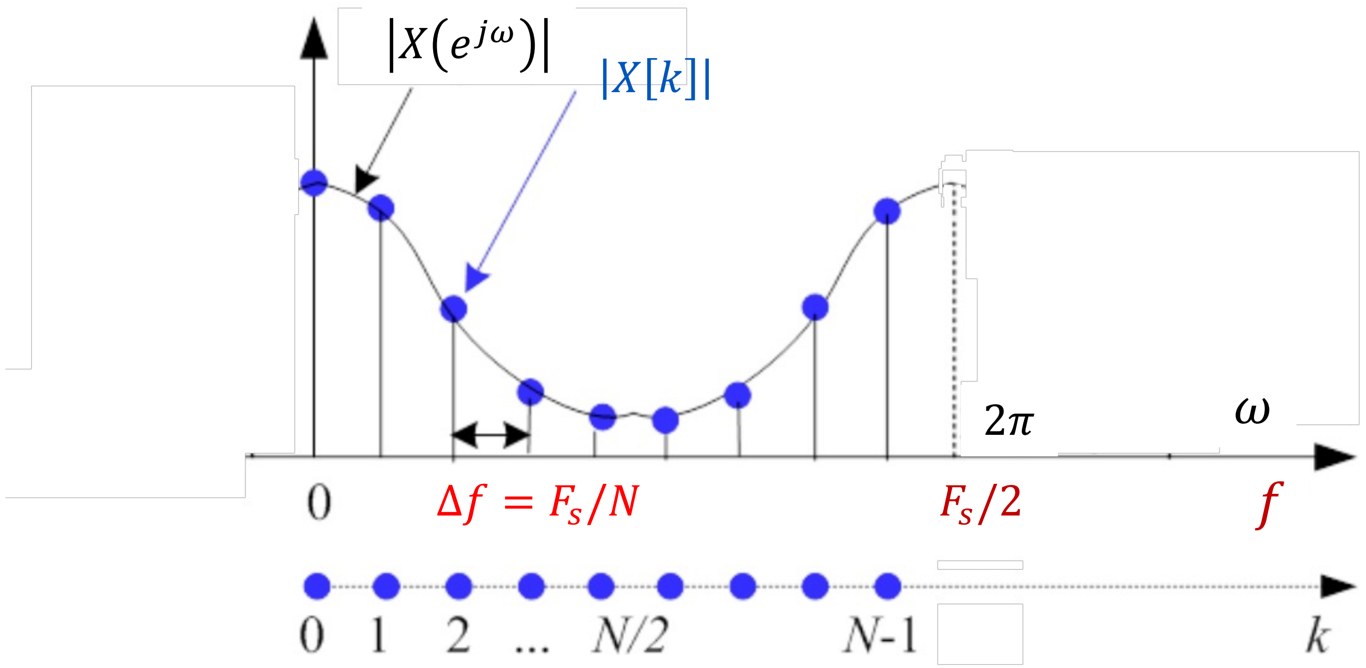
$$|H[k]| = \frac{1}{3} \left| [1 + 2\cos(\frac{2\pi k}{N})] \right| \qquad \text{where } k = 0,1,2$$

DFT Example 2

- Find the DTFT and DFT of a truncated exponential sequence :
  - $x[n] = x_a(nT) = a^{nT}, \; n = 0, 1, \ldots, N-1, \quad |a| < 1$
- This has DTFT :
  - $X(e^{j\omega}) = \frac{1-a^{NT}e^{-j\omega N}}{1-a^{T}e^{-j\omega}}$
- And the DFT :
  - $X[k] = \frac{1-a^{NT}}{1-a^{T}e^{-j\frac{2\pi}{N}k}}$



Relationship Between DTFT and DFT



- Note that the DFT $X[k]$ is just the N-sampled version of the DTFT $X(e^{j\omega})$ between the angular frequency ω of 0 to 2π

Inverse Discrete Fourier Transform (IDFT)

- The inverse DFT (IDFT) equation is

$$x[n] = \begin{cases} \frac{1}{N}\sum_{n=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}, & 0 \leq k \leq N-1 \\ 0, & otherwise \end{cases}$$

- However, the sequence which it produces will be a finite-duration sequence.

