Frequency Response Analysis

EE4015 Digital Signal Processing

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Content

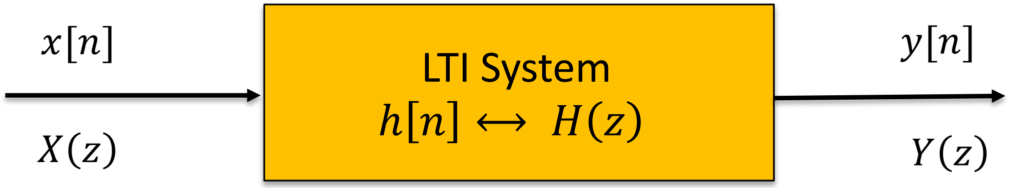
Frequency Response Analysis

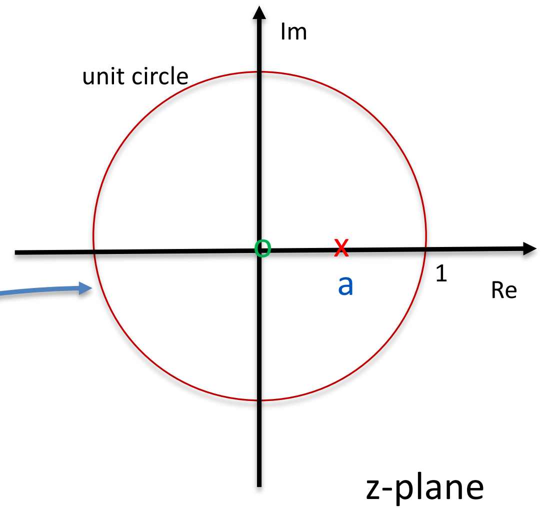
- Frequency Response Estimation
- Magnitude and Phase Responses
- Frequency Response of FIR Systems
- All-Pass Filters
- Second Order Resonant Filter
- Notch Filter Design using Pole-Zero Placement

Structures for Discrete-Time Systems

- Block Diagram Representation
- Signal Flow Graph Representation
- Non-Recursive Structures for FIR System
  - Direct Form and Cascade From
- Recursive Structures for IIR System
  - Direct Form, Canonic Form, Cascade Form and Parallel Form
- Comparison of Different Structures

Frequency Response Estimation



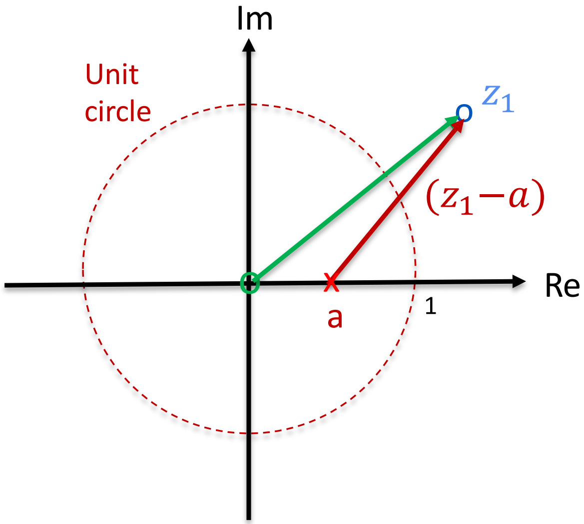


- $Y(z) = X(z)\,H(z)$
- $H(z)$ is referred as Transfer Function of the system.
- Frequency Response $H(e^{j\omega})$ of the transfer function corresponds to the unit circle
  - $H(z)|_{z=e^{j\omega}} = H(e^{j\omega})$

Geometry Interpretation in z-plane

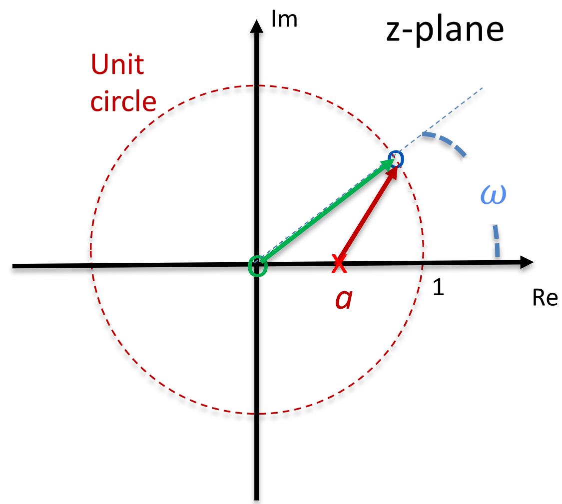
- For example, $H(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$
  - It has a zero at 0 and a pole at a
- Given a point z_1 on the z-plane,
  - The vector of z_1 corresponds to the vector from zero to the point z_1
  - The vector of (z_1-a) corresponds to the vector from the pole at a to the point z_1
  - The magnitude $|X(z_1)| = \frac{|z_1|}{|z_1-a|}$
  - The angle $\angle X(z_1) = \angle z_1 - \angle(z_1 - a)$

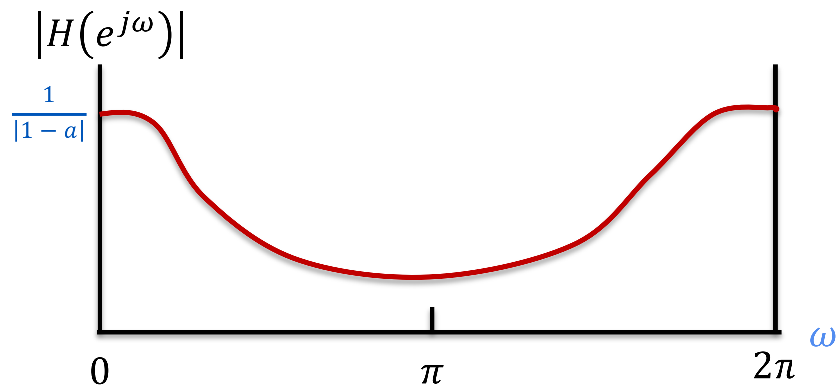
z-plane



Geometry Interpretation of Frequency Response

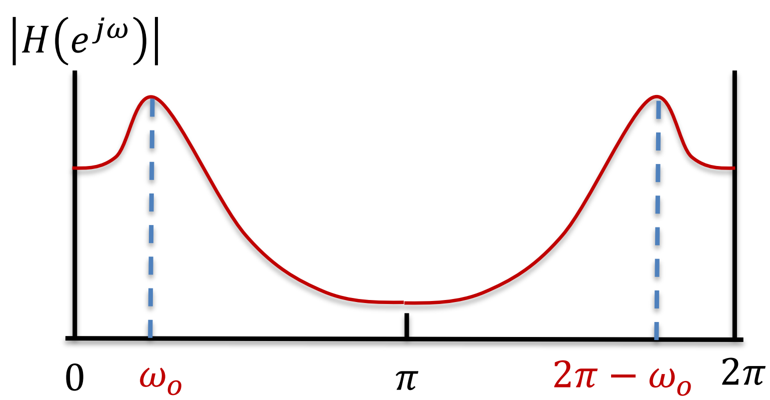
- $H(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$
- The frequency response $H(e^{j\omega})$ corresponds to all the points on the unit circle
  - The magnitude $\left|H(e^{j\omega})\right| = \frac{1}{|e^{j\omega}-a|}$
  - The angle $\angle H(e^{j\omega}) = \angle e^{j\omega} - \angle(e^{j\omega} - a)$



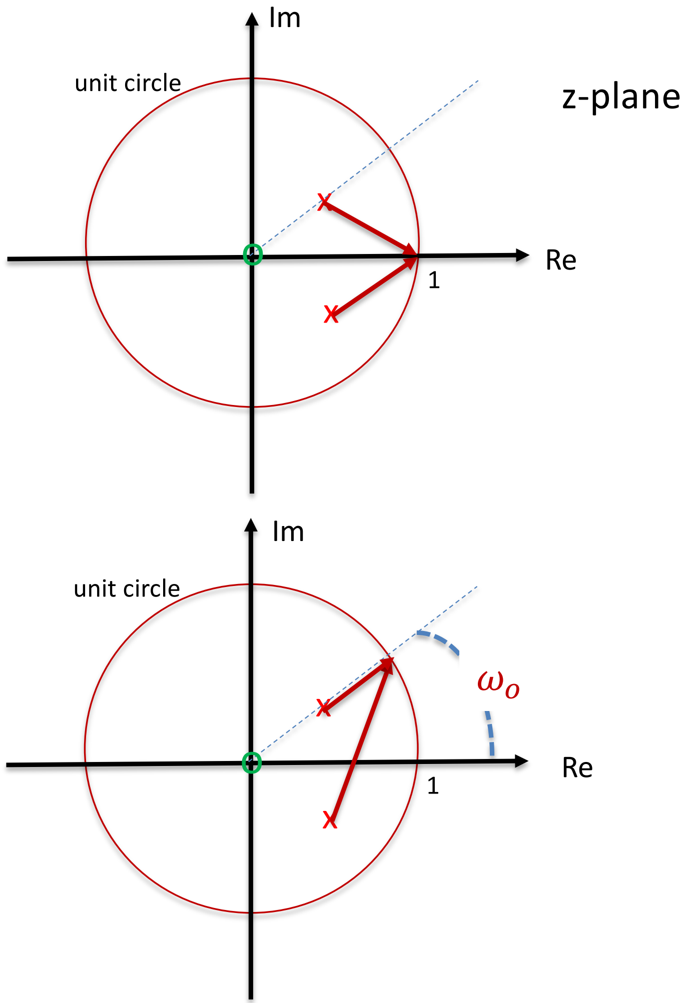


Two Poles Example

- $\left|H\left(e^{j\omega}\right)\right| = \dfrac{\prod legnth\ \overrightarrow{zeror}}{\prod legnth\ \overrightarrow{pole}}$
- $\angle H\left(e^{j\omega}\right) = \sum \angle\overrightarrow{zeror} - \sum \angle\overrightarrow{pole}$



This transfer function has two poles (complex conjugate poles)

