

Analog Filter Design

EE4015 Digital Signal Processing

Dr. Lai-Man Po

Department of Electrical Engineering
City University of Hong Kong

EE4015 Face-to-Face Mid-Term Exam

- The Face-to-Face Mid-Term Exam will be held on **November 8, 2022 (Tuesday of Week 11)**.
- The exam time is **2 hours**. Students should arrive at the venue at least 5 minutes before the start of the exam.
- The Mid-Term Exam is an **open-note exam**. Students can use "Scientific Calculator" and "All Handouts", including exercises and assignments.
 - In addition to hard copies of handouts, students can also use smartphones, tablets or iPads to read notes, but the electronic device must be set to airplane mode. During the exam, you are not allowed to communicate with others and search on the Internet. During the exam, investigators will check from time to time whether your electronic device is in airplane mode.
- Students need to use their own answer sheets (such as A4 paper) to answer the questions.
- The mid-term exam will cover up to week 8 (**Lecture L07B will not be covered in Mid-Term Exam**).

Group Project Oral Presentation

- Students are reminded to well prepare their group-project presentations on week 12 or 13 sessions to show your project findings and results with use of PPT for a presentation of approximately 10 minutes.
- Live demonstration or video demonstration are highly encouraged.
- All members of the project team must participate in its presentation.

Content

- Analog Filter Design Procedure
- Analog Lowpass Filter Specification
- Types of Analog Lowpass Filter
- Design of Butterworth Filters
- Design of Chebyshev I, Chebyshev II and Elliptic Filters (Optional)
- Frequency Transformations
 - Lowpass to Lowpass Transformation
 - Lowpass to Highpass Transformation
 - Lowpass to Bandpass Transformation
 - Lowpass to Bandstop Transformation

IIR Filter Design based on Analog Filter

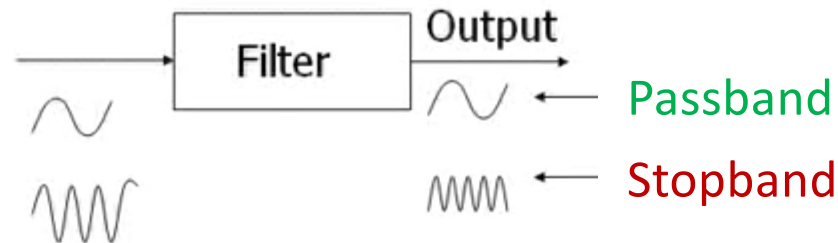
- Some of the IIR filters design methods are to apply a transformation to an existing analog filter to obtain the digital filter transfer functions, such as

$$H_a(s) = \frac{1}{s - a} \xleftrightarrow{\text{Transformation}} H(z) = \frac{1}{1 - bz^{-1}}$$

- Before introducing the IIR filter design methods, we first review the traditional analog filters design method.

Analog Filter Design Procedure

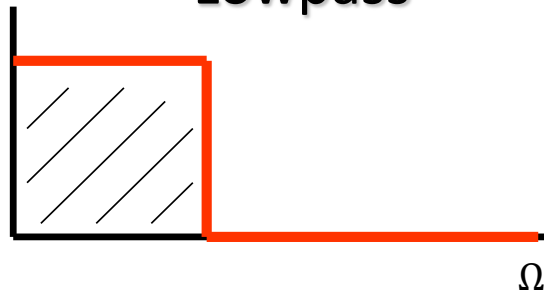
- The analog filter design procedure normally begins with **a specification of the frequency response** for describing how filter reacts to sinusoid inputs
- If an input sinusoid is not attenuated or attenuated less than a specific tolerance as it goes through the system, it is said to be in a **passband** of the filter.



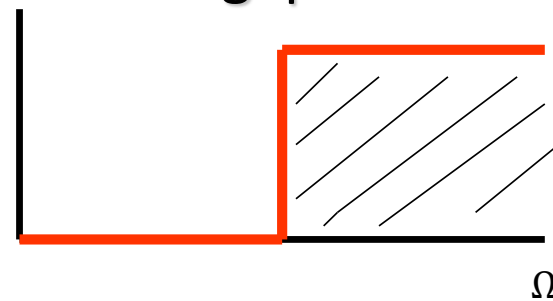
- If it is attenuated more than a specified value, it is said to be stopped and within the **stopband** of the filter.

Four types of Ideal Filters

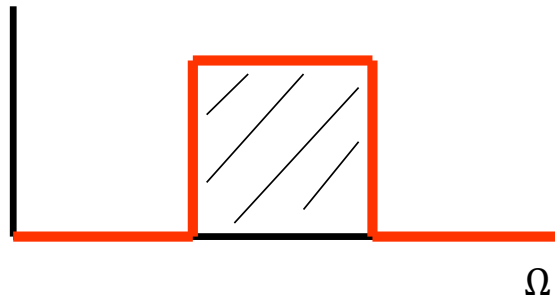
Lowpass



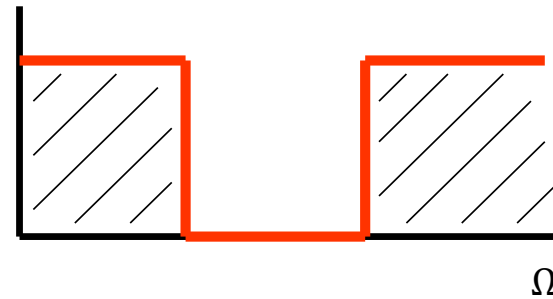
Highpass



Bandpass

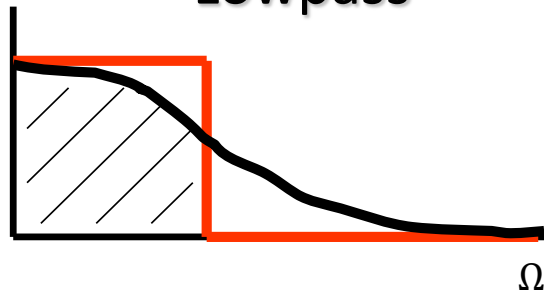


Bandstop

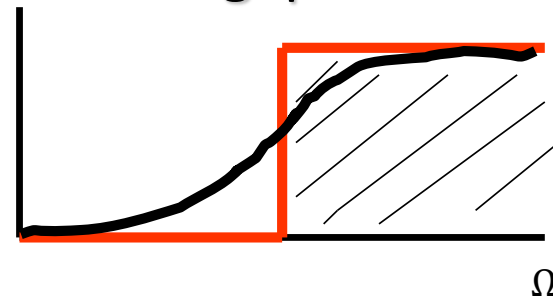


Realistic Frequency Response Analog Filters

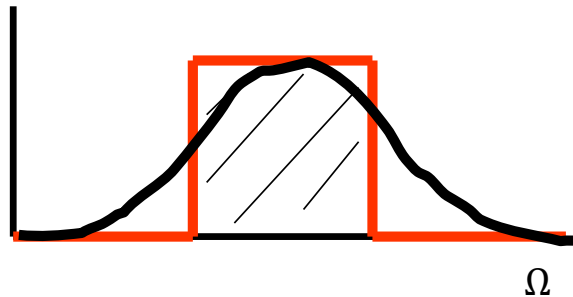
Lowpass



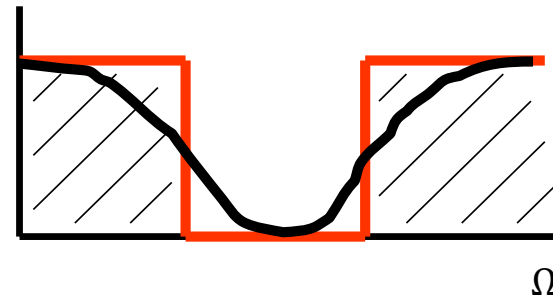
Highpass



Bandpass

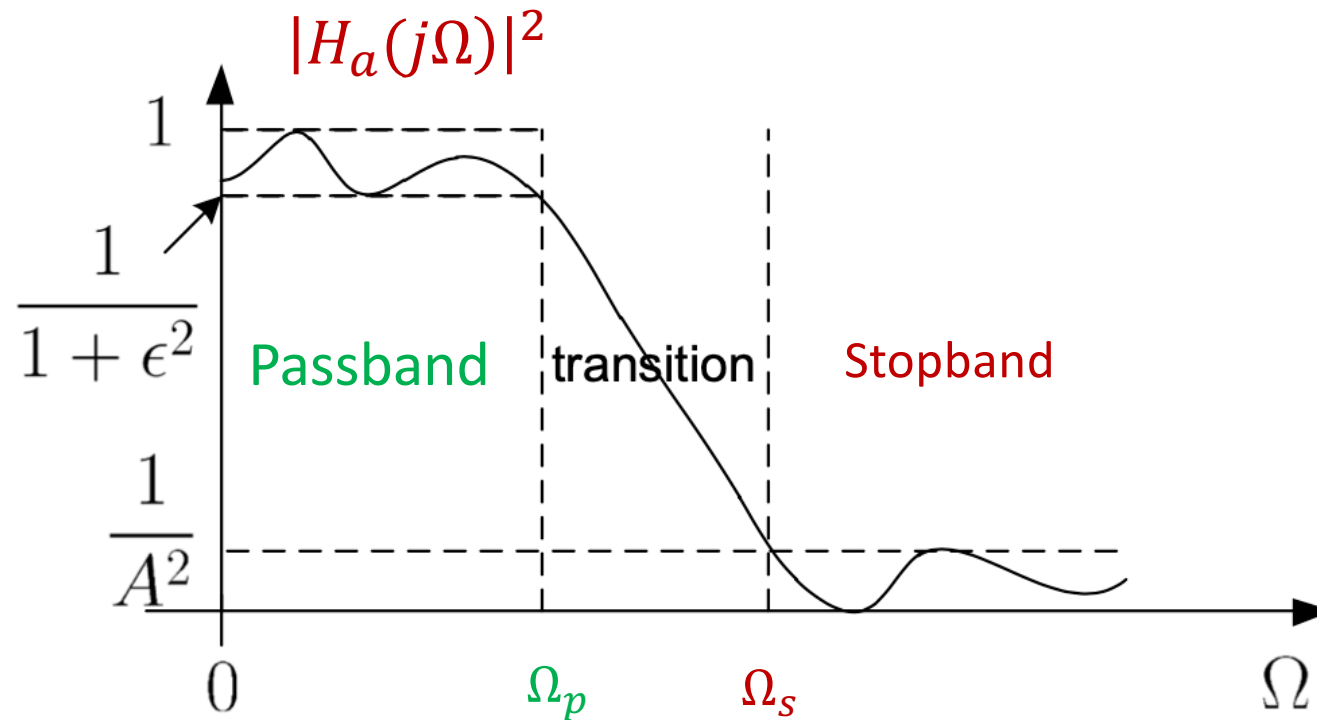


Bandstop



Analog Lowpass Filter Specification

- In analog lowpass filter design, we can only specify the magnitude of $H_a(j\Omega)$. Typically, we employ the magnitude square response, that is, $|H_a(j\Omega)|^2$:



Key Parameters

- **Passband** corresponds to $\Omega \in [0, \Omega_p]$ where Ω_p is the passband frequency and ε is called the **passband ripple**
- **Stopband** corresponds to $\Omega \in [\Omega_s, \infty]$ where Ω_s is the stopband frequency and A is called the **stopband attenuation**
- **Transition band** corresponds to $\Omega \in [\Omega_p, \Omega_s]$
- The specifications are represented as the two inequalities:

$$\frac{1}{1 + \varepsilon^2} \leq |H_a(j\Omega)|^2 \leq 1, \quad 0 \leq \Omega \leq \Omega_p$$

$$0 \leq |H_a(j\Omega)|^2 \leq \frac{1}{A^2}, \quad \Omega > \Omega_s$$

Passband and Stopband Frequencies

- In particular, at $\Omega = \Omega_p$ and $\Omega = \Omega_s$, we have:

$$|H_a(j\Omega_p)|^2 = \frac{1}{1 + \varepsilon^2} \quad \text{and} \quad |H_a(j\Omega_s)|^2 = \frac{1}{A^2}$$

- Apart from ε and A , it is also common to use their respective dB versions, denoted by R_p and A_s :

$$R_p = -10 \log_{10} \left(\frac{1}{1 + \varepsilon^2} \right) \Rightarrow \varepsilon = \sqrt{10^{R_p/10} - 1}$$

and

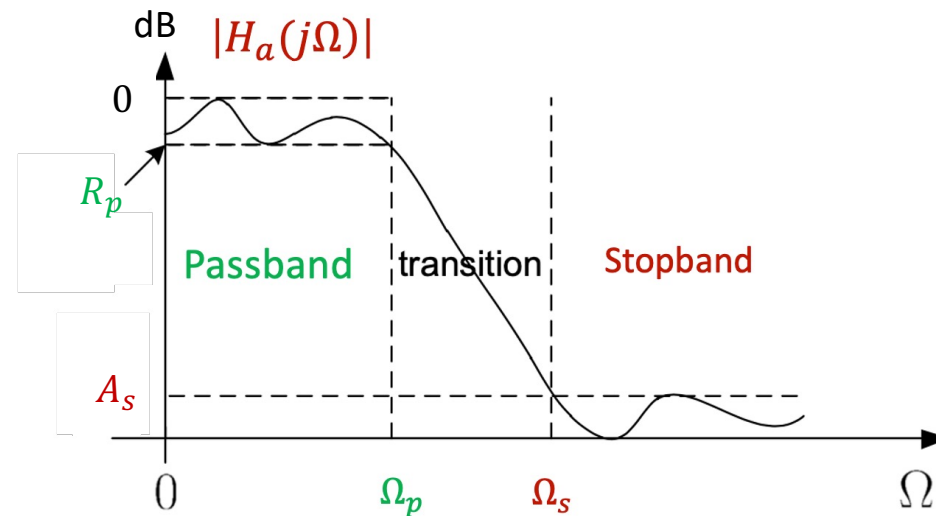
$$A_s = -10 \log_{10} \left(\frac{1}{A^2} \right) \Rightarrow A = 10^{A_s/20}$$

Analog Lowpass Filter Specification in dB

- Passband Gain R_p in dB at passband frequency Ω_p
- Stopband Attenuation A_s in dB at stopband frequency Ω_s

$$R_p = -10 \log_{10} \left(\frac{1}{1 + \varepsilon^2} \right)$$

$$A_s = -10 \log_{10} \left(\frac{1}{A^2} \right)$$

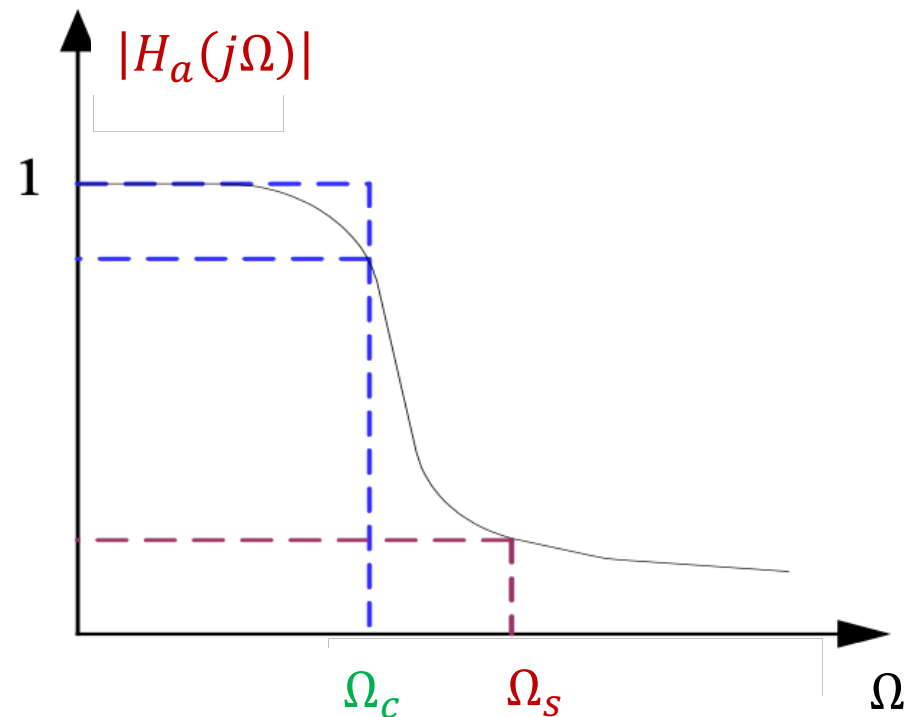


Types of Analog Lowpass Filter

- The first step of analog filter design is to select a low-pass filter type:
 - **Butterworth Lowpass Filter**
 - Maximal flat in passband
 - **Chebyshev I Lowpass Filter**
 - Maximal ripple in passband
 - **Chebyshev II Lowpass Filter**
 - Maximal ripple in stopband
 - **Elliptic Lowpass Filter**
 - Minimal transition width
 - Ripple in both passband and stopband
 - **Bessel Filter**
 - Maximally constant group delay

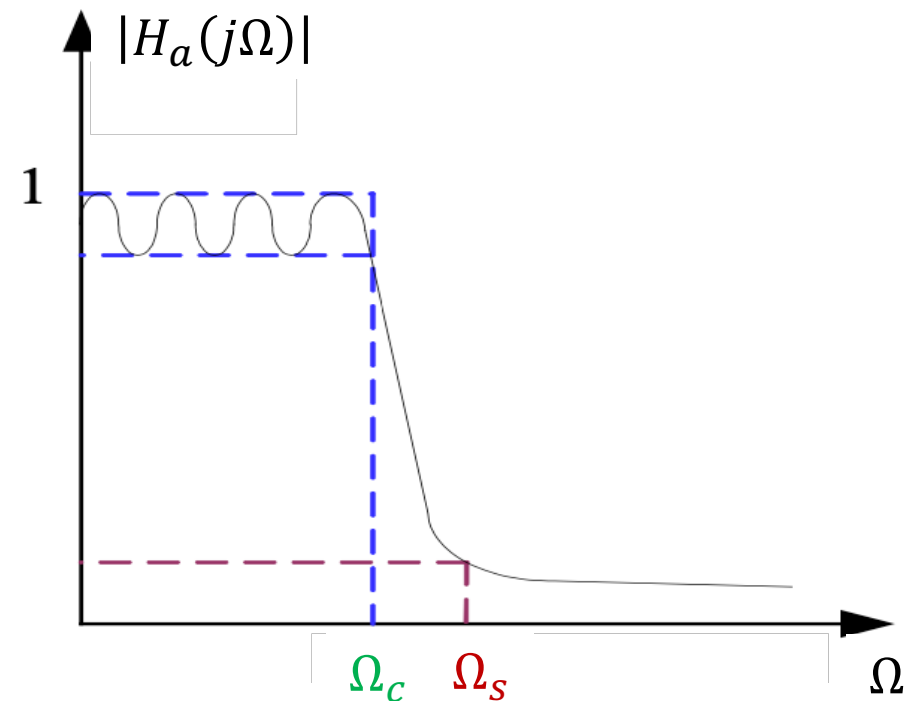
Butterworth Lowpass Filter

- The magnitude response $|H_a(j\Omega)|$ is **monotone decreasing** as Ω
- **Maximally flat in passband and stopband** but **attenuation relatively poor**