IDFT Example

- If $X[k] = \{4, -j2, 0, j2\}$, find its IDFT for $N = 4$

$$x[n] = \frac{1}{4}\sum_{n=0}^{3} X[k]e^{j\frac{2\pi}{4}kn} = \frac{1}{4}\sum_{n=0}^{3} X[k]e^{j\frac{\pi}{2}kn}$$

$$n = 0 \quad x[0] = \frac{1}{4}[4 - j2 + 0 + j2] = 1$$

$$n = 1 \quad x[1] = \frac{1}{4}\left[4 - j2e^{j\frac{\pi}{2}} + 0 + j2e^{j\frac{3\pi}{2}}\right] = 2$$

$$n = 2 \quad x[2] = \frac{1}{4}\left[4 - j2e^{j\pi} + 0 + j2e^{j3\pi}\right] = 1$$

$$n = 3 \quad x[3] = \frac{1}{4}\left[4 - j2e^{j\frac{3\pi}{2}} + 0 + j2e^{j\frac{9\pi}{2}}\right] = 0$$

- The IDFT is $x[n] = \{1, 2, 1, 0\}$,

Properties of the DFT

$$x[n] \leftrightarrow X[k] \quad and \quad y[n] \leftrightarrow Y[k]$$

1. **Linearity** : $ax[n] + b\tilde{y}[n] \leftrightarrow aX[k] + bY\,[k]$

2. **Circular Shift** : $x[(n-m)\;mod\;(N)] \leftrightarrow e^{-j\frac{2\pi}{N}m}X\,[k]$

3. **Symmetry** : $x[n] \leftrightarrow X[k]$, then $x^*[n] \leftrightarrow X^*[(-k)\;mod\;(N)]$ and

   $x^*[(-n)\;mod\;(N)] \leftrightarrow X^*[k]$

4. **Circular Convolution** :

$$x[n] \otimes_N y[n] = x[m]y\,[(n-m)\;mod\;(N)] \leftrightarrow X[k]\,Y[k]$$

Circular Shift of Sequence

- If $x[n] \leftrightarrow X[k]$, then

$$x[(n-m)\ mod\ (N)] \leftrightarrow e^{-j\frac{2\pi}{N}m} X\ [k]$$

- Note that in order to make sure that the resultant time index is within the interval of $[0, N-1]$, we need circular shift, which is defined as

$$(n-m)\ mod\ (N) = n - m + r \cdot N$$

where the integer r is chosen such that

$$0 \leq n - m + r \cdot N \leq N - 1$$

Example of Circular Shifted Sequence (1)

- Determine $x_1[n] = x[(n-2)\ mod\ (4)]$ where $x[n]$ is of length 4 and has the form of :

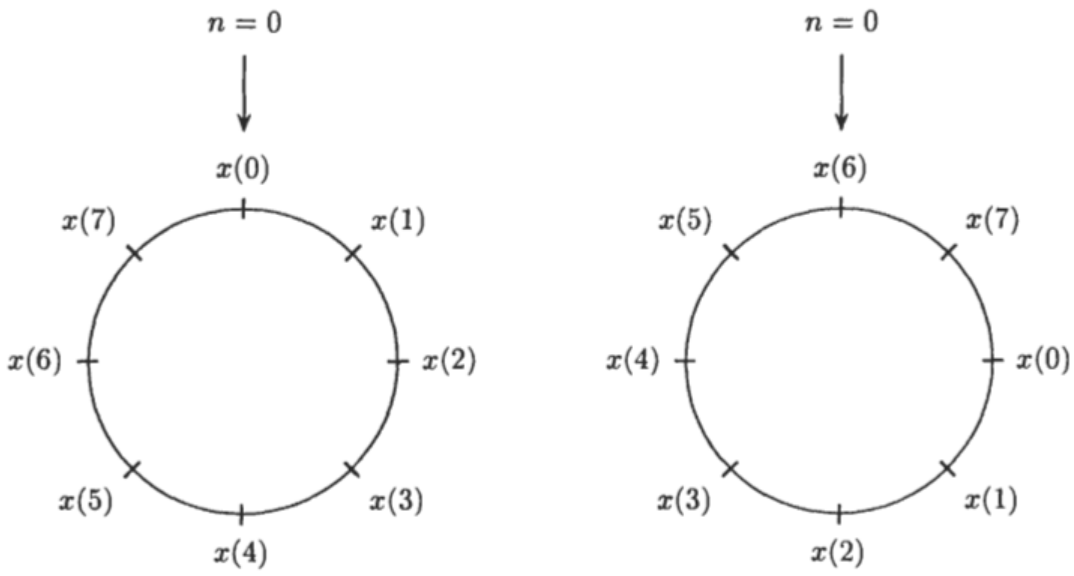
$$x[n] = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ 3, & n = 2 \\ 4, & n = 3 \end{cases}$$