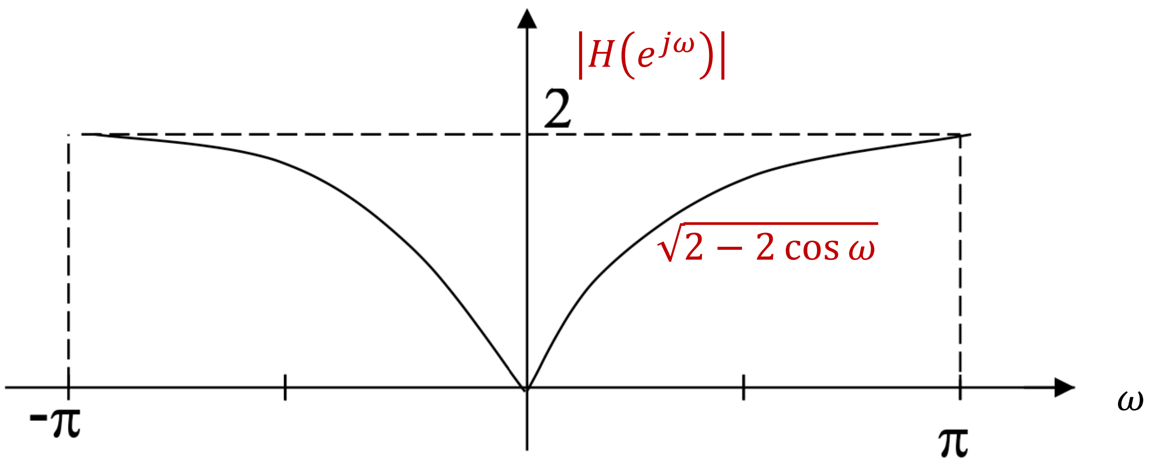


Example of $H(\mathrm{z})$ with only one zero

- Sketch the magnitude response of $H(\mathrm{z}) = 1 - \mathrm{z}^{-1}$

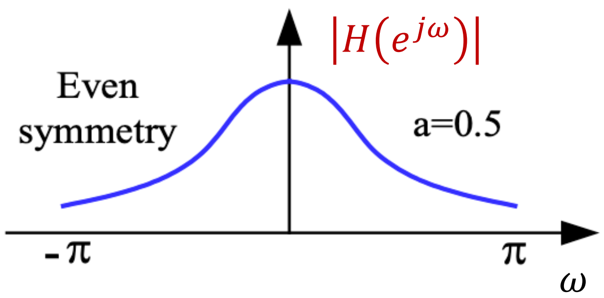
$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = 1 - e^{-j\omega} = (1 - \cos\omega) - j\sin\omega$$

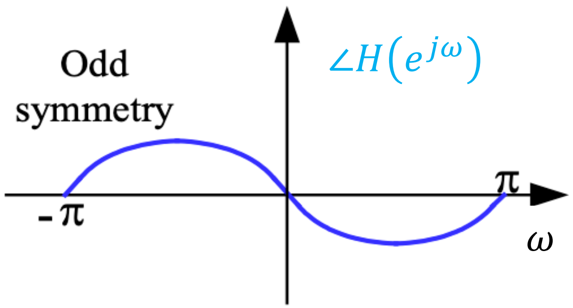
$$\left|H(e^{j\omega})\right| = \sqrt{(1-\cos\omega)^2 + (-\sin\omega)^2} = \sqrt{2 - 2\cos\omega}$$



Magnitude and Phase Responses

- We can show that the magnitude response $|H(e^{j\omega})|$ is an **even function** of frequency
- The phase response $\angle H(e^{j\omega})$ is an **odd function** of frequency

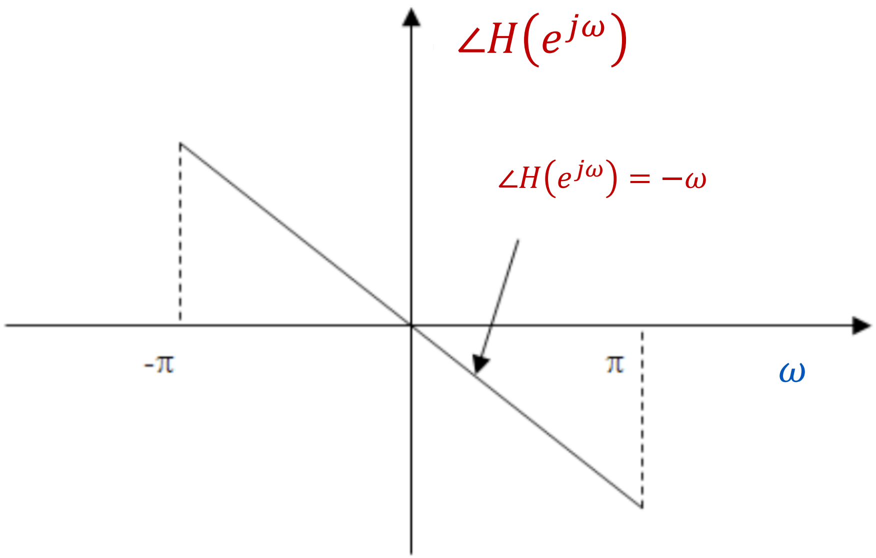




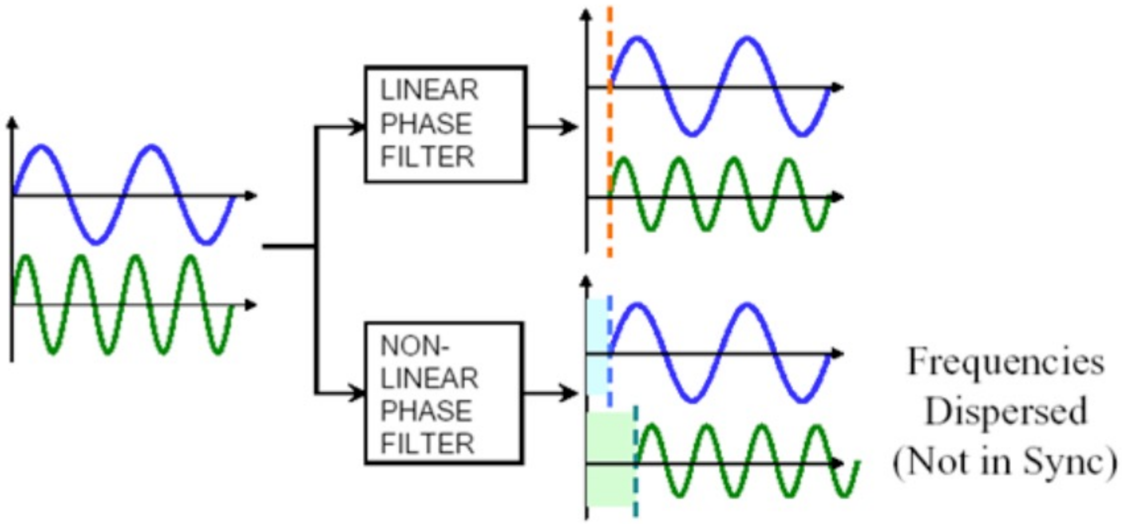
Group Delay

Learn how to calculate the group delay a Discrete-Time system

Phase Response of a Linear-Phase Filter

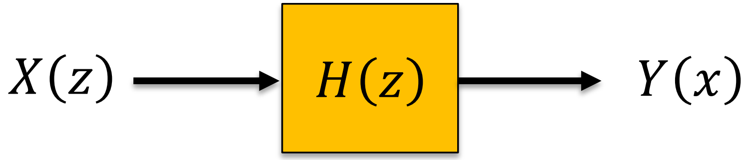


Phase Response of a Linear-Phase Filter



A diagram comparing the performance of a linear phase filter and a non-linear phase filter.

Group Delay



- Frequency response:

Magnitude Response — Phase Response

Phase shift is due to a delay through the system

$$H(z)\Big|_{z=e^{j\omega}} = H(e^{j\omega}) = |H(e^{j\omega})| \angle H(e^{j\omega})$$

- **Group delay** (Delay generally varies with frequency):

$$\tau(\omega) = grad\{H(e^{j\omega})\} = -\frac{d\{\angle H(e^{j\omega})\}}{d\omega}$$

Negative slope of phase response

- Note: Phase plots normally limited in range to $\pm\pi$
  - Ignore discontinuities when evaluating derivative

Group Delay Example 1

- Determine the group delay of a DT system with unit impulse response of $h[n] = \delta[n-5]$. This system is an ideal delay of 5 sample times.

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n-5]z^{-n} = z^{-5}$$

$$H\left(e^{j\omega}\right) = \left(e^{j\omega}\right)^{-5} = 1 \cdot e^{-j5\omega}$$

- Phase Response : $\angle H\left(e^{j\omega}\right) = -5\omega$
- **Group Delay :**

$$\tau(\omega) = -\frac{d\{\angle H\left(e^{j\omega}\right)\}}{d\omega} = -\frac{d\{-5\omega\}}{d\omega} = 5$$

- $\tau(\omega)$ = 5 samples

Group Delay Example 2

- Determine the group delay of a causal 5-point moving average with unit impulse response of $h[n] = \left[\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right]$ with the first sample at $n = 0$.

$$h[n] = \frac{1}{5}(\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]) \Rightarrow H(z) = \frac{1}{5}(z^{-0} + z^{-1} + z^{-2} + z^{-3} + z^{-4})$$

$$H(e^{j\omega}) = \frac{1}{5}(e^{j0} + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega}) = \frac{1}{5}e^{-j2\omega}(e^{j2\omega} + e^{j\omega} + e^{-j0} + e^{-j\omega} + e^{-j2\omega})$$

$$H(e^{j\omega}) = e^{-j2\omega}\boxed{\frac{1}{5}(1 + 2\cos 2\omega + 2\cos\omega)}$$

Real value function

- Phase Response : $\angle H(e^{j\omega}) = -2\omega$
- **Group Delay :**

$$\tau(\omega) = -\frac{d\{-2\omega\}}{d\omega} = 2$$ => $\tau(\omega)$ = 2 samples

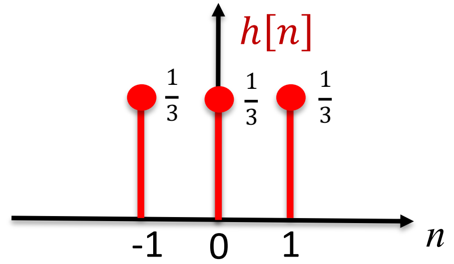
Frequency Response of FIR Systems

Frequency Response of FIR Systems

- Determine the magnitude and phase response of the 3-sample averager given by

$$h[n] = \begin{cases} \frac{1}{3} & -1 \le n \le 1 \\ 0 & otherwise \end{cases}$$

Non-casual System



$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-k} = \sum_{n=1}^{1} h[n]z^{-k} = \frac{1}{3}z^{-1} + \frac{1}{3}z^{0} + \frac{1}{3}z^{1} = \frac{1}{3}[z^{-1} + z + z^{1}]$$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{1}{3}[e^{-j\omega} + e^{j(0)} + e^{j\omega}] = \frac{1}{3}[1 + e^{-j\omega} + e^{j\omega}] = \frac{1}{3}[1 + 2\cos\omega]$$

- Precautions must be taken when determining the phase response of a filter having a **real-valued** transfer function, because negative real values produce an additional phase of π radians.

Linear Phase Response Characteristics

- **A linear-phase transfer function** can be expressed as

$$H(e^{j\omega}) = e^{-jk\omega}B(e^{j\omega}) = [B(e^{j\omega})\cos(-k\omega)] - j[B(e^{j\omega})\sin(k\omega)]$$

- Real-valued function $B(e^{j\omega})$ of that can take positive and negative values.
- Let phase angle is θ

$$\tan\theta = -\frac{B(e^{j\omega})\sin(k\omega)}{B(e^{j\omega})\cos(k\omega)} = -\tan(k\omega)$$

➔ Phase Response

$$\theta = -k\omega$$

$$\angle H(e^{j\omega}) = -k\omega$$

The phase function includes linear phase term and accommodates for the sign changes in $B(e^{j\omega})$. Since -1 can be expressed as phase jumps of ±π, This will occur at frequencies where $B(e^{j\omega})$ changes sign.