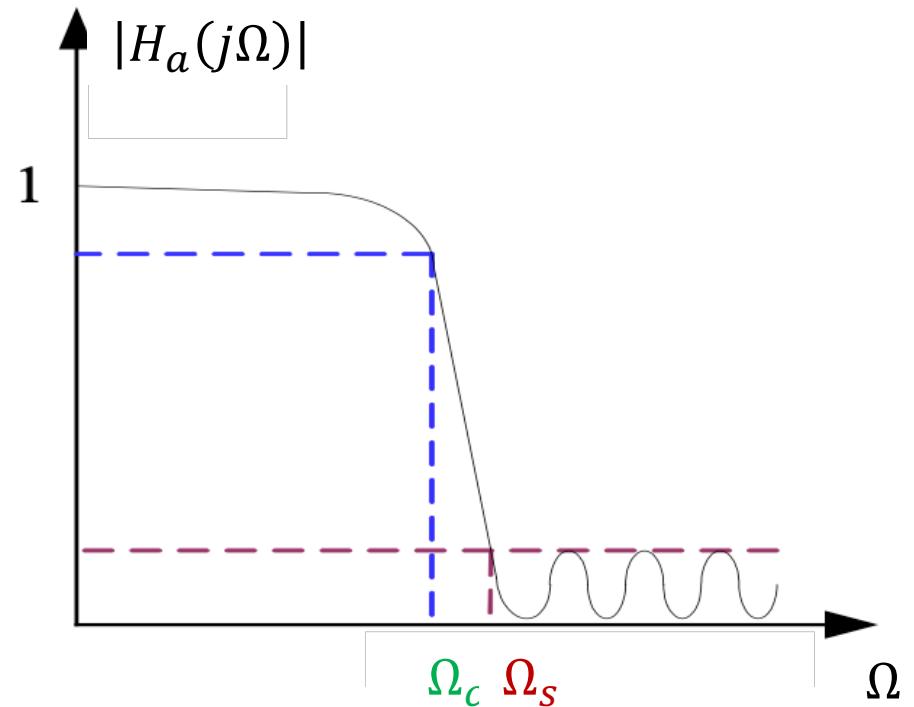
Chebyshev I Lowpass Filter

- Ripple in passband, monotonic in stopband
- Better match with ideal filter characteristic but maximal ripple in passband



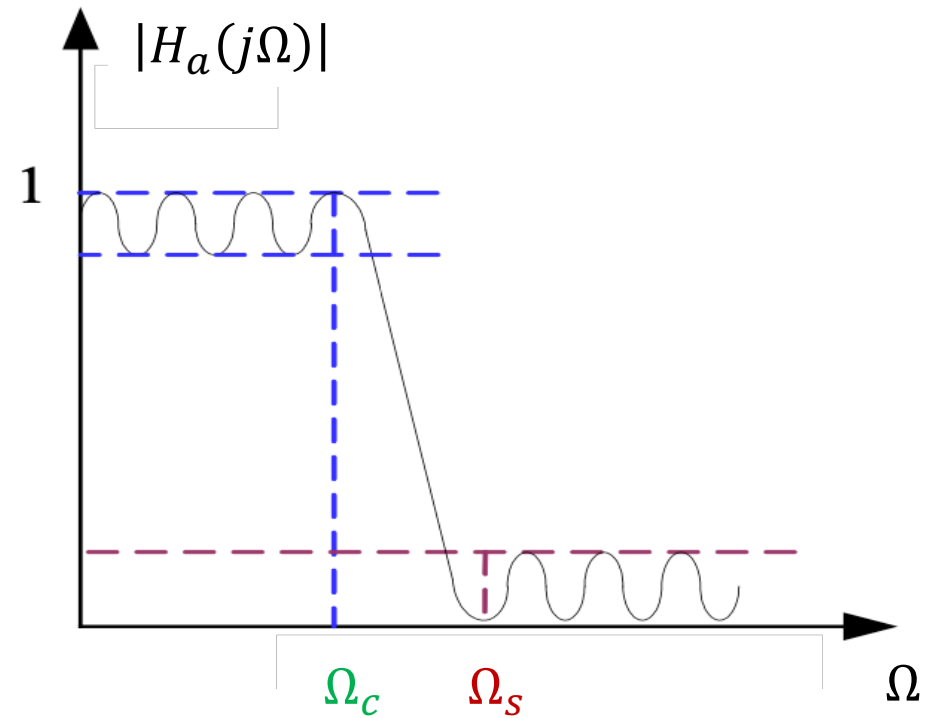
Chebyshev II Lowpass Filter

- Monotonic in passband, **ripple in stopband**
- Better match with ideal filter characteristic but **maximal ripple in stopband**



Elliptic Lowpass Filter

- Ripple in both passband and stopband
- Minimal transition width



Butterworth Lowpass Filter Design

Design of Lowpass Butterworth Filter

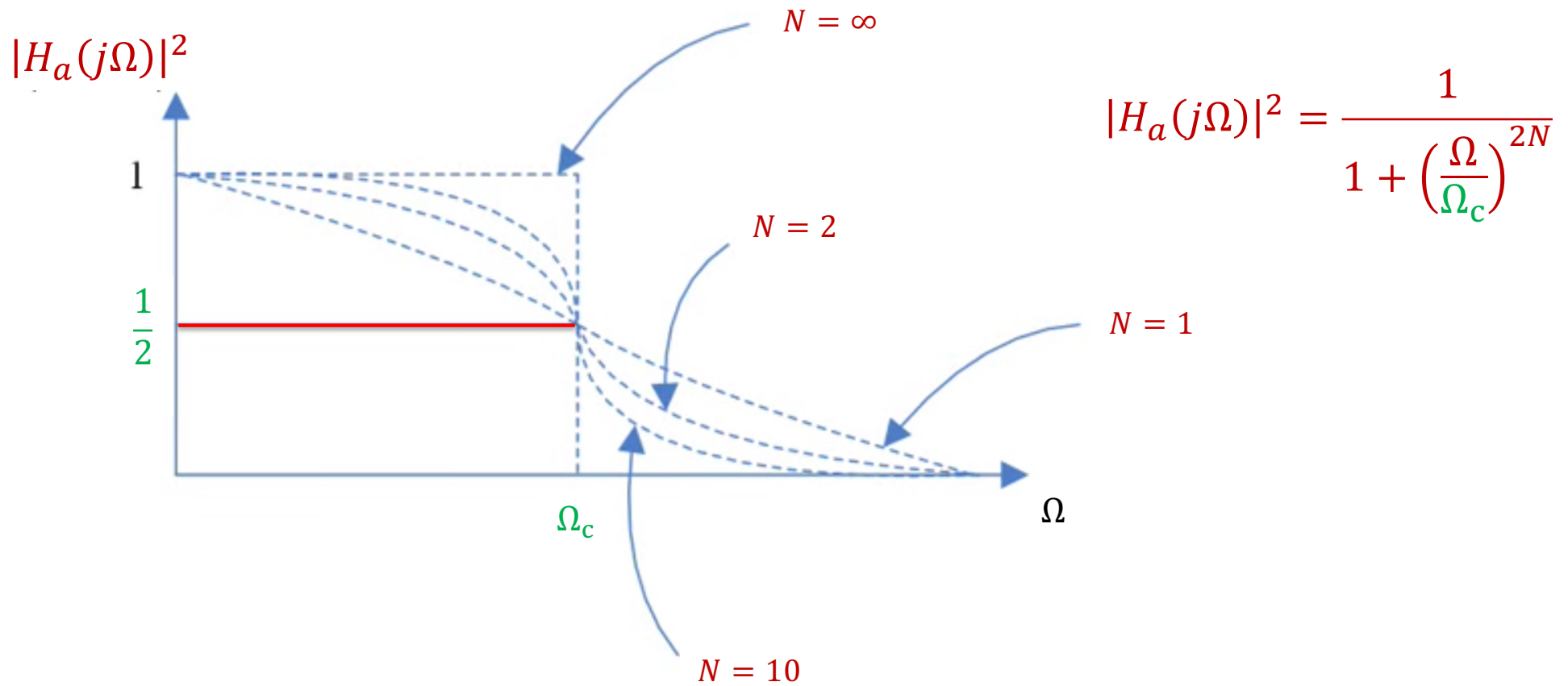
The magnitude square response of a N th-order Butterworth lowpass filter is:

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

The filter is characterized by **cutoff frequency** Ω_c and **filter order** N :

- $|H_a(j\Omega)|^2 = 1$ at $\Omega = 0$ and $|H_a(j\Omega)|^2 = 0.5$ at $\Omega = \Omega_c$ for all N .
- $|H_a(j\Omega)|^2$ is a monotonically decreasing function of frequency which indicates that there is **no ripple**
- Filter shape is **closer to the ideal response** as N increases, although the filter with order of $N \rightarrow \infty$ is not realizable.

Butterworth Filter Magnitude Frequency Response



Maximally Flat Property of Butterworth Filter

- $|H_a(j\Omega)|^2$ is called **maximally flat at the origin** since all order derivatives exist with respect to Ω are zero at the origin.