- According the definition with $N = 4$, $x_1[n]$ is determined as:

$$x_1[0] = x[(0-2)\ mod\ (4)] = x[-2\ mod\ (4)] = x[2] = 3, \qquad r = 1$$
$$x_1[1] = x[(1-2)\ mod\ (4)] = x[-1\ mod\ (4)] = x[3] = 4, \qquad r = 1$$
$$x_1[2] = x[(2-2)\ mod\ (4)] = x[0\ mod\ (4)] = x[0] = 1, \qquad r = 0$$
$$x_1[3] = x[(3-2)\ mod\ (4)] = x[1\ mod\ (4)] = x[1] = 2, \qquad r = 0$$

Example of Circular Shifted Sequence (2)

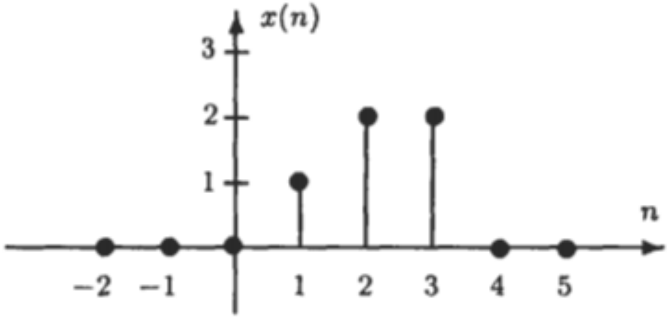
$N = 8$



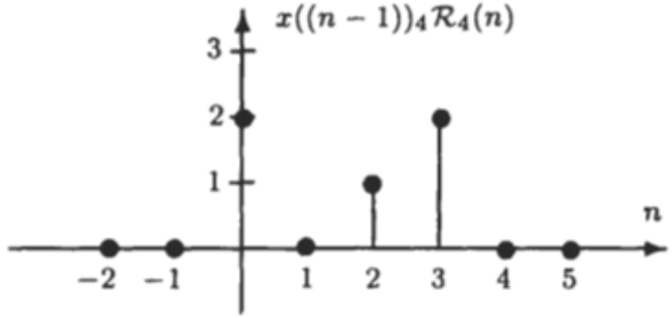
(a) An eight-point sequence. (b) Circular shift by two.

$x[n]$ $\qquad$ $x_1[n] = x[(n-2)\ mod\ (8)]$

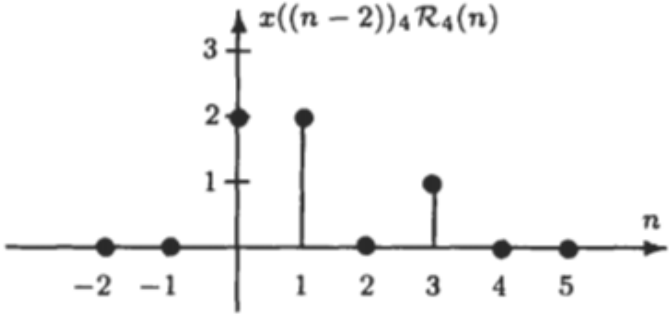
Example of Circular Shifted Sequence (3)



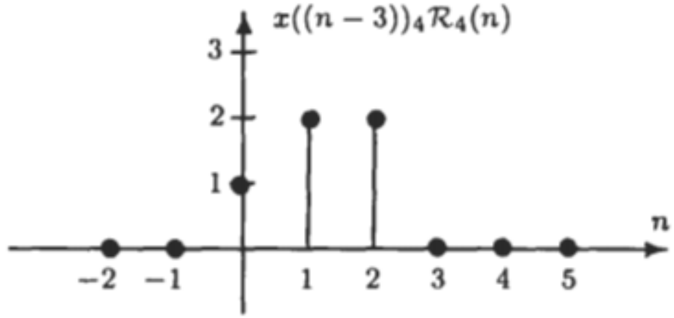
(a) A discrete-time signal of length $N = 4$.