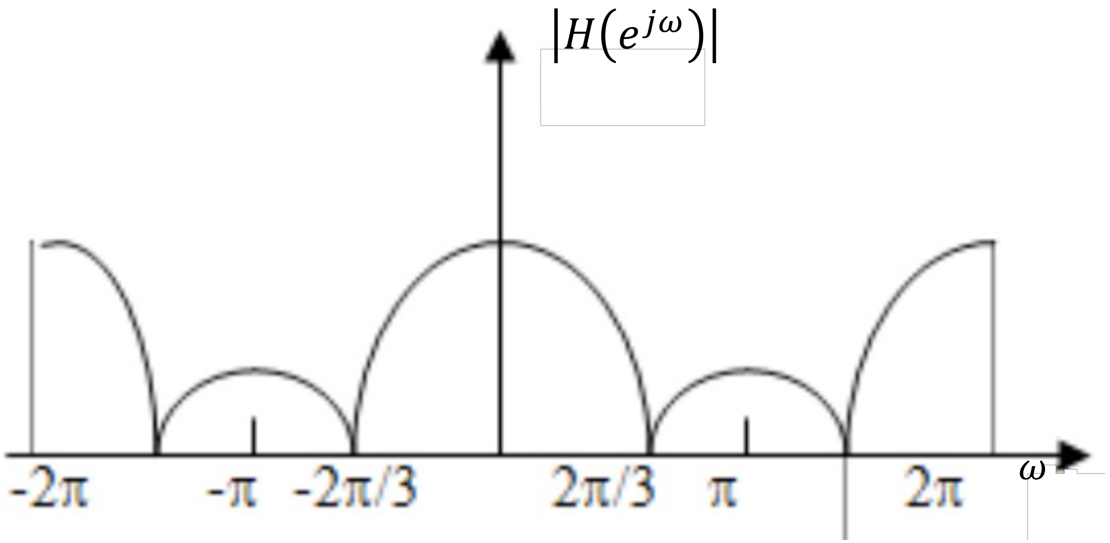
If $B(e^{j\omega}) > 0$, the $\angle H(e^{j\omega}) = -k\omega$ If $B(e^{j\omega}) < 0$, then $\angle H(e^{j\omega}) = -k\omega \pm \pi$

Magnitude Response of the 3-Sample Averager

$$H(e^{j\omega}) = \frac{1}{3}[1 + 2\cos\omega]$$

Magnitude Response $|H(e^{j\omega})|$:

$$|H(e^{j\omega})| = \left|\frac{1}{3}[1 + 2\cos\omega]\right|$$



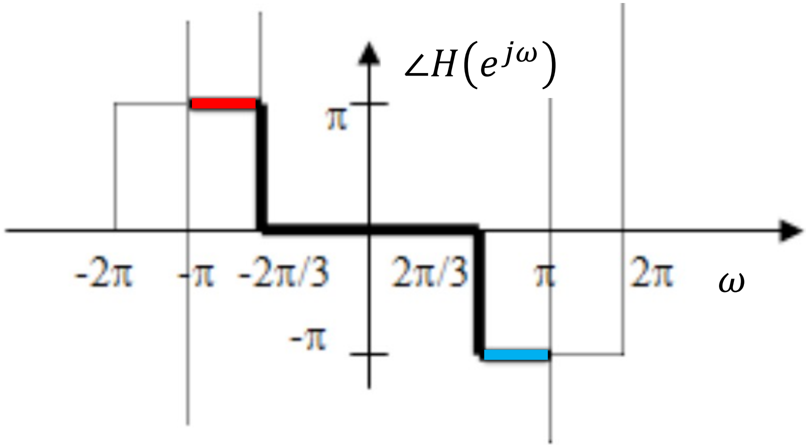
Even Function

Zero Phase Response of the 3-Sample Averager

$$H(e^{j\omega}) = e^{j(0)\omega}\frac{1}{3}[1 + 2\cos\omega] = e^{j(0)\omega}B(e^{j\omega})$$

Zero Phase Response $\angle H(e^{j\omega})$:

$$\angle H(e^{j\omega}) = \begin{cases} 0 & B(e^{j\omega}) > 0 & -\frac{2\pi}{3} < \omega < \frac{2\pi}{3} \\ 0 \pm \pi & B(e^{j\omega}) < 0 & -\pi \le \omega \le -\frac{2\pi}{3} \ and \ \frac{2\pi}{3} < \omega < \pi \end{cases}$$

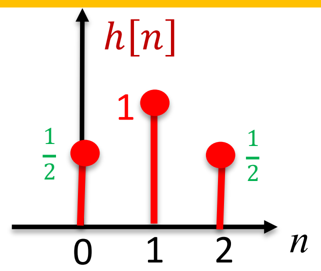


Odd Function

Casual 3-Point Weighted Averager Example

- Find the magnitude and phase responses of the 3-point weighted average with the impulse response as

Casual System



$$h[0] = \frac{1}{2}, h[1] = 1, h[2] = \frac{1}{2}$$

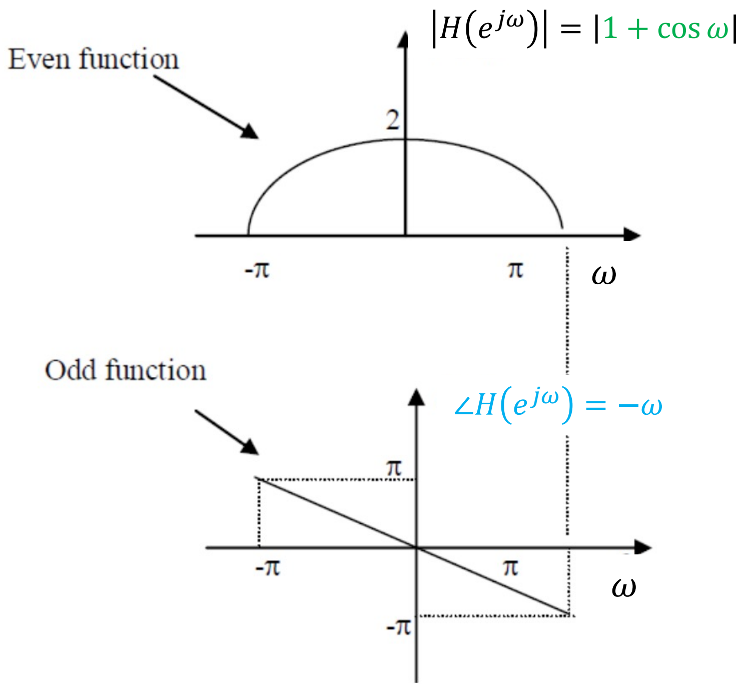
$$H(z) = \frac{1}{2}z^{0} + z^{-1} + \frac{1}{2}z^{-2}$$

$$H(e^{j\omega}) = \frac{1}{2} + e^{-j\omega} + \frac{1}{2}e^{-2j\omega} = e^{-j\omega}(\frac{1}{2}e^{j\omega} + 1 + \frac{1}{2}e^{-j\omega})$$

$$H(e^{j\omega}) = e^{-j\omega}\underbrace{[1 + \cos\omega]}_{B(e^{j\omega})}$$

$$\angle H(e^{j\omega}) = -\omega$$

The amplitude function is never negative (therefore there is no phase jumps of $\pm\pi$)

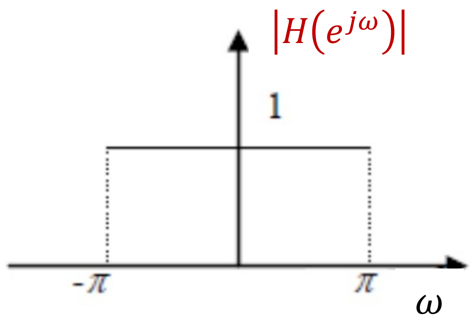


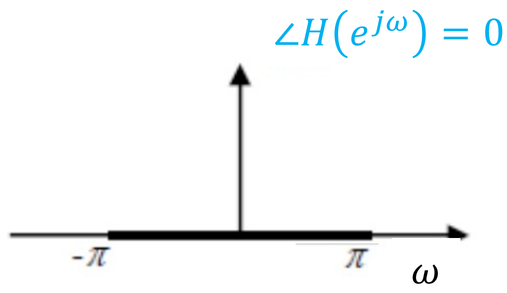
Linear Phase

Magnitude and Phase Responses of Unit Sample

Case 1

$$h[n] = \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$$



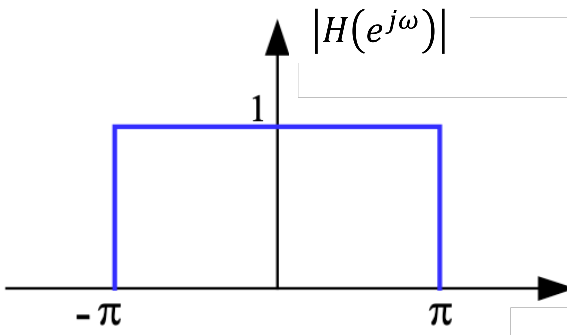


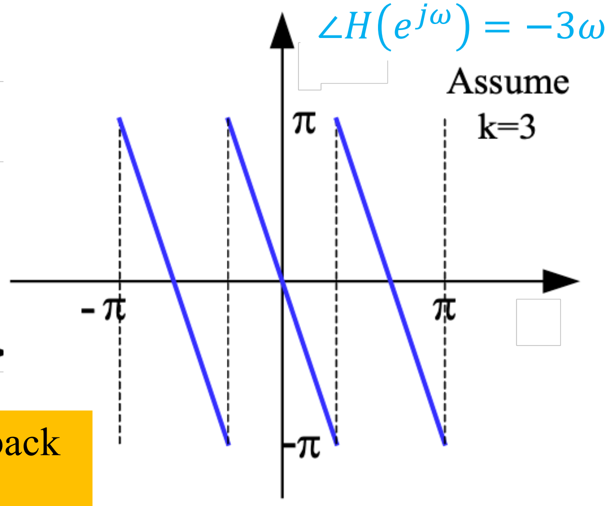
Case 2

$$h[n] = \delta[n-k]$$

$$H(z) = z^{-k}$$

$$H(e^{j\omega}) = e^{-jk\omega}$$





Note: When phase exceeds ±π range a jump of ±2π is needed to bring the phase back into ±π range.

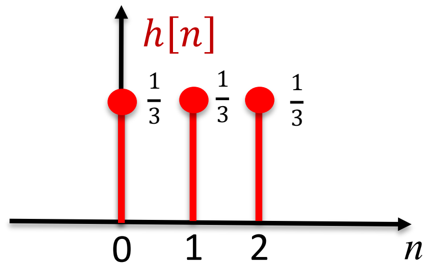
Phase Jumps

- From the previous examples, we note that there are two occasions for which the phase function experiences discontinuities or jumps.
  1. A jump of ±2π occurs to maintain the phase function within the principal value range of [-π and π]
  2. A jump of ± π occurs when $B\left(e^{j\omega}\right)$ undergoes a change of sign
- The sign of the phase jump is chosen such that the resulting phase function is odd and, after the jump, lies in the range [-π and π].