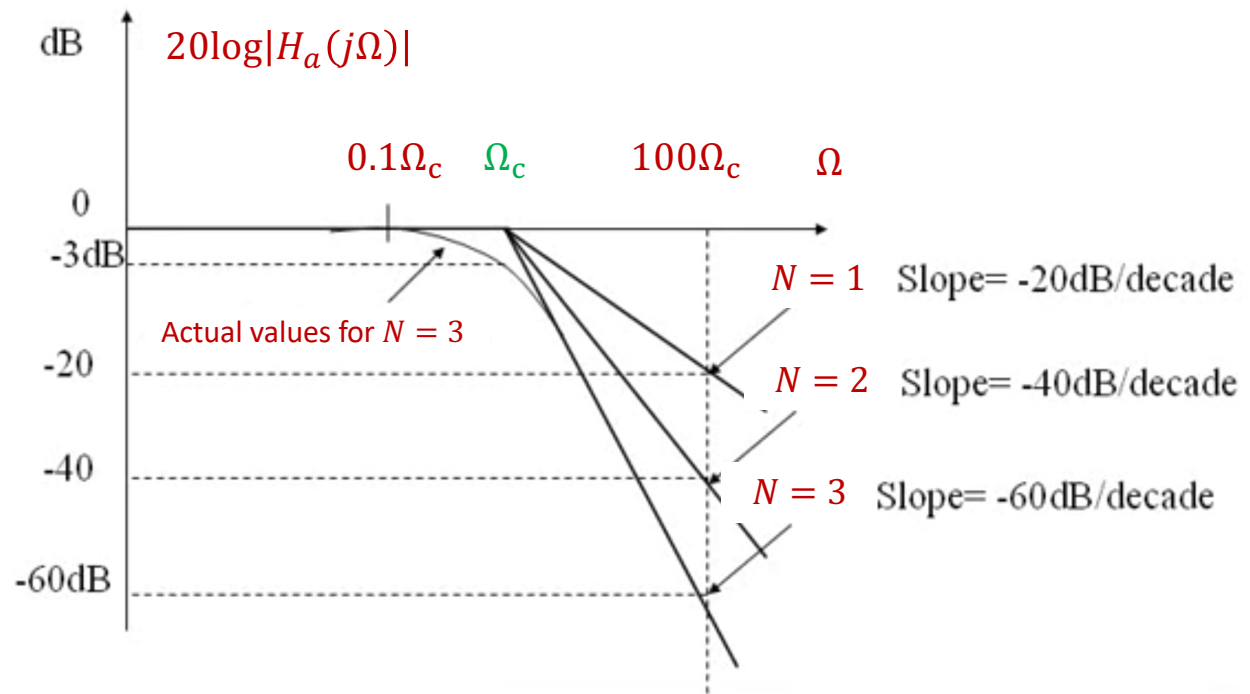
$$|H_a(j\Omega)| = \frac{1}{\sqrt{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}} = \left(1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}\right)^{-1/2}$$
$$= 1 - \frac{1}{2} \left(\frac{\Omega}{\Omega_c}\right)^{2N} + \frac{3}{8} \left(\frac{\Omega}{\Omega_c}\right)^{4N} - \frac{5}{16} \left(\frac{\Omega}{\Omega_c}\right)^{6N} + \dots$$

Power series expansion

1st 2N-1 derivatives
are zero at $\Omega=0$

Filter Gain plot

- It is convenient in many cases to look at the frequency response in decibels, that is, plot $20\log|H_a(j\Omega)|$ versus Ω .
- This is a straight-line approximation of the frequency response in decibels (dBs) for the Butterworth filters.



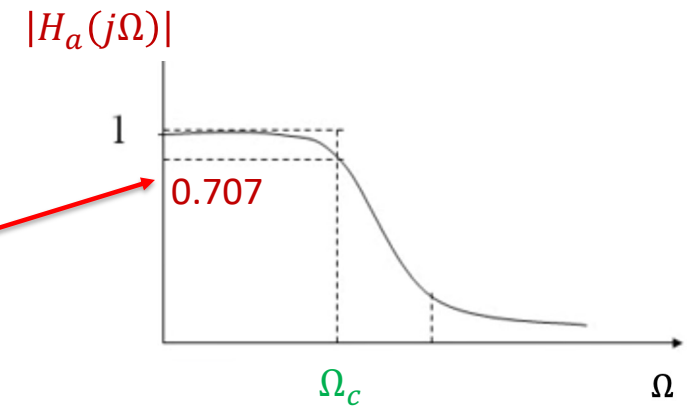
Cutoff Frequency Ω_c Properties

The following properties are easily determined:

- $|H_a(j\Omega)|_{\Omega=0}^2 = 1$, for all N .
- $|H_a(j\Omega)|_{\Omega=\Omega_c}^2 = \frac{1}{2}$, for all N , this implies
 - $|H_a(j\Omega)|_{\Omega=\Omega_c} = \frac{1}{\sqrt{2}} = 0.707$

and

- $20 \log |H_a(j\Omega)|_{\Omega=\Omega_c} = -3.0dB$
- Ω_c is also called half-power frequency as
 - $|H_a(j\Omega)|_{\Omega=\Omega_c}^2 = \frac{1}{2}$



$$|H_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

Transfer Function $H_a(j\Omega)$ of Butterworth Filter

- To find the transfer function $H_a(s)$, we first make use of its relationship with $H_a(j\Omega)$

$$H_a(s) \Big|_{s=j\Omega} = H_a(j\Omega)$$

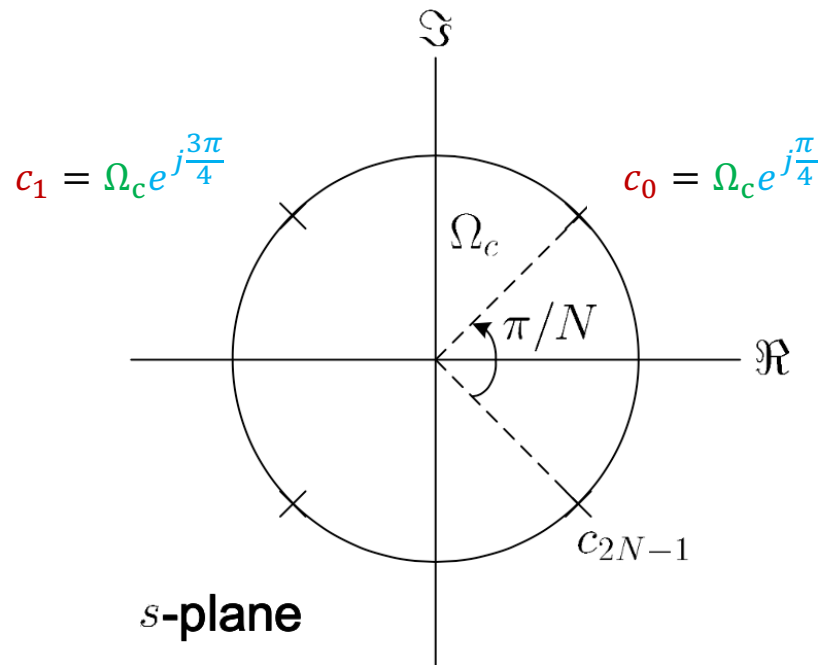
$$H_a(s)H_a(-s) = |H_a(j\Omega)|^2 \Big|_{\Omega=\frac{s}{j}} = \frac{(j\Omega_c)^{2N}}{s^{2N} + (j\Omega_c)^{2N}}$$

- The $2N$ poles of $H_a(s)H_a(-s)$, can be solved by

$$\left(\frac{c_k}{j\Omega_c}\right)^{2N} = -1 \quad \longrightarrow \quad = \begin{cases} \Omega_c e^{j\frac{k\pi}{N}}, & \text{odd } N \\ \Omega_c e^{j\frac{k\pi}{N}} \cdot e^{j\frac{\pi}{2N}}, & \text{even } N \end{cases} \quad k = 0, 1, 2, \dots, 2N - 1$$

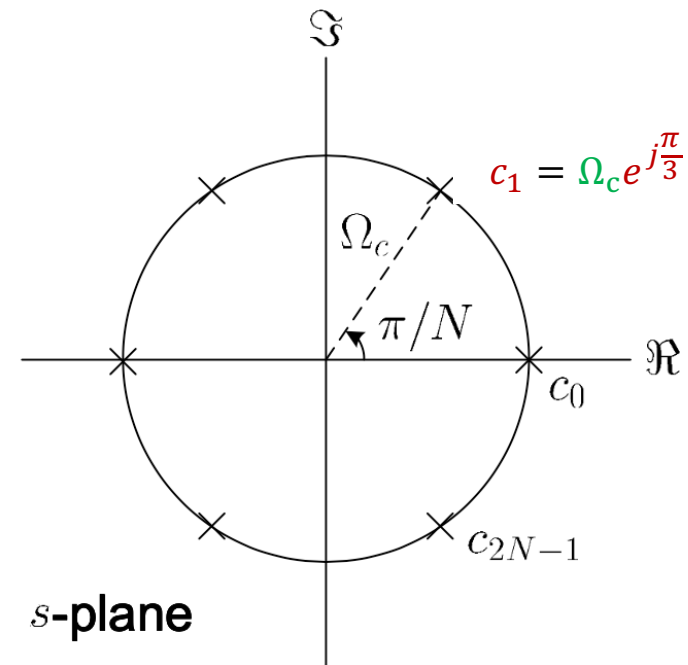
$2N$ poles spaced by $(2\pi)/2N$ at radius Ω_c