(b) Circular shift by one.



(c) Circular shift by two.



(d) Circular shift by three.

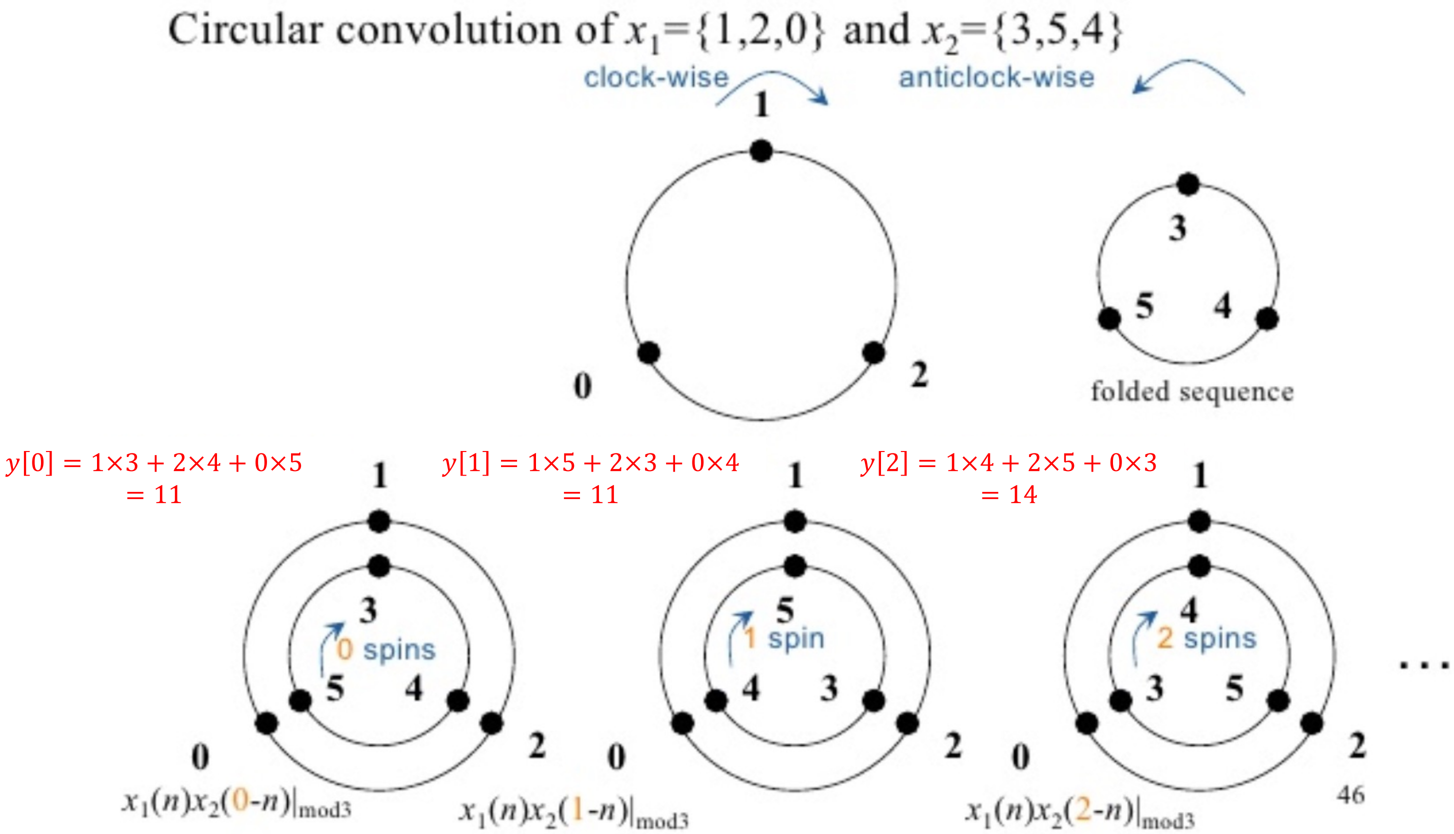
Circular Convolution

- Let $(x_1[n], X_1[k])$ and$(x_2[n], X_2[k])$ be two DFT pairs with same duration of N. We have

$$x_1[n] \otimes_N x_2[n] = \sum_{m=0}^{N-1} x_1[m]\, x_2[(n-m)\, mod(N)] \leftrightarrow X_1[k]X_2[k]$$

where $\otimes_N$ is the circular convolution operator.