Causal 3-Sample Averager

- Determine the magnitude and phase response of the 3-sample averager given by

$$h[n] = \begin{cases} \frac{1}{3} & 0 \le n \le 2 \\ 0 & otherwise \end{cases}$$

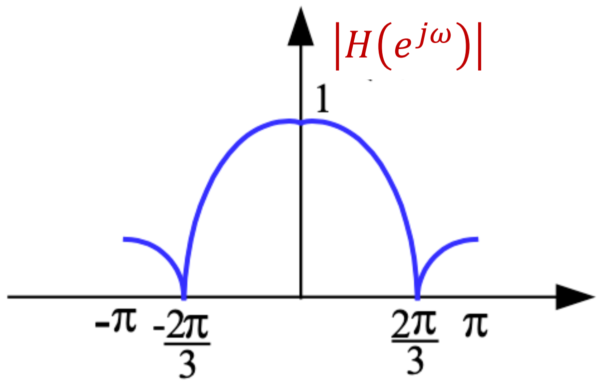


$$H(z) = \frac{1}{3}z^0 + \frac{1}{3}z^{-1} + \frac{1}{3}z^{-2} = \frac{1}{3}[1 + z^{-1} + z^{-2}]$$

$$H(e^{j\omega}) = \frac{1}{3}[1 + e^{-j\omega} + e^{-j2\omega}] = e^{-j\omega}\frac{1}{3}[1 + e^{j\omega} + e^{-j\omega}] = e^{-j\omega}\underbrace{\frac{1}{3}[1 + 2\cos\omega]}_{B(e^{j\omega})}$$

Magnitude Response

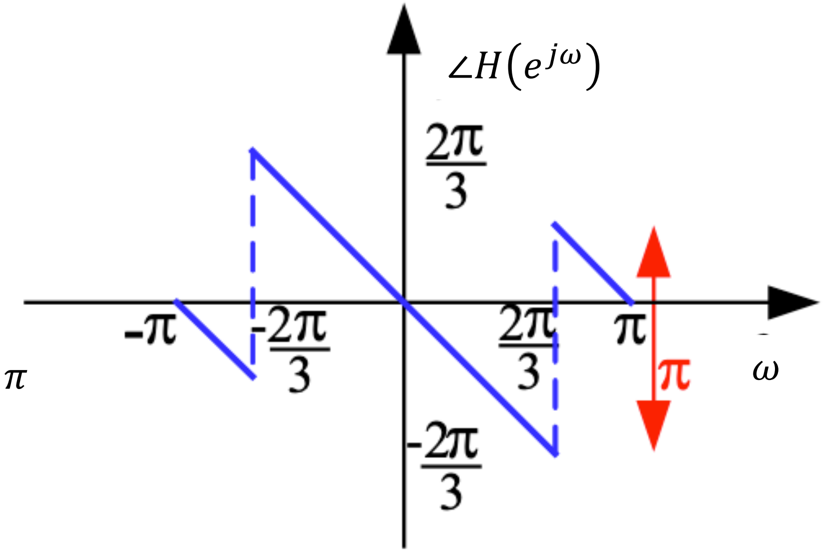
$$|H(e^{j\omega})| = \left|\frac{1}{3}[1 + 2\cos\omega]\right|$$



Linear Phase Response of the Causal 3-Sample Averager

$$H(e^{j\omega}) = e^{-j\omega}\underbrace{\frac{1}{3}[1 + 2\cos\omega]}_{B(e^{j\omega})}$$

$$\angle H(e^{j\omega}) = \begin{cases} -\omega & B(e^{j\omega}) > 0 \quad -\frac{2\pi}{3} < \omega < \frac{2\pi}{3} \\ -\omega \pm \pi & B(e^{j\omega}) < 0 \quad -\pi \le \omega \le -\frac{2\pi}{3} \; and \; \frac{2\pi}{3} < \omega < \pi \end{cases}$$



Note: Phase is undefined at points $|H(e^{j\omega})| = 0$ or $B(e^{j\omega})$=0.

Four Types of Causal Linear Phase FIR Systems

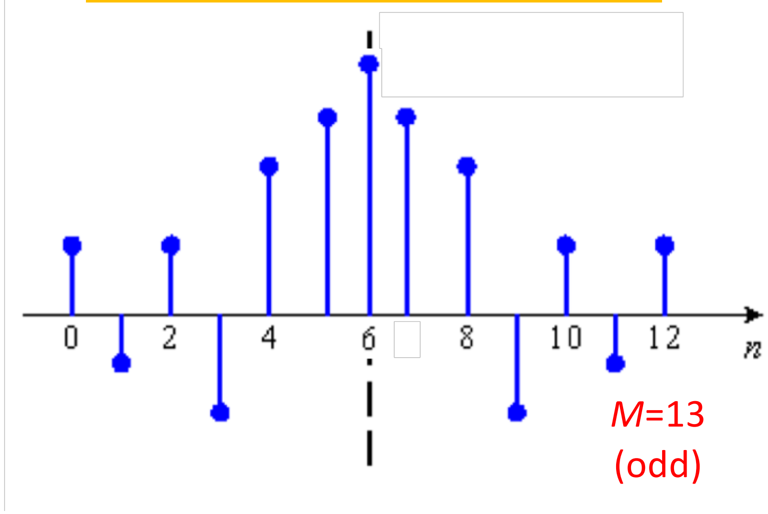
- For casual FIR systems, if their impulse response $h[n]$ satisfied the symmetrical property, then the systems will have linear phase responses.
- The symmetrical impulse response property is defined as

$$h[n] = \pm h[M-1-n], \qquad n = 0,1,\ldots,M-1$$

- **There 4 types of linear phase FIR systems**:
  - **Type I : Odd Positive Symmetric** – M is odd and $h[n] = h[M-1-n]$
  - **Type II : Even Positive Symmetric** – M is even and $h[n] = h[M-1-n]$
  - **Type III : Odd Negative Symmetric** – M is odd and $h[n] = -h[M-1-n]$
  - **Type IV : Even Negative Symmetric** – M is even and $h[n] = -h[M-1-n]$

Positive Symmetry Impulse Responses

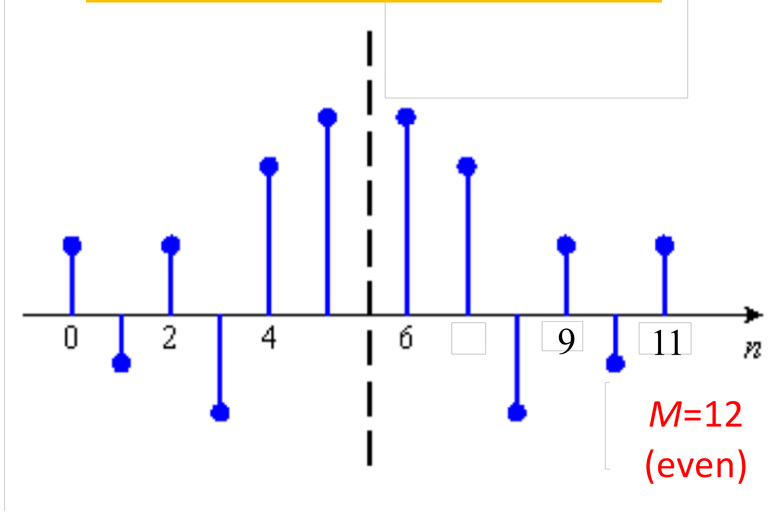
Odd Positive Symmetry (Type I)



$$h[n] = h[M-1-n]$$

$$H(e^{j\omega}) = e^{-j(\frac{\omega(M-1)}{2})}\left(h\left[\frac{M-1}{2}\right] + 2\sum_{k=1}^{(M-3)/2} h\left[\frac{M-1}{2}-k\right]\cos k\omega\right)$$

Even Positive Symmetry (Type II)

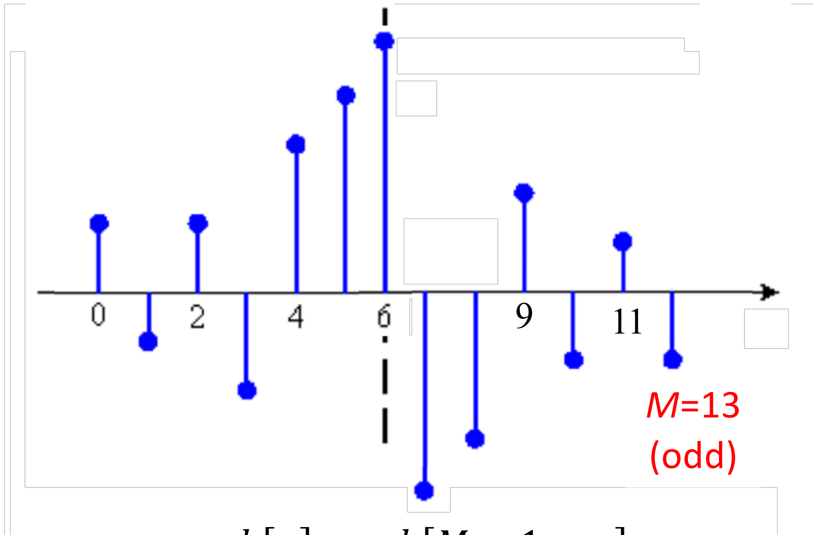


$$h[n] = h[M-1-n]$$

$$H(e^{j\omega}) = e^{-j(\frac{\omega(M-1)}{2})}\left(2\sum_{k=1}^{(M-3)/2} h\left[\frac{M-1}{2}-k\right]\cos((k-\frac{1}{2})\omega)\right)$$

Negative Symmetry Impulse Responses

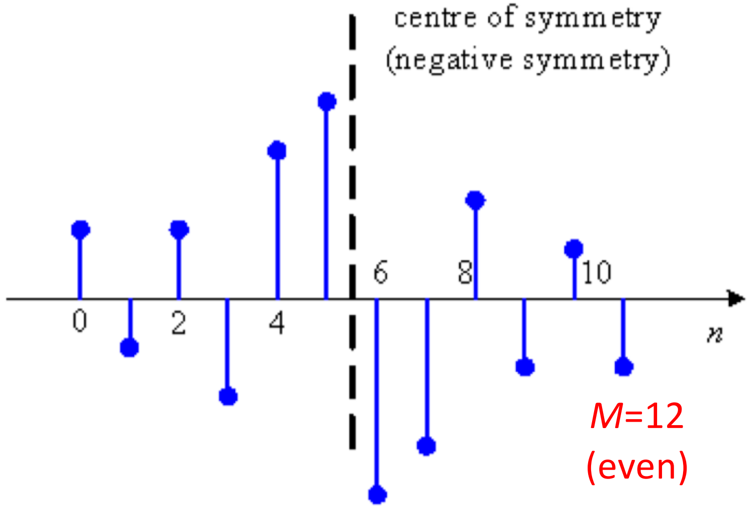
Odd Negative Symmetry
(Type III)



$$h[n] = -h[M-1-n]$$

$$H(e^{j\omega}) = e^{-j\left(\frac{\omega(M-1)}{2}-\frac{\pi}{2}\right)}\left(2\sum_{k=1}^{(M-1)/2} h\left[\frac{M-1}{2}-k\right]\sin k\omega\right)$$

Even Negative Symmetry
(Type IV)