Poles of Butterworth Lowpass Filter



$$N = 2$$

$$c_k = \Omega_c e^{j\frac{k\pi}{2}} \cdot e^{j\frac{\pi}{4}}$$

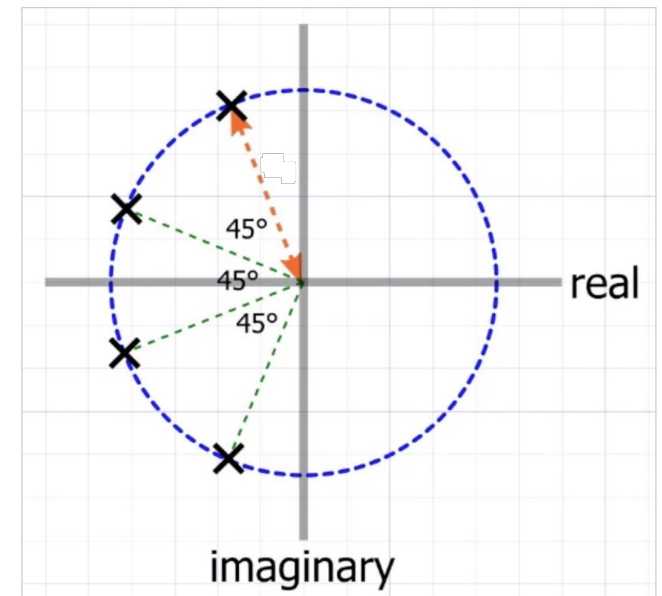


$$N = 3$$

$$c_k = \Omega_c e^{j\frac{k\pi}{3}}$$

- $\{c_k\}$ are uniformly distributed on a circle of radius Ω_c with angular spacing of π/N in the s-plane
- Poles are symmetrically located with respect to the imaginary axis
- there are two real-valued poles when N is odd
- To extract $H_a(s)$, we utilize the knowledge that all poles of a stable and causal analog filter should be on the **left half** of the s-plane. As a result, is:

$$H_a(s) = \frac{\Omega_c^N}{\prod_{\Re\{c_k\} < 0} (s - c_k)}$$

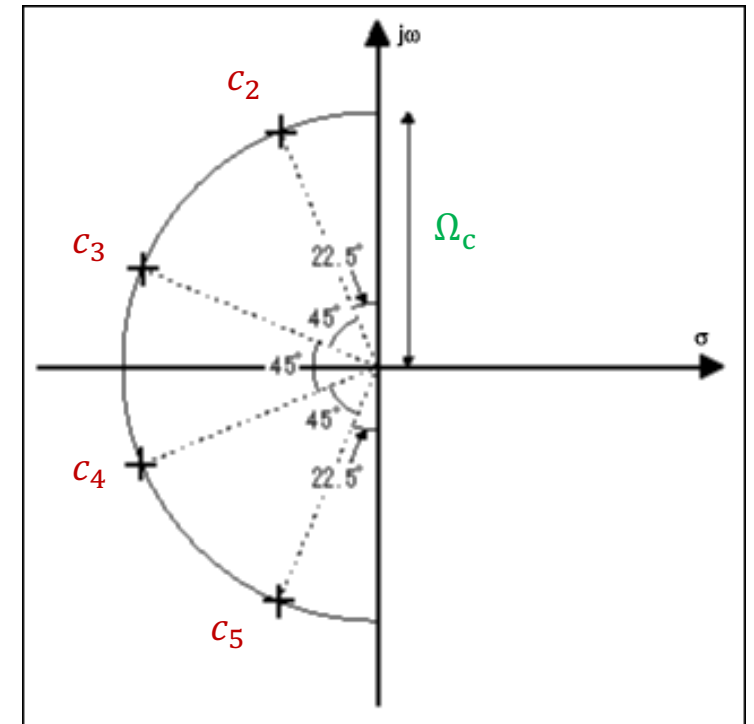


Pole locations for N=8

Example : $H_a(s)$ of 4th Order Butterworth Filter

- If $N = 4$, the $c_k = \Omega_c e^{j\frac{k\pi}{4}} \cdot e^{j\frac{\pi}{8}}$ $k = 0,1,2,\dots,7$
- For stable and causal, choose poles in the left half plan. Then, we only select the left half plane poles of
 - $k = 2 \Rightarrow c_2 = \Omega_c e^{j\frac{2\pi}{4}} \cdot e^{j\frac{\pi}{8}} = \Omega_c e^{j\frac{5\pi}{8}}$
 - $k = 3 \Rightarrow c_3 = \Omega_c e^{j\frac{3\pi}{4}} \cdot e^{j\frac{\pi}{8}} = \Omega_c e^{j\frac{7\pi}{8}}$
 - $k = 4 \Rightarrow c_4 = \Omega_c e^{j\frac{4\pi}{4}} \cdot e^{j\frac{\pi}{8}} = \Omega_c e^{j\frac{9\pi}{8}} = \Omega_c e^{-j\frac{7\pi}{8}}$
 - $k = 5 \Rightarrow c_5 = \Omega_c e^{j\frac{5\pi}{4}} \cdot e^{j\frac{\pi}{8}} = \Omega_c e^{j\frac{11\pi}{8}} = \Omega_c e^{-j\frac{5\pi}{8}}$
- The transfer function of the filter is

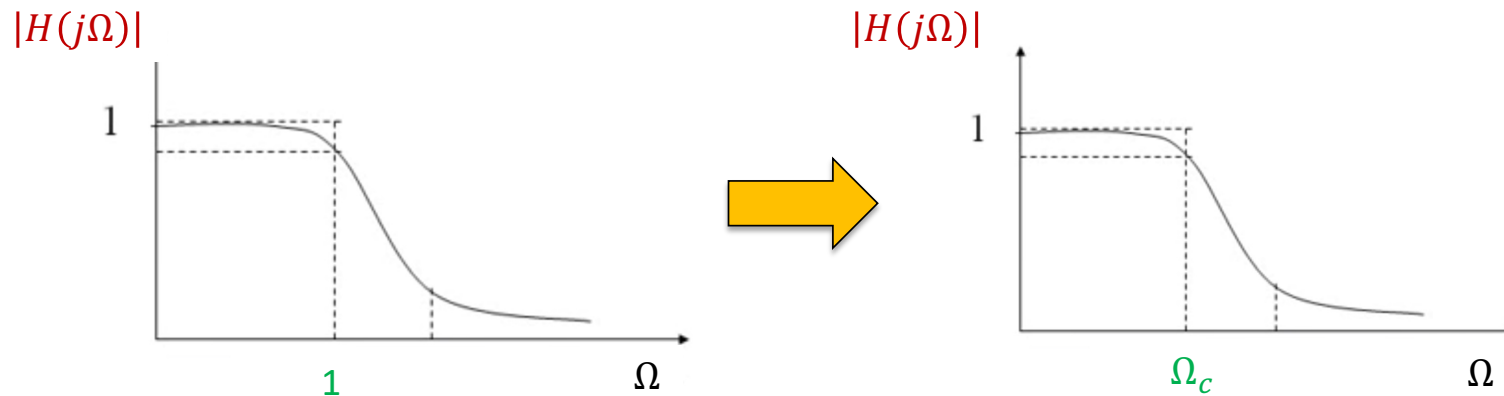
$$H_a(s) = \frac{\Omega_c^4}{(s - \Omega_c e^{j\frac{5\pi}{8}})(s - \Omega_c e^{-j\frac{5\pi}{8}})(s - \Omega_c e^{j\frac{7\pi}{8}})(s - \Omega_c e^{-j\frac{7\pi}{8}})}$$



Pole locations for $N=4$

Prototype Lowpass Filter Function

- The technique used to design analogue filters is to specify a **prototype lowpass filter function** which is **normalized** to provide a cutoff frequency Ω_c at **1 rad/sec**.
- Then apply transformations to achieve the actual desired cutoff frequencies and filter type.



Normalized Butterworth Filter Transfer Functions

For $\Omega_c = 1$, we have the transfer functions of the Butterworth Filters in different order n.

N	Butterworth Transfer Functions
1	$\frac{1}{s + 1}$
2	$\frac{1}{s^2 + \sqrt{2}s + 1}$
3	$\frac{1}{(s^2 + s + 1)(s + 1)}$
4	$\frac{1}{(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)}$
5	$\frac{1}{(s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)}$

https://en.wikipedia.org/wiki/Butterworth_filter#Normalized_Butterworth_polynomials

Example 1

- The magnitude square response of a Butterworth lowpass filter has the form of:

$$|H_a(j\Omega)|^2 = \frac{1}{1 + 0.00001\Omega^6}$$

- Determine the filter transfer function $H_a(s)$.

Solution of Example 1

- Expressing $|H_a(j\Omega)|^2$ as:

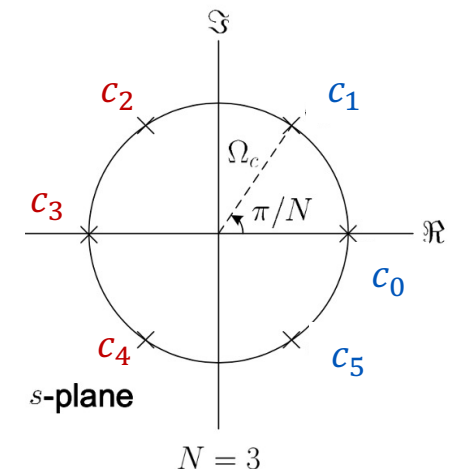
$$|H_a(j\Omega)|^2 = \frac{1}{1 + 0.000001\Omega^6} = \frac{1}{1 + \left(\frac{\Omega}{10}\right)^{2 \cdot 3}}$$

- This Butterworth filter's cutoff frequency $\Omega_c = 10$ rad/sec, and order $N = 3$, the poles of the transfer function are

$$c_k = 10e^{j\frac{k\pi}{3}}, \quad k = 2,3,4$$

- Then, the transfer function can be obtained by

$$\begin{aligned} H_a(s) &= \frac{\Omega_c^3}{(s - c_2)(s - c_3)(s - c_4)} = \frac{1000}{(s - 10e^{j\frac{2\pi}{3}})(s + 10)(s - 10e^{j\frac{2\pi}{3}})} \\ &= \frac{1000}{(s + 10)(s^2 + 10s + 100)} \end{aligned}$$