Example of Circular Convolution (1)



Example of Circular Convolution (2)

Circular convolution of $x_1 = \{1, 2, 0\}$ and $x_2 = \{3, 5, 4\}$.

clock-wise

anticlock-wise

k	-2	-1	0	1	2	3	
$x_1(k)$			1	2	0		
$x_2(-k)\vert_3$	4	5	3	4	5	3	$y(0) = 11$
$x_2(1-k)\vert_3$	3	4	5	3	4	5	$y(1) = 11$
$x_2(2-k)\vert_3$	5	3	4	5	3	4	$y(2) = 14$
$x_2(3-k)\vert_3 = x_2(-k)\vert_3$	4	5	3	4	5	3	$y(3) = 11$

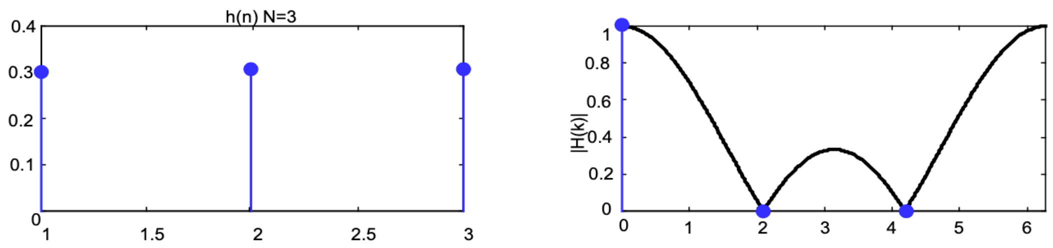
Padding with Zeros and Frequency Resolution

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} & 0 \le k \le N-1 \\ 0 & otherwise \end{cases}$$

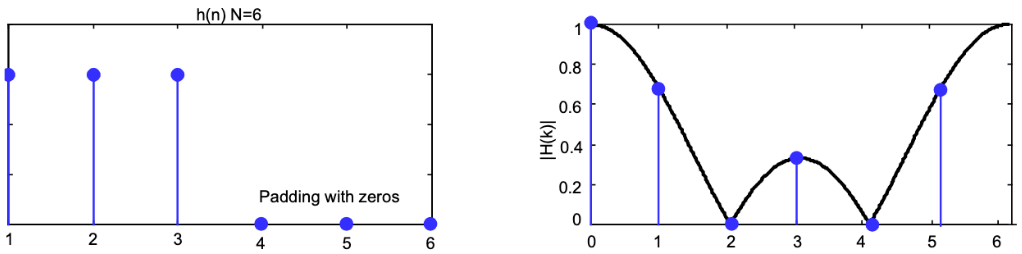
- To obtain more points in the DFT sequence, we can always increase the duration of $x[n]$ by adding additional zero-valued elements. This procedure is called **padding with zeros**.
- These zero-valued elements contribute noting to the sum in the above equation, but act to decrease the frequency spacing ($2\pi/N$).
- The zero padding gives us a high-density spectrum and provided a better displayed version for plotting. But it does not give us a high-resolution spectrum because no new information is added to the signal. Only additional zeros are added in the data.