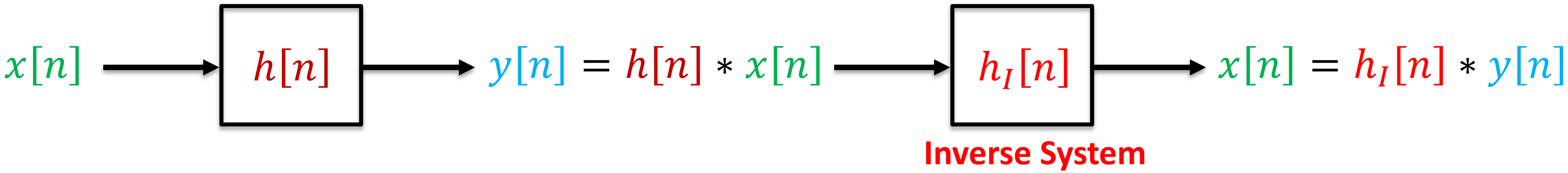


$$h[n] = -h[M-n-1]$$

$$H(e^{j\omega}) = e^{-j\left(\frac{\omega(M-1)}{2}-\frac{\pi}{2}\right)}\left(2\sum_{k=1}^{(M-1)/2} h\left[\frac{M}{2}-k\right]\sin\left(\left(k-\frac{1}{2}\right)\omega\right)\right)$$

Inverse Systems for LIT Systems

Inverse Systems for LIT Systems



- In terms of system functions in z-transforms:

  $Y(z) = H(z)X(z)$ and $X(z) = H_I(z)Y(z)$ => $H(z)\,H_I(z) = 1$ z-plane

$$\Rightarrow H_I(z) = \frac{1}{H(z)}$$

- For a stable inverse system, ROC of $H_I(z)$ must include the unit circle ($|z| = 1$)
  - For causal system, the poles of the $H_I(z)$ must inside the unit circle
  - The poles of $H_I(z)$ are the zeros of $H(z)$
- **For a stable system with inverse system exit:**
  - **Both of the zeros and poles have to be insider the unit circle.**

Inverse Systems for LIT Systems

- Rational Transfer functions of LTI systems can be expressed as

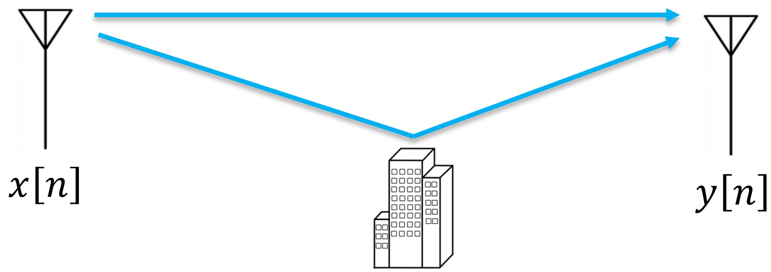
$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_{M-1} z^{M-1} + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{N-1} + a_N z^{-N}} = K \frac{\prod_{k=1}^{M} \left(1 - \beta_k z^{-k}\right)}{\prod_{k=1}^{N} \left(1 - \alpha_k z^{-k}\right)}$$

  - β_k are zeros and the α_k are poles of the system $H(z)$
- The inverse system $H_I(z) = \frac{1}{K} \frac{\prod_{k=1}^{N} \left(1 - \alpha_k z^{-k}\right)}{\prod_{k=1}^{M} \left(1 - \beta_k z^{-k}\right)}$
  - β_k become the poles and the α_k become zeros of the **inverse system**
- Stable/Causal $H(z)$ => $|\alpha_k| < 1$
- Stable/Causal $H_I(z)$ => $|\beta_k| < 1$

For a stable/causal system with an inverse system, both zeros and poles must be inside the unit circle.

Inverse System Example 1

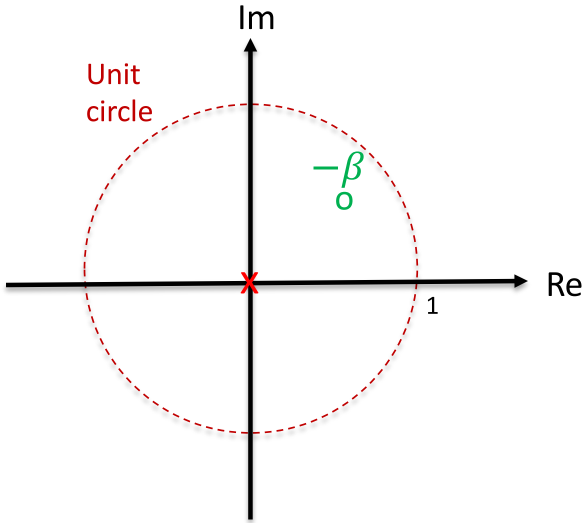
- Multipath Communication:
  - Difference Equation Model
  - $y[n] = x[n] + \beta x[n-1]$



- **Does a stable/causal inverse system exist?**

$$H(z) = 1 + \beta z^{-1}$$

with Pole at $z = 0$ and Zero at $z = -\beta$



- If $|\beta| < 1$ (Zero of $H(z)$ is inside the unit circle)
  - The inverse system exit

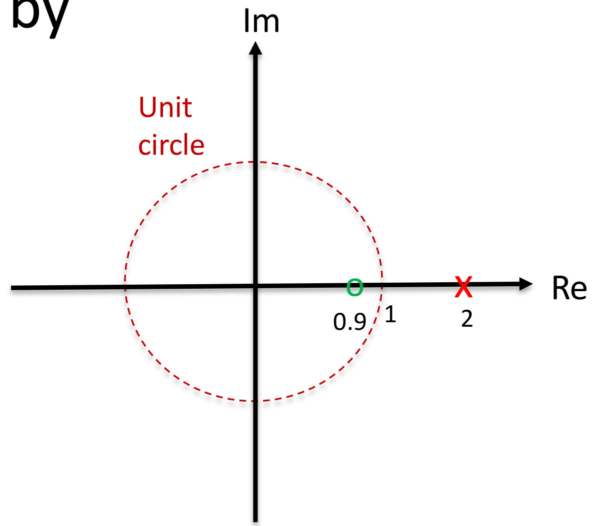
$$H_I(z) = \frac{1}{1+\beta z^{-1}} \Rightarrow y[n] = x[n] - \beta y[n]$$

Inverse System Example 2

- **Does a stable/causal inverse system exist?** $H(z) = \dfrac{z^{-1} - 0.5}{1 - 0.9z^{-1}}$

- The transfer function of the inverse system is given by

$$H_I(z) = \frac{1 - 0.9z^{-1}}{z^{-1} - 0.5} = -2\frac{1 - 0.9z^{-1}}{1 - 2z^{-1}}$$



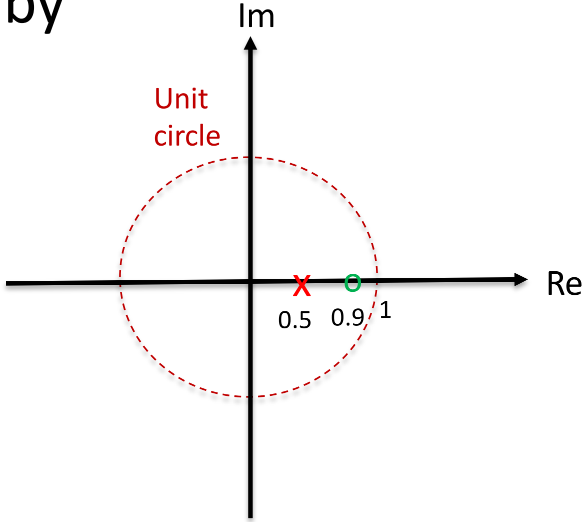
- For ROC $|z| < 2$, it is stable but non-causal
- For ROC $|z| > 2$, it is causal but unstable
- A stable/causal inverse system does not exist.

Inverse System Example 3

- **Does a stable/causal inverse system exist?** $H(z) = \frac{z^{-1} - 2}{1 - 0.9z^{-1}}$

- The transfer function of the inverse system is given by

$$H_I(z) = \frac{1 - 0.9z^{-1}}{z^{-1} - 2} = -\frac{1}{2}\frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}}$$



- For ROC $|z| < 0.5$, it is unstable and non-causal
- For ROC $|z| > 0.5$, it is causal and stable
- **A stable/causal inverse system exist.**

Minimum Phase Systems

- A stable/causal system has a stable/causal inverse system **if and only if** all poles and zeros are inside unit circle.
  - **This is called Minimum Phase System**.
- Can show that phase lag of a system with poles/zero inside the unit circle is less than that of any other system with identical magnitude response
- Any rational system function

$$H(z) = H_{min}(z)\, H_{ap}(z)$$

Minimum Phase — All Pass

All-Pass Systems

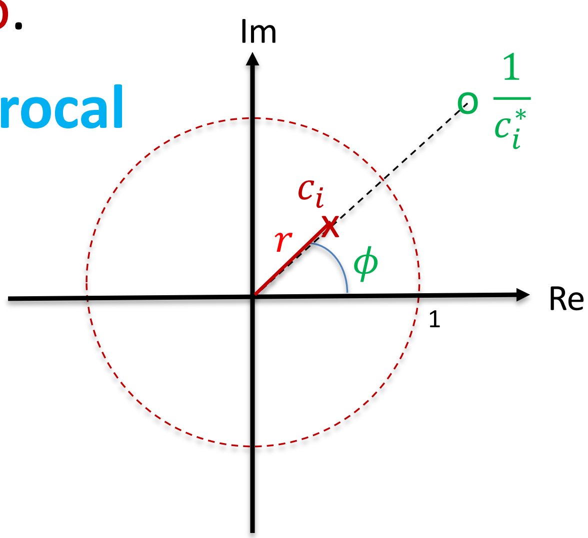
All-Pass Systems

- An all-pass filter is one whose magnitude response $|H_{ap}(e^{j\omega})|$ is **constant for all frequencies**:
  - All pass : $\left|H_{ap}(e^{j\omega})\right| = \mathbf{1} \;\; or \;\; Constant$
  - However, the phase response is not identically zero.
- Poles and Zeros of all-pass systems in **conjugate reciprocal** pairs

$$H_{ap}(z) = \prod_{i=1}^{P} \frac{z^{-1} - c_i^*}{1 - c_i z^{-1}}$$

$$Poles : c_i = re^{j\phi}$$

$$Zeros: \frac{1}{c_i^*} = \frac{1}{r} e^{j\phi}$$



Magnitude Response of All-Pass Systems

$$H_{ap}(z) = \prod_{i=1}^{P} \frac{z^{-1} - c_i^*}{1 - c_i z^{-1}}$$

$$Poles: c_i = r e^{j\phi}$$

$$Zeros: c_i^* = \frac{1}{r} e^{j\phi}$$

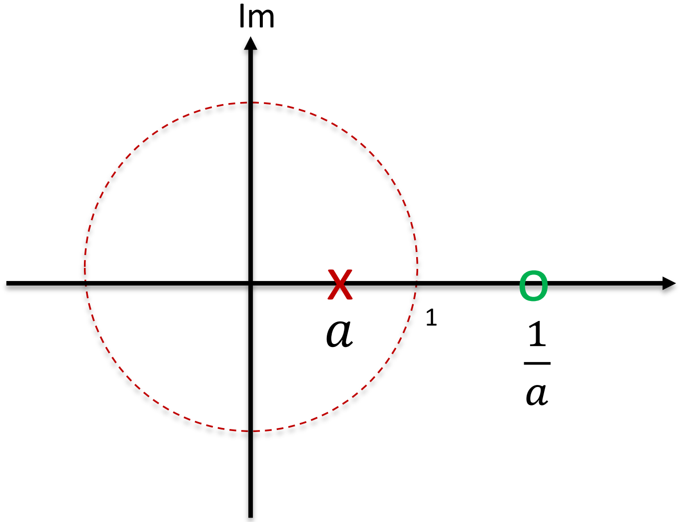
- To show : $\left|H_{ap}\left(e^{j\omega}\right)\right| = 1$, consider $P = 1$

$$\left|H_{ap}\left(e^{j\omega}\right)\right| = \left|\frac{e^{-j\omega} - c^*}{1 - ce^{-j\omega}}\right| = \left|\frac{e^{-j\omega}\left(1 - c^* e^{j\omega}\right)}{1 - ce^{-j\omega}}\right| = \frac{\left|e^{-j\omega}\right|\left|1 - c^* e^{j\omega}\right|}{\left|1 - ce^{-j\omega}\right|}$$