Gain Equations in dB

- The Gain G_x in dB at any frequency Ω_x of the Butterworth filter is

$$\begin{aligned}G_x &= 20 \log_{10} |H_a(j\Omega_x)| \\&= 20 \log_{10} \left(1 / \sqrt{1 + \left(\frac{\Omega_x}{\Omega_c}\right)^{2N}} \right) \\&= 0 - 20 \log_{10} \left(\sqrt{1 + \left(\frac{\Omega_x}{\Omega_c}\right)^{2N}} \right) \\&= 10 \log_{10} \left(1 + \left(\frac{\Omega_x}{\Omega_c}\right)^{2N} \right)\end{aligned}$$

Gain Equations in dB

- So, Gains at frequency Ω_p and Ω_s are

$$R_p = 10 \log_{10} \left(1 + \left(\frac{\Omega_p}{\Omega_c} \right)^{2N} \right) \Rightarrow \left(\frac{\Omega_p}{\Omega_c} \right)^{2N} = 10^{R_p/10} - 1$$

$$A_s = 10 \log_{10} \left(1 + \left(\frac{\Omega_s}{\Omega_c} \right)^{2N} \right) \Rightarrow \left(\frac{\Omega_s}{\Omega_c} \right)^{2N} = 10^{A_s/10} - 1$$

- Dividing these two equations by each other yields

$$\left(\frac{\Omega_p}{\Omega_s} \right)^{2N} = \frac{10^{R_p/10} - 1}{10^{A_s/10} - 1}$$

Determination of N from Ω_p , Ω_s , R_p and A_s

- The previous equations can easily be solved in terms of Ω_p , Ω_s , R_p and A_s for the filter order N as

$$N = \left\lceil \frac{\log_{10} \left[\frac{(10^{R_p/10} - 1)}{(10^{A_s/10} - 1)} \right]}{2 \log_{10}(\Omega_p / \Omega_s)} \right\rceil$$

- Where $\lceil u \rceil$ rounds up u to the nearest integer.

Determine the Range of Cutoff Ω_c

- Based on previous equations, Ω_c can be computed by

$$\Omega_c = \frac{\Omega_p}{\left(10^{\frac{R_p}{10}} - 1\right)^{1/(2N)}} \quad \Omega_c = \frac{\Omega_s}{\left(10^{\frac{A_s}{10}} - 1\right)^{1/(2N)}}$$

- As a result, the admissible range of Ω_c is:

$$\Omega_c \in \left[\frac{\Omega_p}{\left(10^{\frac{R_p}{10}} - 1\right)^{1/(2N)}}, \frac{\Omega_s}{\left(10^{\frac{A_s}{10}} - 1\right)^{1/(2N)}} \right]$$

Example 2

Determine the transfer function of a Butterworth lowpass filter whose magnitude requirements are $\Omega_p = 4\pi \text{ rads}^{-1}$, $\Omega_s = 6\pi \text{ rads}^{-1}$, $R_p = 8 \text{ dB}$ and $A_s = 16 \text{ dB}$

Solution:

- Based on the filter order equation, we can determine the filter order as

$$N = \left\lceil \frac{\log_{10} \left[\frac{(10^{R_p/10} - 1)}{(10^{A_s/10} - 1)} \right]}{2 \log_{10}(\Omega_p / \Omega_s)} \right\rceil$$
$$= \left\lceil \frac{\log_{10} \left[\frac{(10^{8/10} - 1)}{(10^{16/10} - 1)} \right]}{2 \log_{10}(4\pi / 6\pi)} \right\rceil = [2.45] = 3$$

Python Code

```
import numpy as np
Rp=8
As=16
Wp=4.0*np.pi
Ws=6.0*np.pi

N=(np.log((pow(10,Rp/10)-1)/(pow(10,As/10)-1)))/(2*np.log(Wp/Ws))

N
2.4529668274738468
```

$$N = \left\lceil \frac{\log_{10}[(10^{8/10} - 1)/(10^{16/10} - 1)]}{2 \log_{10}(4\pi/6\pi)} \right\rceil = [2.45] = 3$$

- Put $N = 3$, the range of cutoff frequency Ω_c is

$$\Omega_c \in \left[\frac{\Omega_p}{\left(10^{\frac{R_p}{10}} - 1\right)^{1/(2N)}}, \frac{\Omega_s}{\left(10^{\frac{A_s}{10}} - 1\right)^{1/(2N)}} \right]$$

$$= \left[\frac{4\pi}{\left(10^{\frac{8}{10}} - 1\right)^{1/(2 \cdot 3)}}, \frac{6\pi}{\left(10^{\frac{16}{10}} - 1\right)^{1/(2 \cdot 3)}} \right] = [9.5141, 10.2441]$$

- For simplicity, we select $\Omega_c = 10$. Using Example 1, the filter transfer function is:

$$H_a(s) = H_a(s) \Big|_{s=\frac{s}{10}} = \frac{1}{(s+1)(s^2+s+1)} \Big|_{s=\frac{s}{10}}$$

$$H_a(s) = \frac{1000}{(s+10)(s^2+10s+100)} = \frac{100}{s^3+20s^2+200s+1000}$$

Python Code

```
import numpy as np
Rp=8
As=16
Wp=4.0*np.pi
Ws=6.0*np.pi
N=3
Wc1 = Wp/(pow((pow(10,Rp/10)-1),(1/(2*N))))
Wc2 = Ws/(pow((pow(10,As/10)-1),(1/(2*N))))

Wc1
9.514075465781463
Wc2
10.244148025477667
```

$$\Omega_c \in \left[\frac{4\pi}{\left(10^{\frac{8}{10}} - 1\right)^{1/(2 \cdot 3)}}, \frac{6\pi}{\left(10^{\frac{16}{10}} - 1\right)^{1/(2 \cdot 3)}} \right] = [9.5141, 10.2441]$$

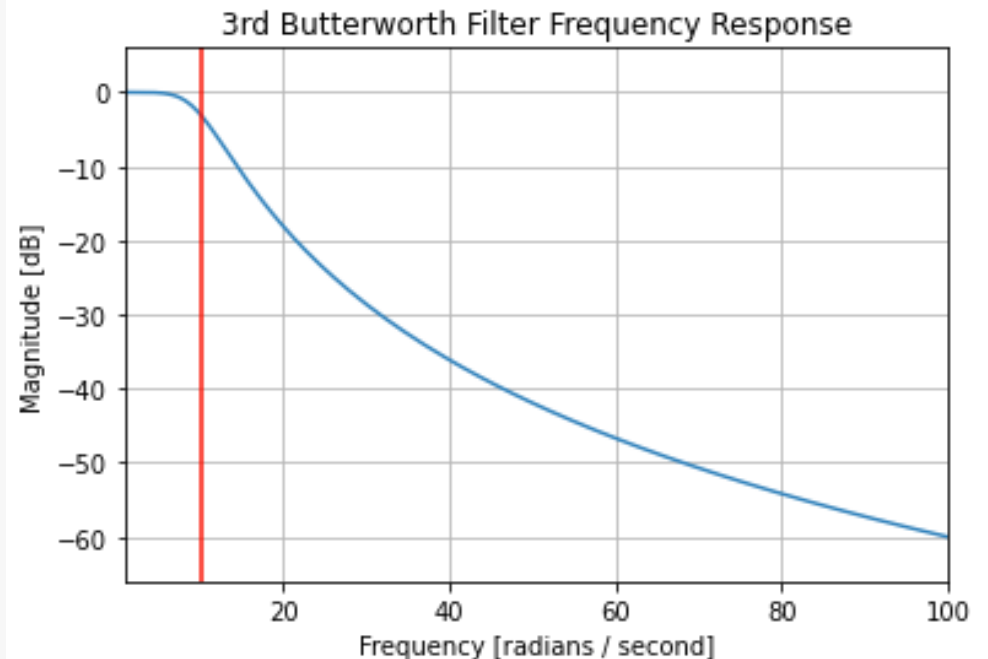
Python: Analog Butterworth Filter Design

```
from scipy import signal
import matplotlib.pyplot as plt

b, a = signal.butter(3, 10, 'low', analog=True)
w, h = signal.freqs(b, a)

print("b = ", b)
print("a = ", a)

plt.plot(w, 20 * np.log10(abs(h)))
plt.title('3rd Butterworth Filter Frequency Response')
plt.xlabel('Frequency [radians / second]')
plt.ylabel('Magnitude [dB]')
plt.margins(0, 0.1)
plt.grid(which='both', axis='both')
plt.axvline(10, color='red') # cutoff frequency
plt.show()
```



```
b = [1000.]
a = [ 1.  20. 200. 1000.]
```

➔ $H_a(s) = \frac{1000}{(s + 10)(s^2 + 10s + 100)} = \frac{1000}{s^3 + 20s^2 + 200s + 1000}$

Chebyshev I Filters (Optional)

- There are two types Chebyshev filters, one containing a **ripple in the passband (type I)** and the other containing **a ripple in the stopband (Type II)**.
- A type-I Lowpass normalized (unit bandwidth) Chebyshev filter with a ripple in the passband is characterized by the magnitude squared frequency response as

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 \cdot T_N^2(\Omega/\Omega_p)}$$

where $T_N(\Omega)$ is the N^{th} order Chebyshev polynomial.