Zero-Padding on Three-Point Average System

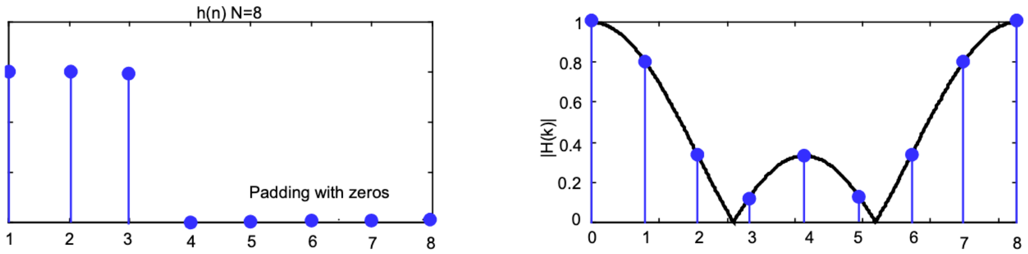
N=3



N=6

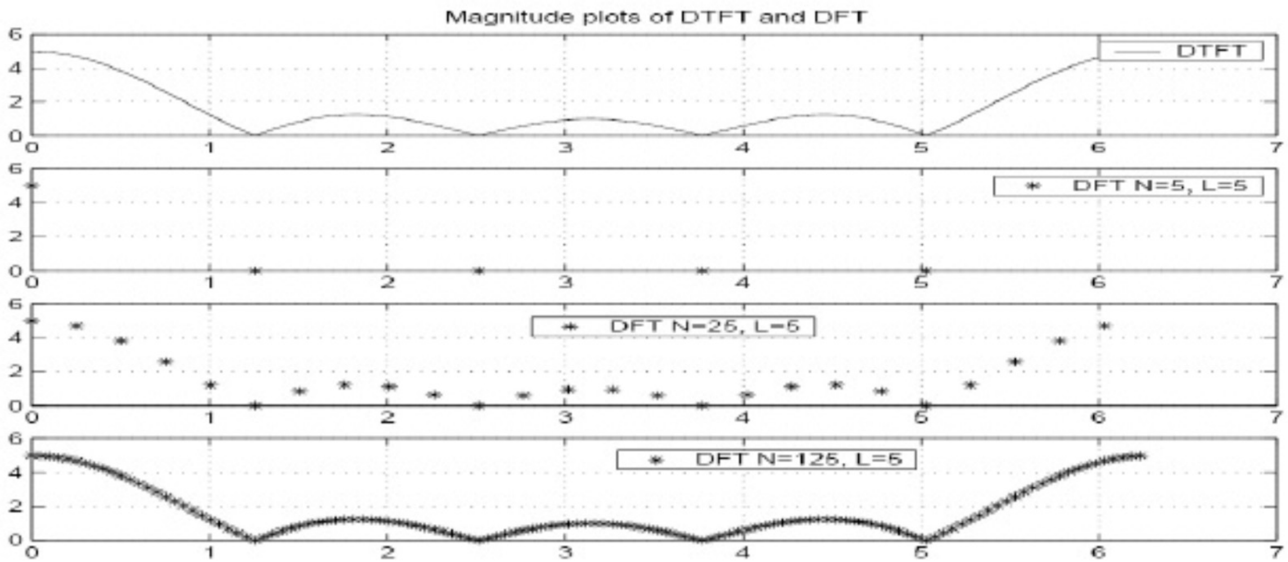


N=8



Zero-Padding Effects

- Padding is often used to make the length of sequence a power of 2
  - Good for Fast computing of DFT using Fast Fourier Transform (FFT) Algorithms



A $L-$point DFT is sufficient to represent $x(n)$ but visually it does not give a good picture of the spectral characteristics. A better picture is obtained by interpolating $X(\omega)$ at more closely spaced frequencies; this is achieved by passing $\{x(n)\}$ with $N-L$ zeros; see figure

Linear Convolution using Circular Convolution

- It can be shown that circular convolution of the padded sequence corresponds to the linear convolution
- Finite-duration sequences of $x_1[n]$ length M and $x_2[n]$ length L.
  - Output sequence $y[n] = \sum_{k=0}^{M-1} x_1[k]x_2[n-k]$ of length $L + M - 1$
- With this relationship, we can use circular convolution to compute Linear convolution by DTF and IDFT
  - Let $N \geq L + M - 1$, pad $x_1[n]$ and $x_2[n]$ with zeros to length N. (Zero Padding)
  - Perform their *N*-point DFTs $x_1[n]$ and $x_2[n]$, set $Y[k] = X_1[k]X_2[k]$
  - Compute the *N*-point IDFT of $Y[k] \Rightarrow y[n] = \sum_{k=0}^{N-1} x_1[k]x_2[n-k]$

Circular Convolution of the Padded Sequence corresponds to the Linear Convolution

$$y(n) = \sum_{k=0}^{N-1} x_1(k)\, x_2(n-k)\big|_{\text{modulo } N} = \sum_{k=0}^{M-1} x_1(k)\, x_2(n-k)\big|_{\text{modulo } N}$$

$$= \sum_{k=0,k\le n}^{M-1} x_1(k)\, x_2(n-k) + \sum_{k=0,k>n}^{M-1} x_1(k)\, x_2(n-k+N).$$