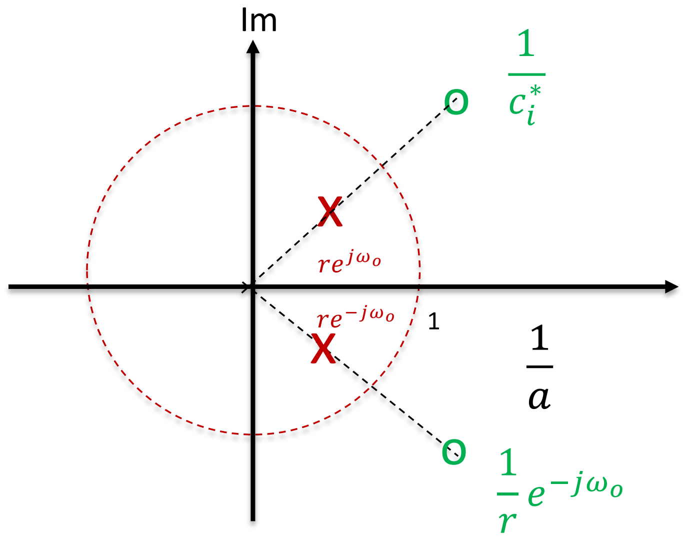
$$= \frac{\left|1 - c^* e^{j\omega}\right|}{\left|1 - ce^{-j\omega}\right|} = \frac{\left|\left(1 - ce^{-j\omega}\right)^*\right|}{\left|1 - ce^{-j\omega}\right|} = \frac{|b^*|}{|b|} = 1$$

Pole-Zero Patterns of All-Pass Systems

- If $|z_0|$ is the modulus of a pole of $H(z)$, then $1/|z_0|$ is the modulus of a zero of $H(z)$ {i.e. the modulus of poles and zeros are reciprocals of one another}.



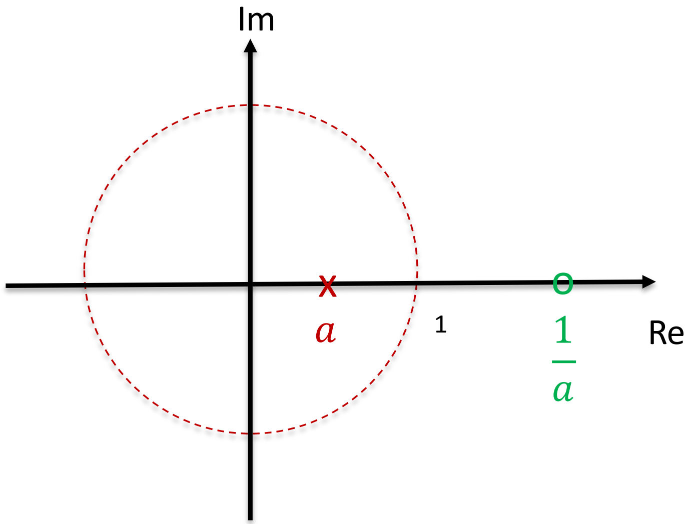
A Single Pole All-Pass System



A Two-Pole All-Pass System

Example of All-Pass System

- $H(z) = \frac{z^{-1} - a}{1 - az^{-1}}$
- **Magnitude Response**



$$|H(e^{j\omega})| = H(z)\Big|_{z=e^{j\omega}}$$

$$|H(e^{j\omega})| = \left|\frac{e^{-j\omega} - a}{1 - ae^{-j\omega}}\right| = \left|\frac{e^{-j\omega}(1 - ae^{j\omega})}{1 - ae^{-j\omega}}\right|$$

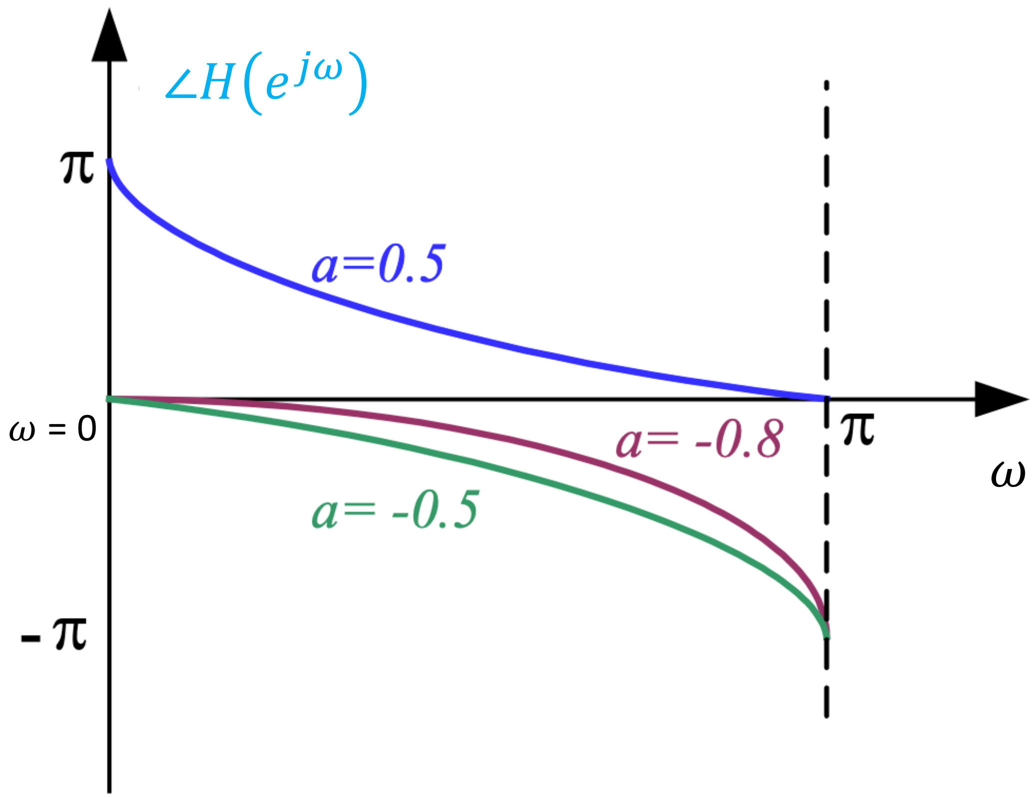
$|e^{-j\omega}| = 1$

$$= \left|\frac{1 - ae^{j\omega}}{1 - ae^{-j\omega}}\right| = 1$$

The Magnitude Response is constant

Phase Responses of All-Pass Systems

- When $0 < a < 1$, the zero lies on the positive real axis. The phase over $0 \le \theta \le \pi$ is positive, at $\omega = 0$ it is equal to π and decreases until $\omega = \pi$, where it is zero.
- When $-1 < a < 0$, the zero lies on the negative real axis. The phase over $0 \le \omega \le \pi$ is negative, starting at 0 for $\omega = 0$ and decreases to $-\pi$ at $\omega = \pi$.



The Transfer Function of All-Pass Systems

- A more interesting all-pass filter is one that is described by

$$H_{ap}(z) = \frac{a_L + a_{L-1}z^{-1} + \cdots + a_1 z^{-L+1} + a_0 z^{-L}}{1 + a_1 z^{-1} + \cdots + a_{L-1}z^{-L+1} + a_L z^{-L}}$$

where $a_0 = 1$

- If we define the polynomial $A(z)$ as

$$A(z) = \sum_{k=0}^{L} a_k z^{-k} \qquad a_0 = 1$$

$$H_{ap}(z) = z^{-L}\frac{A(z^{-1})}{A(z)} \Rightarrow \left|H\left(e^{j\omega}\right)\right|^2 = H(z)\cdot H(z^{-1})\Big|_{z=e^{j\omega}} = 1$$

- i.e. all-pass filter

All-Pass System Example

Show that the following transfer function $H(z)$ can be obtained using a parallel connection of two all-pass filters.

$$H(z) = \frac{10 - 6z^{-1}}{3 + z^{-1}}$$

$$H(z) = \frac{9 + 3z^{-1} + 1 + 3z^{-1}}{3 + z^{-1}} = 3 + \frac{1 + 3z^{-1}}{3 + z^{-1}}$$

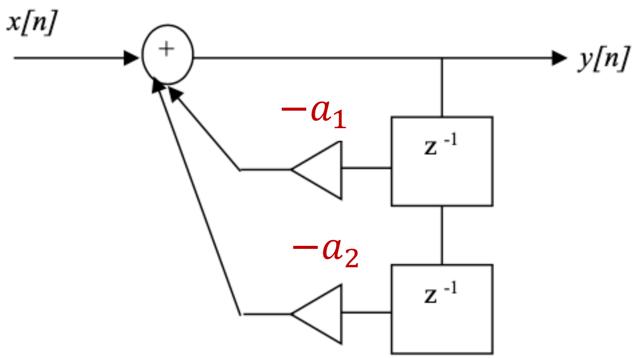
$$H(z) = \underbrace{3}_{\text{All-pass Filter}} + \left(\frac{1}{3}\right)\underbrace{\frac{1 + 3z^{-1}}{1 + \frac{1}{3}z^{-1}}}_{\text{All-pass Filter}}$$

A Second Order Resonant System
(Complex Poles)

A Second Order Resonant System

- The transfer function of a 2nd order resonant system can be expressed as

$$H(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{z^2}{z^2 + a_1 z + a_2}$$

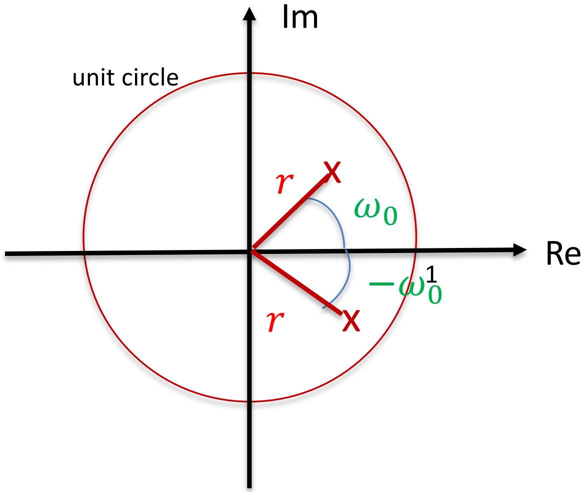


- It has a pair of complex conjugate poles

$$p_1 = re^{j\omega_0} = r\cos\omega_0 + j\,r\sin\omega_0$$

$$p_2 = re^{-j\omega_0} = r\cos\omega_0 - j\,r\sin\omega_0$$

r



- All pole system has poles only (without counting the zeros at the origin)

$$H(z) = \frac{z^2}{z^2+a_1z+a_2} = \frac{z^2}{(z-p_1)(z-p_2)} = \frac{z^2}{\left(z-re^{j\omega_0}\right)\left(z-re^{-j\omega_0}\right)}$$

$$H(z) = \frac{z^2}{z^2-r\left(e^{j\omega_0}+e^{-j\omega_0}\right)z+r^2} = \frac{z^2}{z^2-2r\cos\omega_0 z+r^2}$$