- Chebyshev polynomial can be generated by the following recursive formula:

$$T_N(\Omega) = 2\Omega T_{N-1}(\Omega) - 2T_{N-2}(\Omega) \quad N > 2$$

with $T_0(\Omega) = 1$ and $T_1(\Omega) = \Omega$

N	$T_N(\Omega)$
0	1
1	Ω
2	$2\Omega^2 - 1$
3	$4\Omega^3 - \Omega$
4	$8\Omega^4 - 8\Omega^2 + 1$
5	$16\Omega^5 - 20\Omega^3 + 5\Omega$

Chebyshev I Filter Characteristics (Optional)

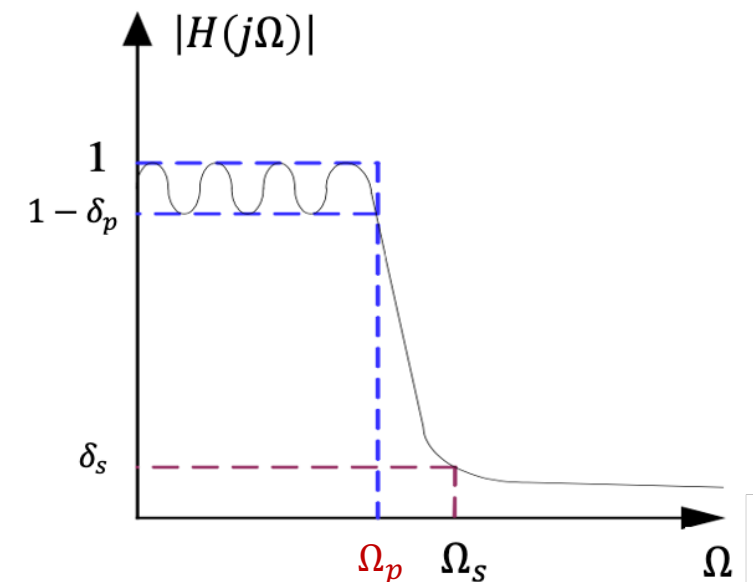
- A type-I Lowpass normalized (unit bandwidth) Chebyshev filter with a ripple in the passband is characterized by the magnitude squared frequency response as

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 \cdot T_N^2(\Omega/\Omega_p)}$$

where $T_N(\Omega)$ is the N^{th} order Chebyshev polynomial.

- For $|\Omega/\Omega_p| < 1$, $T_N(\Omega/\Omega_p)$ oscillates between ± 1 ,
 - $T_N(1) = 1$, $\Rightarrow |H(j\Omega)|^2$ oscillates between $\frac{1}{1+\varepsilon^2}$ and 1.
- For $|\Omega/\Omega_p| > 1$, $T_N(\Omega/\Omega_p)$ monotone increasing
- DC Gain is

$$|H(j0)|^2 = \begin{cases} 1, & N \text{ odd} \\ \frac{1}{1 + \varepsilon^2}, & N \text{ even} \end{cases}$$



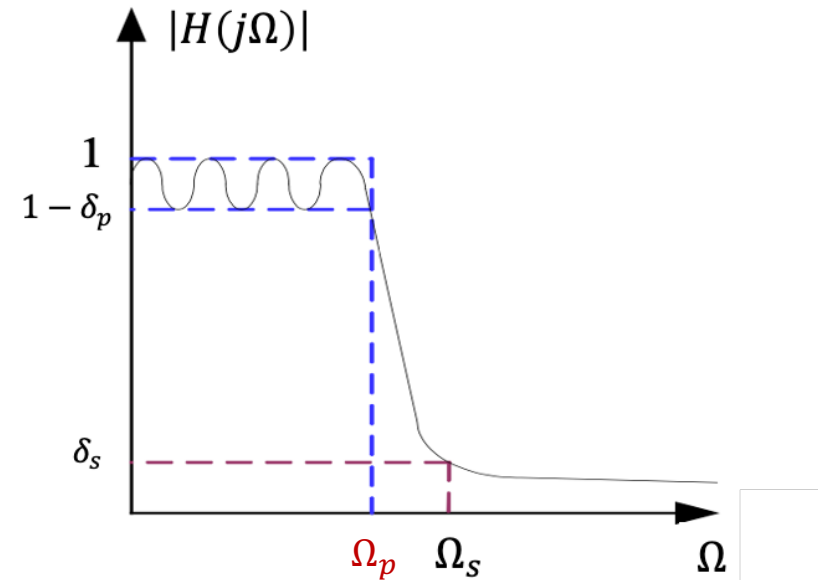
Design of Chebyshev Filter I (Optional)

Three design parameters: Ω_p , ε^2 and N

- Set Ω_p equal to passband edge
- Choose ε^2 so that $1 - \delta_p = \left(\frac{1}{1+\varepsilon^2}\right)^{1/2}$
- Choose N to satisfy stopband constraint

$$\delta_s^2 \geq \frac{1}{1 + \varepsilon^2 (T_N(\Omega_s))^2}$$

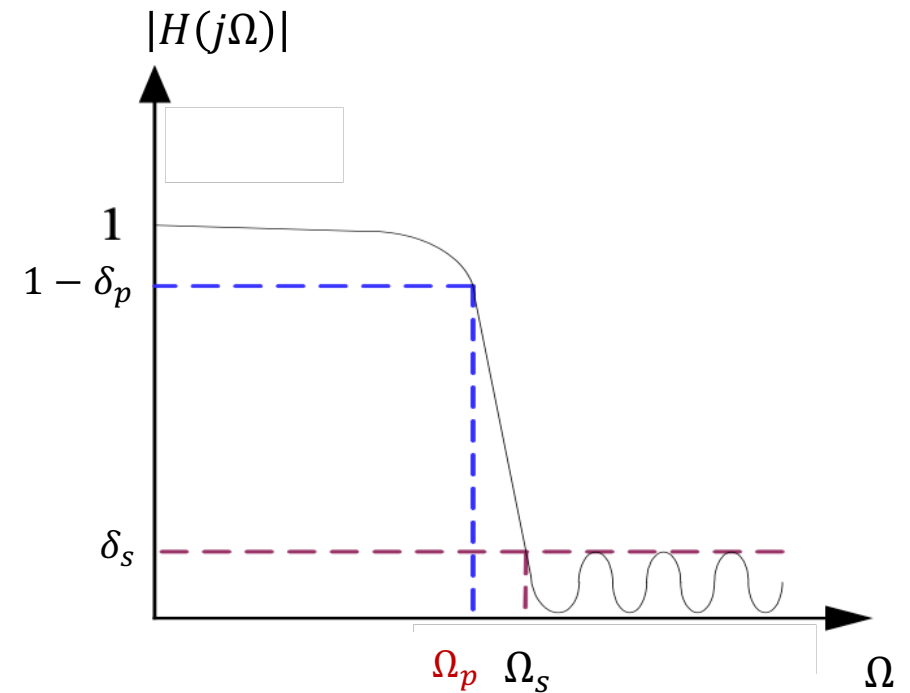
- Filter poles located on an ellipse in the s-plan



Design of Chebyshev Filter II (Optional)

$$|H(j\Omega)|^2 = \frac{1}{1 + \left(\varepsilon^2 \cdot T_N^2\left(\frac{\Omega_p}{\Omega}\right)\right)^{-1}} = \frac{\varepsilon^2 \cdot T_N^2\left(\frac{\Omega_p}{\Omega}\right)}{1 + \varepsilon^2 \cdot T_N^2\left(\frac{\Omega_p}{\Omega}\right)}$$

- For $|\Omega_o/\Omega| < 1$ ($|\Omega| > \Omega_p$), $T_N(\Omega_p/\Omega)$ oscillates between ± 1 ,
 - $\Rightarrow 0 \leq |H(j\Omega)|^2 \leq \frac{\varepsilon^2}{1+\varepsilon^2}, |\Omega| > \Omega_p.$
- For $|\Omega_p/\Omega| > 1$, $T_N(\Omega_p/\Omega)$ increasing as Ω to 0
- DC Gain
 - $|H(j0)|^2 = 1$

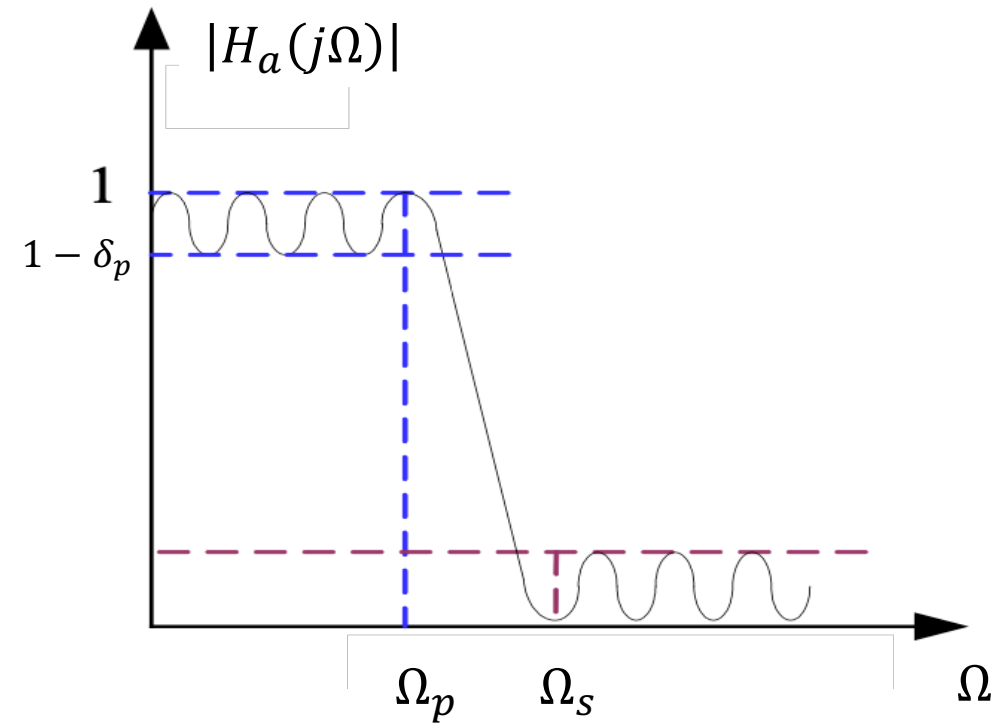


Elliptic Filters (Optional)