<u>If $n \ge M-1$</u>

$$y(n) = \sum_{k=0}^{M-1} x_1(k)\, x_2(n-k).$$

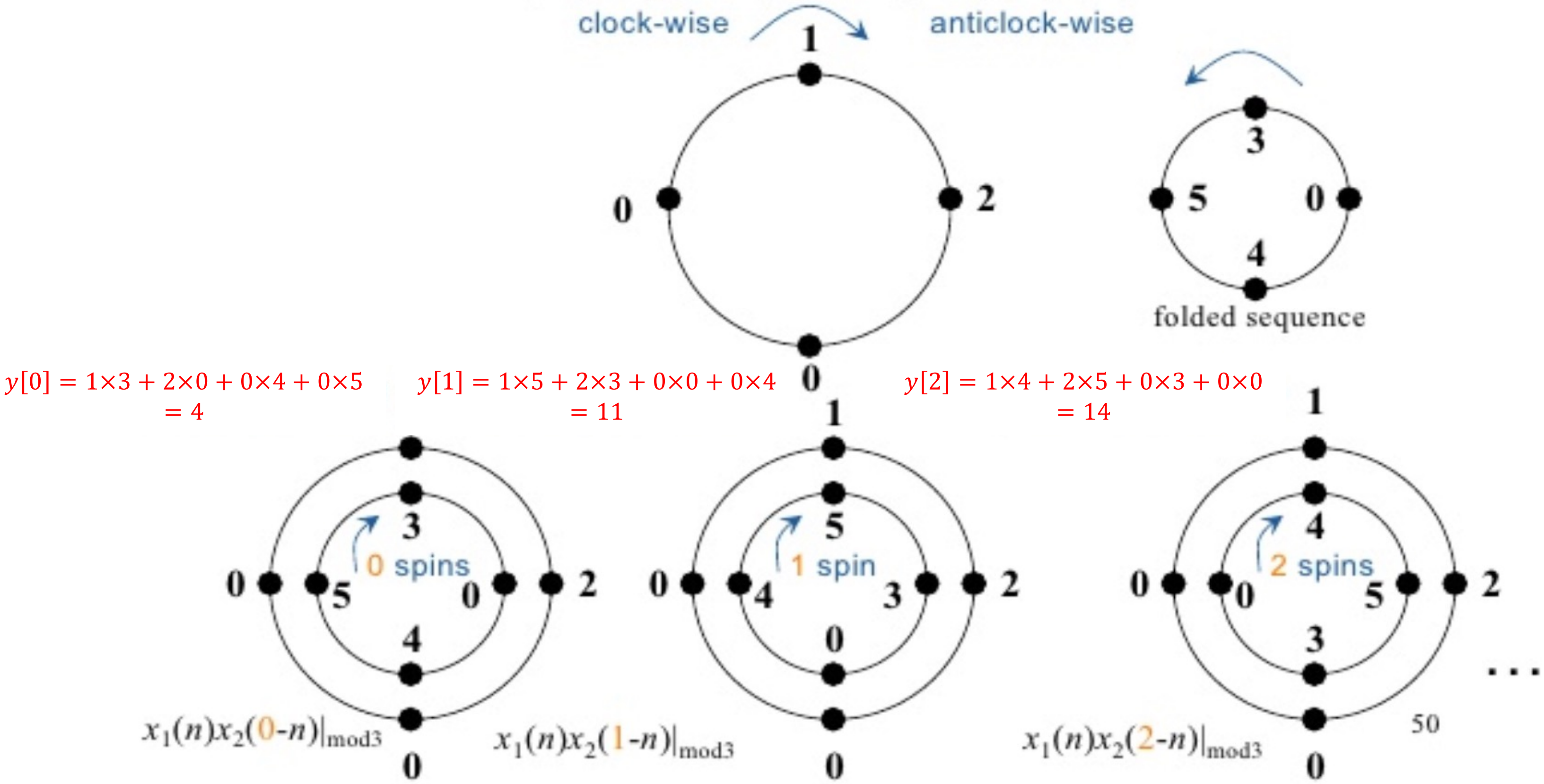
<u>If $n < M-1$</u>

$$y(n) = \sum_{k=0}^{n} x_1(k)\, x_2(n-k) + \sum_{k=n+1}^{M-1} x_1(k)\, x_2(n-k+N) \quad (\text{but } n-k+N \ge L)$$

$$= \sum_{k=0}^{n} x_1(k)\, x_2(n-k) = \sum_{k=0}^{M-1} x_1(k)\, x_2(n-k).$$

Example of Circular Convolution with Different Length

Convolution of $x_1 = \{1, 2\}$ $(M = 2)$ and $x_2 = \{3, 5, 4\}$ $(L = 3)$,
$N = L + M - 1 = 4$, $x_1 = \{1, 2, 0, 0\}$ & $x_2 = \{3, 5, 4, 0\}$