- Comparing with the two equations, we have

$a_1 = -2r\cos\omega_0$ and $a_2 = r^2$

Then, $\cos\omega_0 = -\frac{a_1}{2\sqrt{a_2}}$ $\omega_0 = \frac{2\pi f_0}{F_s}$

- ω_0 is resonant frequency

Magnitude Response of 2nd Order Resonant System

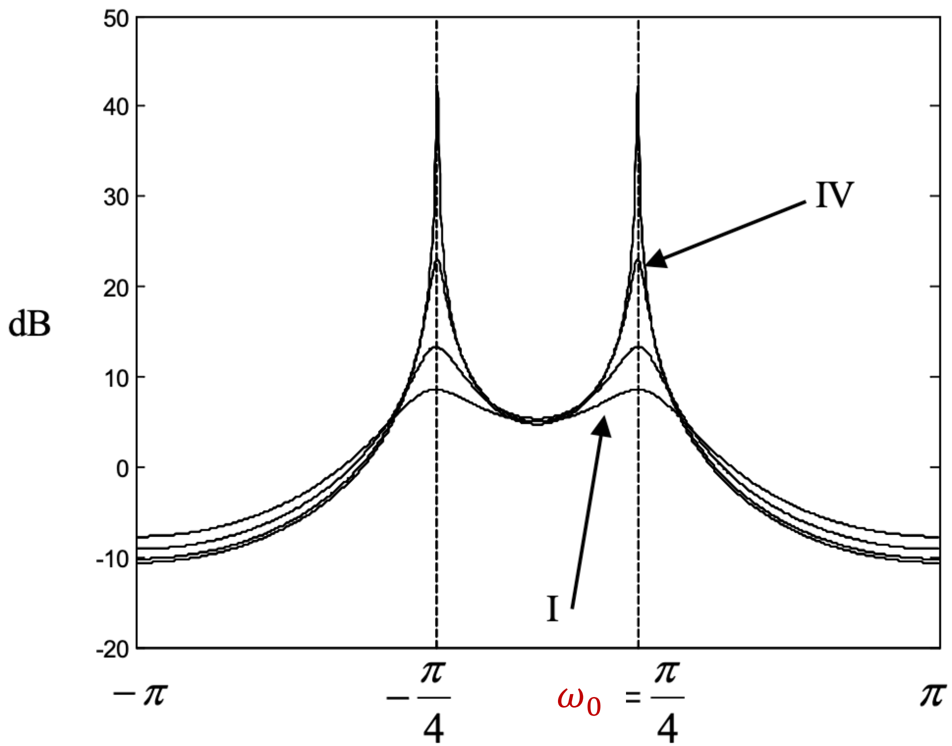
$$H(z) = \frac{z^2}{z^2 + a_1 z + a_2}$$

$$a_1 = -2r\cos\omega_0$$

$$a_2 = r^2$$

$$\omega_0 = \cos^{-1}\left[-\frac{a_1}{2\sqrt{a_2}}\right]$$

	a_1	$a_2 = r^2$
I	-0.94	0.5
II	-1.16	0.7
III	-1.34	0.9
IV	-1.41	0.99

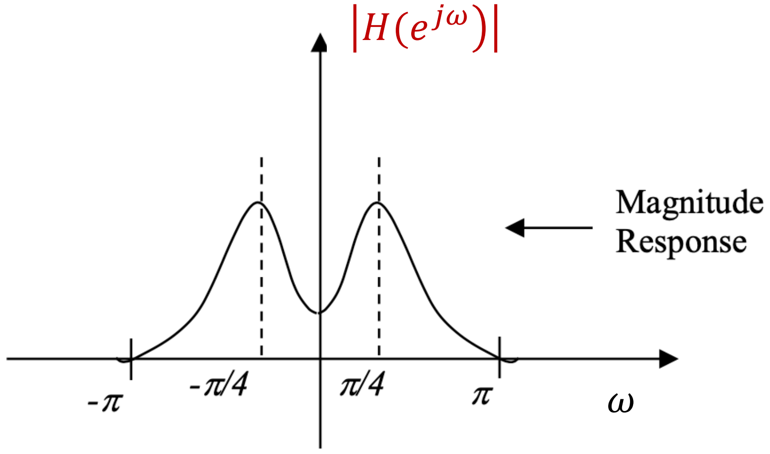


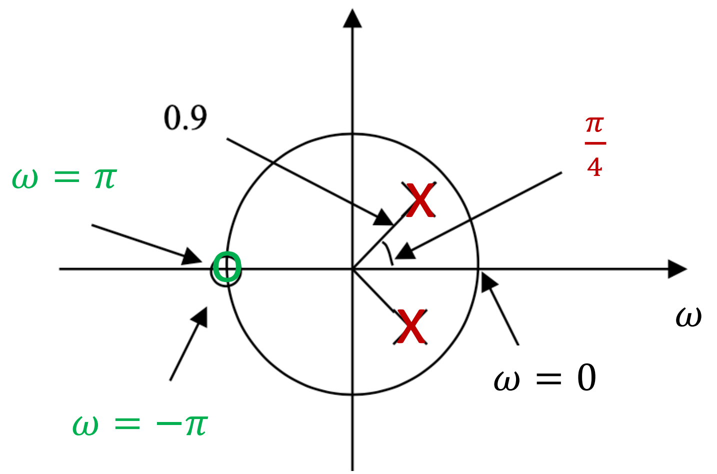
Example 1

Sketch the magnitude response for the system having the transfer function

$$H(z) = \frac{1 + z^{-1}}{\left(1 - 0.9e^{j\frac{\pi}{4}}z^{-1}\right)\left(1 - 0.9e^{-j\frac{\pi}{4}}z^{-1}\right)}$$

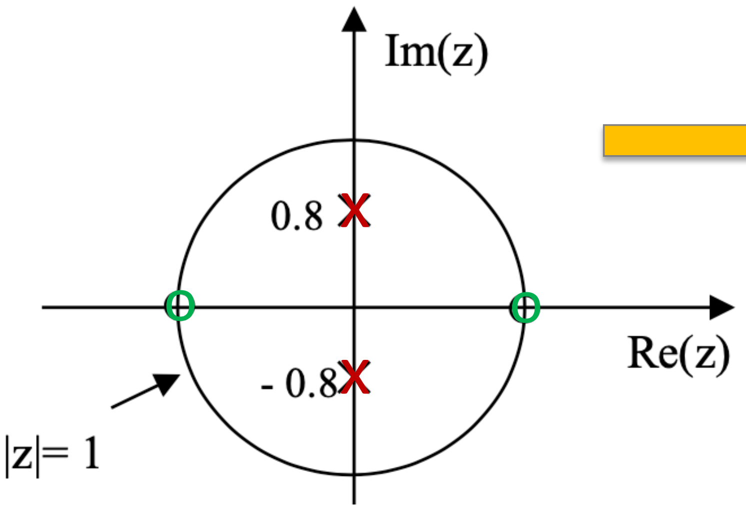
- The system has a zero at $z = -1$ and complex conjugate poles at $z = 0.9e^{\pm j\frac{\pi}{4}}$
- Thus, the magnitude response will be zero at $\omega_0 = \pi$ and large at $\omega_0 = \pm j\frac{\pi}{4}$ because the poles are close to the unit circle.

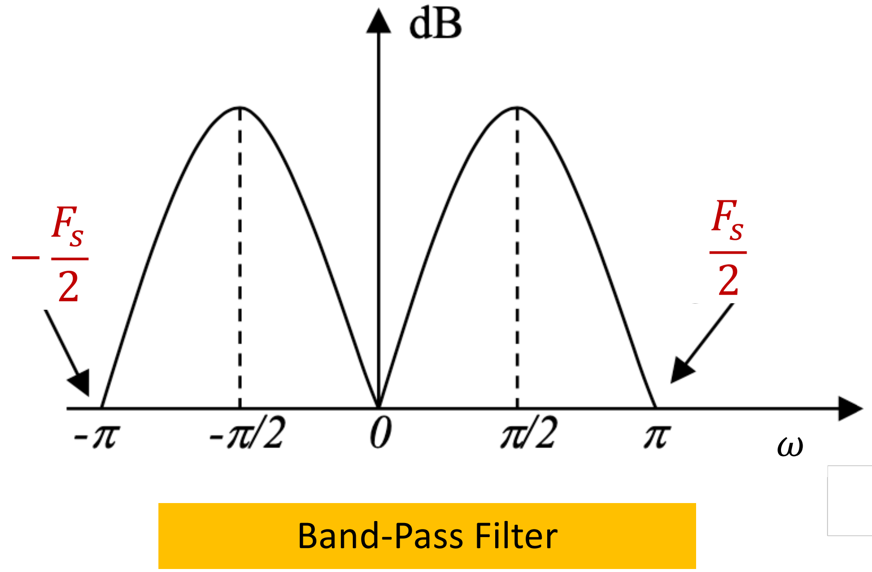




Example 2

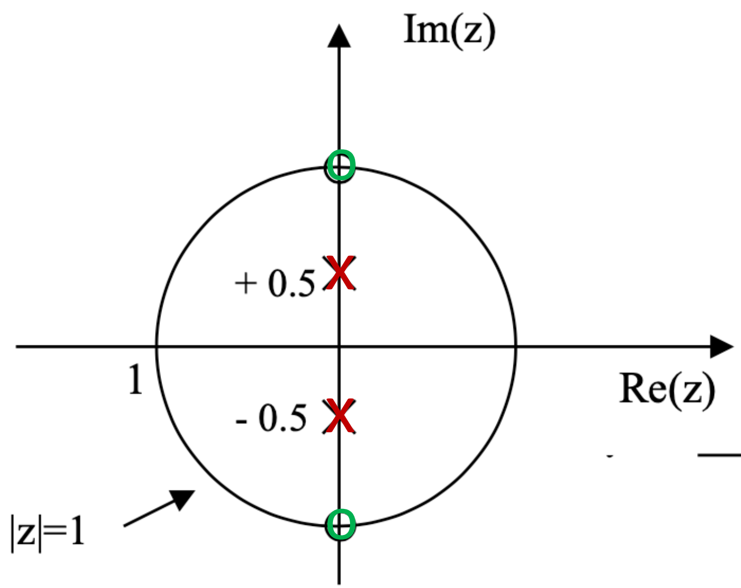
Sketch the approximate magnitude response from the pole-zero map given below:

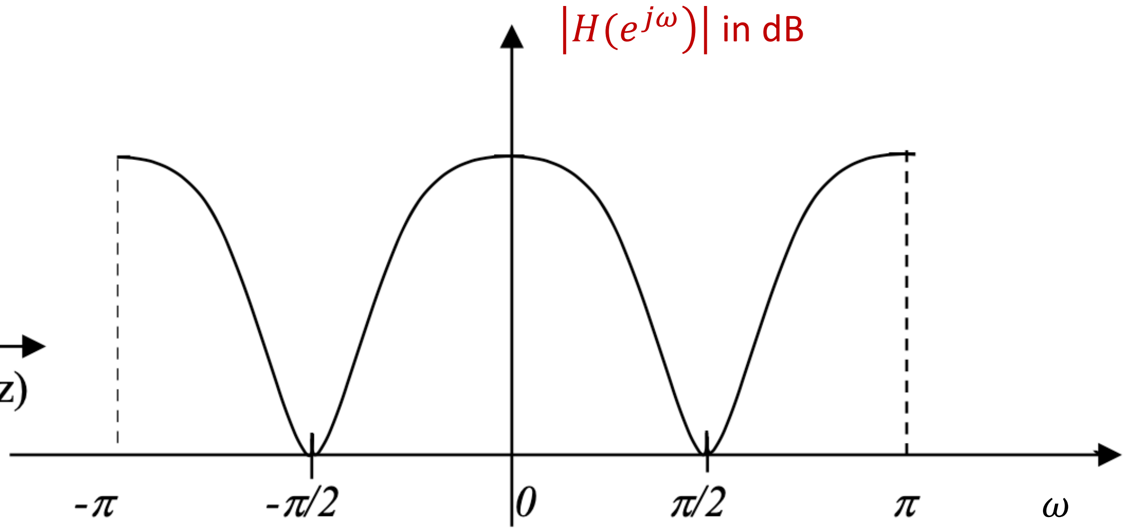




Example 3

Sketch the approximate magnitude response from the pole-zero map given below:



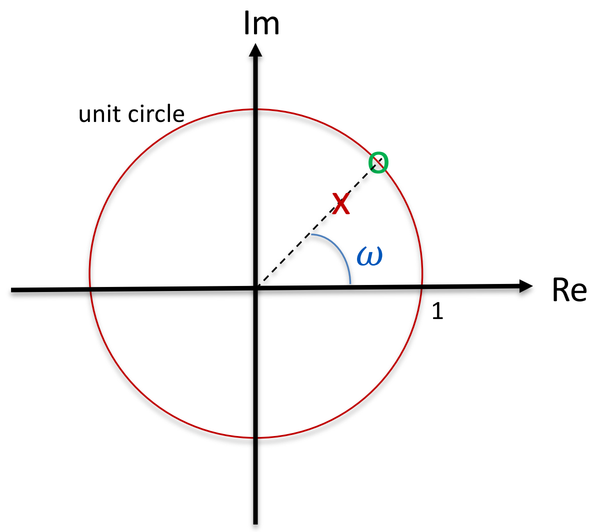


Band-Stop Filter (Notch Filter)

Notch Filter Design Using Pole-Zero Placement

Notch Filters

- When a zero is placed at a given point on the z-plane, the **frequency response will be zero at the corresponding point**.
- A pole on the other hand produces **a peak at the corresponding frequency point.**
- Poles that are close to the unit circle give rise large peaks, whereas zeros close to or on the unit circle produces troughs or minima.
- Thus, by strategically placing poles and zeros on the z-plane, we can obtain sample lowpass or other frequency selective filters such as **notch filters**.



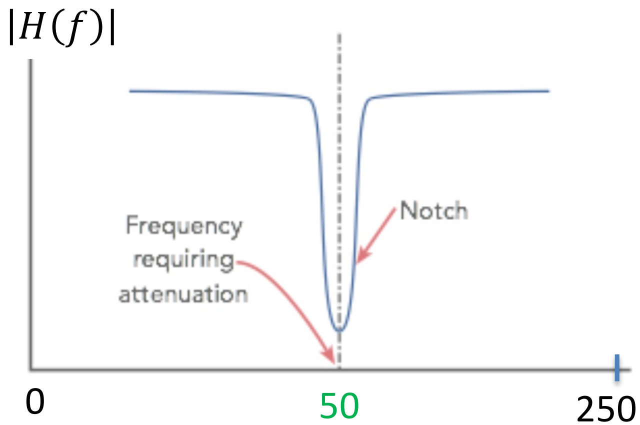
z-plane

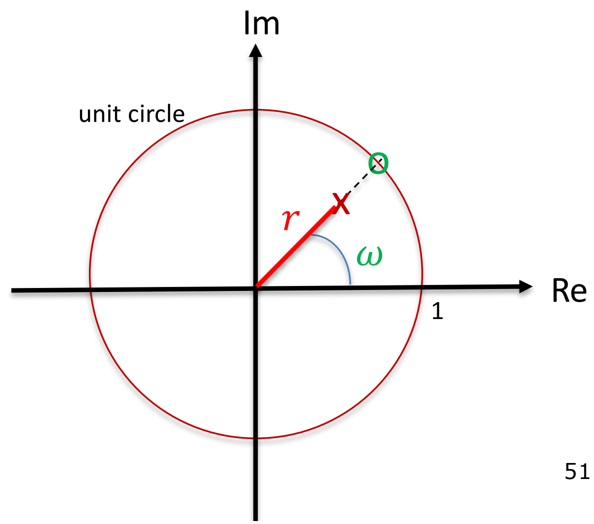
Pole-Zero Placement Notch Filter Design

- Obtain, by the pole-zero placement method, the transfer function of a sample digital notch filter that meets the following specifications:
  - Notch frequency f_{notch} : 50 Hz
  - 3 dB bandwidth of the Notch Δf : ±5Hz
  - Sampling frequency F_s : 500Hz

- The radius, r of the poles is determined by

$$r = 1 - \left(\frac{\Delta f}{F_s}\right)\pi$$

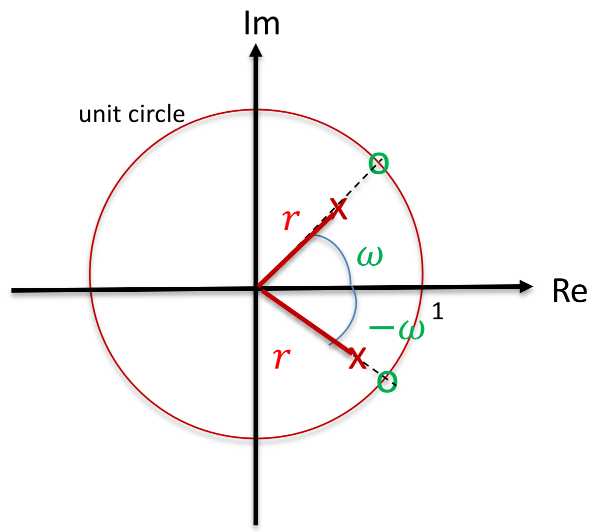




Use of a Pair of Complex Zeros

- To reject the component at 50Hz, place a pair of complex zeros at points on the unit circle corresponds to 50Hz. i.e. at angle of
  - $\omega = \Omega T = 2\pi \cdot 50 \cdot \frac{1}{500} = \pm 0.2\pi$
- To achieve a sharp notch filter and improved amplitude response on either side of the notch frequency, a pair of complex conjugate zeros are placed at a radius $r < 1$.

$$r = 1 - \left(\frac{\Delta f}{F_s}\right)\pi = 1 - \left(\frac{10}{500}\right)\pi = 0.937$$



Notch Filter Transform Function

- Based on the pole-zero locations, we can obtain the transfer function of the notch filter by

$$H(z) = \frac{(z - e^{-j0.2\pi})(z - e^{j0.2\pi})}{(z - 0.937e^{-j0.2\pi})(z - 0.937e^{j0.2\pi})}$$

$$= \frac{z^2 + 1 - (e^{j0.2\pi} + e^{-j0.2\pi})}{z^2 + 0.878 - 0.937(e^{j0.2\pi} + e^{-j0.2\pi})z}$$

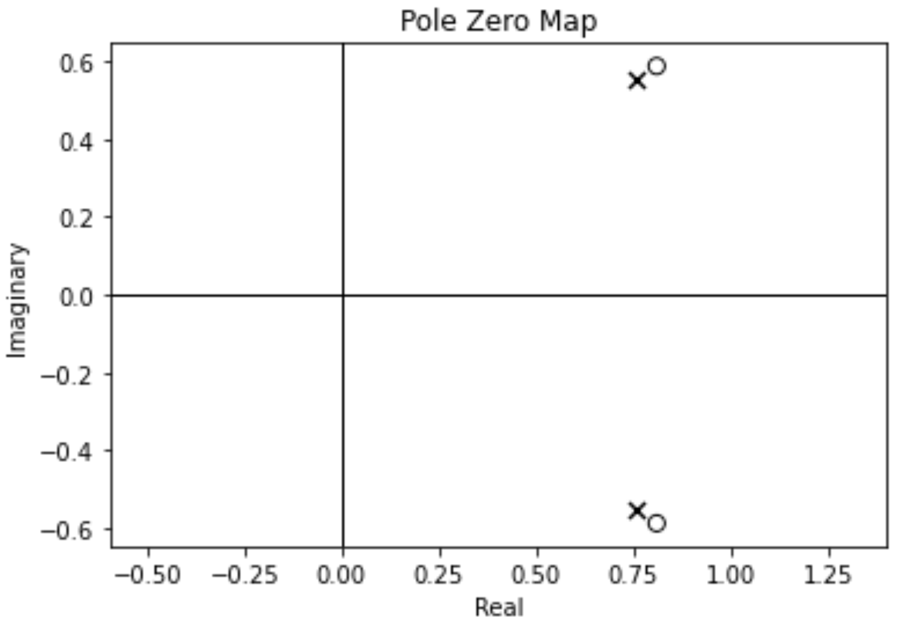
$$= \frac{z^2 + 1 - 2\cos(0.2\pi)}{z^2 + 0.878 - 2\times 0.937\cos(0.2\pi)}$$

$$= \frac{1 - 1.6180z^{-1} + z^{-2}}{1 - 1.5161z^{-1} + 0.878z^{-2}}$$

Python Code : Notch Filter's Pole-Zero Plot

```
import matplotlib.pyplot as plt
import numpy as np
import cmath
import control

# Define the Poles and Zeros of the Notch Filter
p1 = cmath.rect(0.937,np.pi*0.2)
p2 = cmath.rect(0.937,-np.pi*0.2)
z1 = cmath.rect(1,np.pi*0.2)
z2 = cmath.rect(1,-np.pi*0.2)

poles = [p1, p2]
zeros = [z1, z2]


# Determine the polynomial of the transfer function
H(z)=B(z)/A(z) from the poles and zeros
b = np.poly(zeros)
a = np.poly(poles)


tf = control.TransferFunction(b,a)
control.pzmap(tf)
plt.show()
```

$$H(z) = \frac{1 - 1.6180z^{-1} + z^{-2}}{1 - 1.5161z^{-1} + 0.878z^{-2}}$$



Python Code : Notch Filter's Magnitude Response

```
from scipy import signal
import numpy as np

w, h = signal.freqz(b, a, fs=500)

import matplotlib.pyplot as plt
fig = plt.figure()
ax1 = fig.add_subplot(1, 1, 1)
ax1.set_title(' Notch Filter : Magnitude Response')

ax1.plot(w, abs(h), 'r')
ax1.set_ylabel('Magnitude', color='b')
ax1.set_xlabel('Frequency [Hz]')
ax1.grid()

plt.axis('tight')
plt.show()
```