Example of Circular Convolution with Different Length

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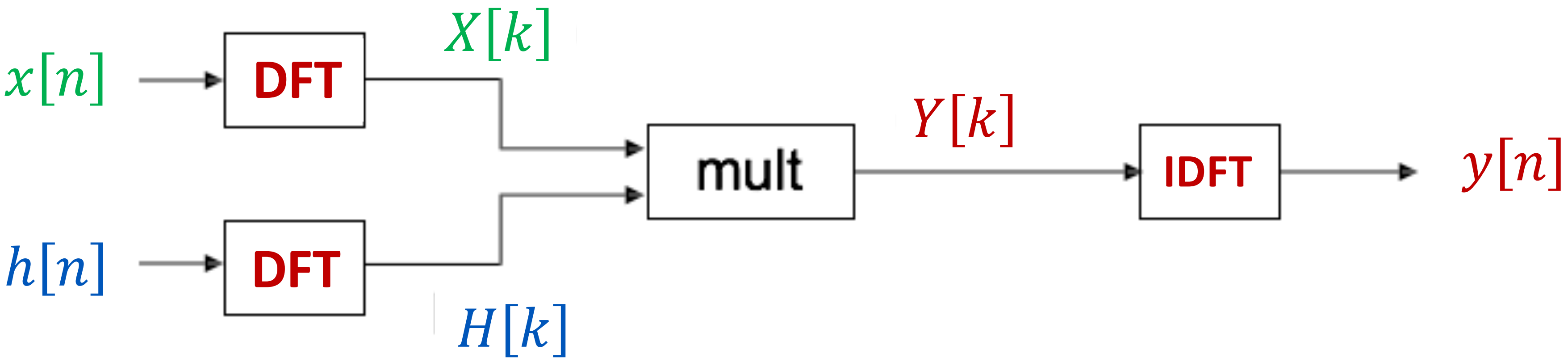
$N = L + M - 1 = 4$, $x_1 = \{1, 2, 0, 0\}$ & $x_2 = \{3, 5, 4, 0\}$

k	-2	-1	0	1	2	3	
$x_1(k)$			1	2	0	0	
$x_2(-k)\|_4$			3	0	4	5	$y(0) = 3$
$x_2(1-k)\|_4$			5	3	0	4	$y(1) = 11$
$x_2(2-k)\|_4$			4	5	3	0	$y(2) = 14$
$x_2(3-k)\|_4$			0	4	5	3	$y(3) = 8$

Linear Convolution using DFT



- If the impulse response is NOT the same size as the input sequence $x[n]$, we have to pad the $h[n]$ with zeros to match the length
- Multiplication is point-by-point, of complex numbers

IDFT for Circular Convolution Computation

- The IDFT of $X_1[n], X_2[k]$ is given by

$$y[n] = \frac{1}{N}\sum_{\mathrm{k}=0}^{N-1} X_1[k]X_2[k]\mathrm{e}^{\frac{j2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1} x_1[m]\mathrm{e}^{-\frac{j2\pi}{N}km}\right)\left(\sum_{l=0}^{N-1} x_2[l]\mathrm{e}^{-\frac{j2\pi}{N}kl}\right)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} x_1[m]\sum_{l=0}^{N-1} x_2[l]\left(\sum_{m=0}^{N-1} x_1[m]\mathrm{e}^{\frac{j2\pi}{N}k(n-m-l)}\right) = x_1[n] \otimes_N x_2[n]$$

- But $\sum_{l=0}^{N-1} e^{\frac{j2\pi}{N}k(n-m-l)} = N$ if $l = n - m$ module N and otherwise

$$\mathrm{e}^{\frac{j2\pi}{N}k(n-m-l)} = \frac{1-\mathrm{e}^{j2\pi k(n-m-l)}}{1-\mathrm{e}^{\frac{j2\pi}{N}k(n-m-l)}} = 0$$