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 \cdot U_N^2(\Omega)}$$

where $U_N(\Omega)$ Jacobian elliptic function

- Equi-ripple error in both pass and stop bands



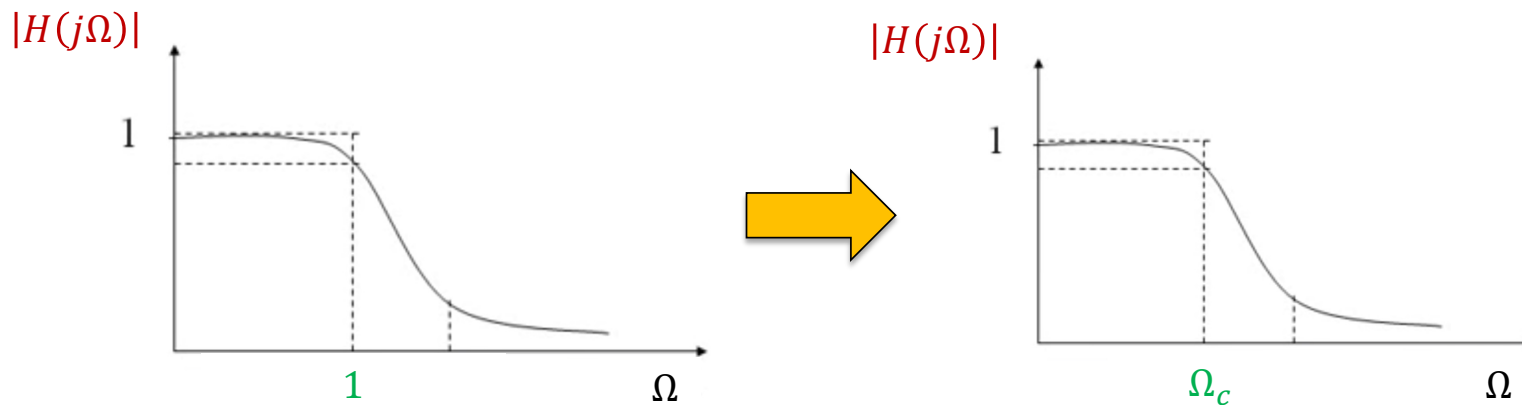
Summary of Popular Analog Filters

- **Butterworth**
 - All Pole, No ripples in passband or stopband
 - Maximally Flat Response
- **Chebyshev (Type I and II)**
 - All Pole, Type I has Ripple in Passband, No Ripple in Stopband
 - Type II has Ripple in Stopband, No Ripple in Passband
 - Shorter Transition bandwidth than Butterworth for given number of poles
- **Elliptic**
 - Has Poles and Zeros, Ripple in both passband and stopband
 - Shorter transition bandwidth than Chebyshev for given number of poles
 - Degraded Phase Response

Frequency Transformations

Prototype Lowpass Filter Function

- The technique used to design analogue filters is to specify a **prototype lowpass filter function** which is normalized to provide a cutoff frequency Ω_c at *1 rad/sec*.
- Then apply transformations to achieve the actual desired cutoff frequencies and filter type.

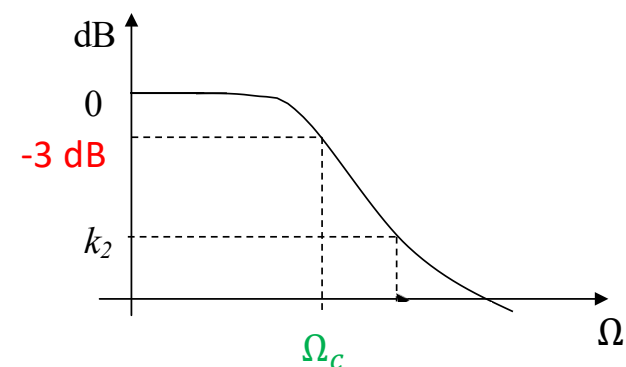


Lowpass to Lowpass Transformation

- To transform analog lowpass filter $H_a(s)$ with unity cutoff frequency to lowpass filter $H_{LP}(s)$ with cutoff frequency Ω_c , we substitute

$$s \rightarrow \frac{s}{\Omega_c}$$

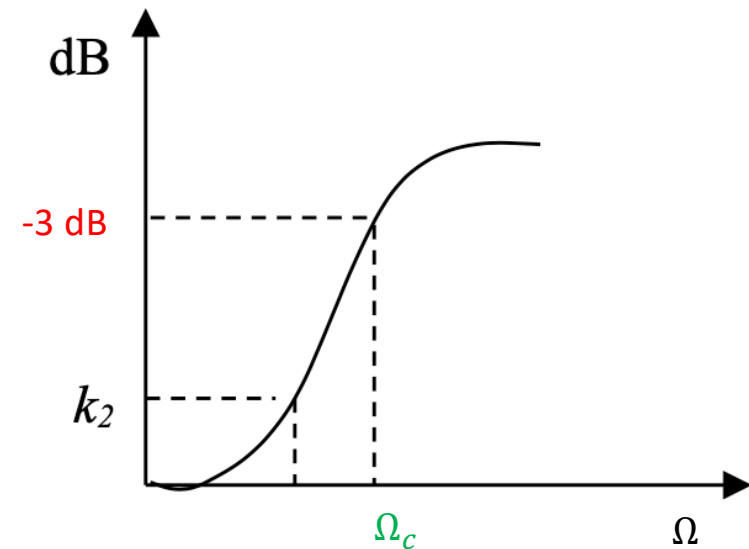
Transformed Filter Response



Lowpass to Highpass Transformation

- To transform analog lowpass filter $H_a(s)$ with unity cutoff frequency to **highpass filter** $H_{HP}(s)$ with cutoff frequency Ω_c , we substitute

$$s \rightarrow \frac{\Omega_c}{s}$$



Highpass Filter Transformation Example 1

- First order Butterworth Lowpass prototype filter is given by

$$H_{LP}(s) = \frac{1}{s+1}$$

- To transform to highpass filter with cutoff frequency $\Omega_c = 5$, we replace s with $\frac{\Omega_c}{s}$

$$H_{HP}(s) = H_{LP}(s) \Big|_{s=\frac{\Omega_c}{s}} = \frac{1}{\frac{\Omega_c}{s}+1} = \frac{s}{s+\Omega_c} = \frac{s}{s+5}$$

Note: Highpass filter contains zeros as well as poles.

Highpass Filter Transformation Example 2

- Determine the transfer function $H_{HP}(s)$ of a 3rd order Butterworth highpass filter with cutoff frequency $\Omega_c = 1$ (normalized form)

$$H_a(s) = \frac{1}{(s+1)(s^2+s+1)}$$

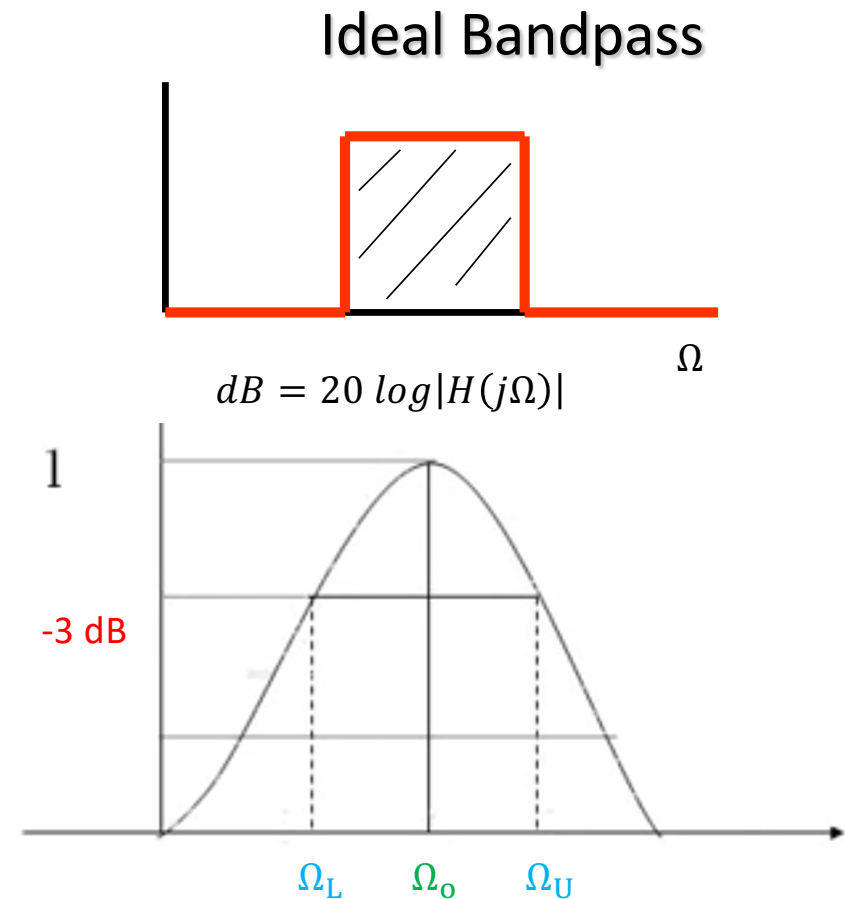
3rd order Butterworth low-pass filter

$$s \rightarrow \frac{\Omega_c}{s} \Rightarrow \frac{1}{s}$$

$$H_{HP}(s) = H_{LP}(s) \Big|_{s=\frac{1}{s}} = \frac{1}{\left(\frac{1}{s}+1\right)\left(\left(\frac{1}{s}\right)^2 + \frac{1}{s} + 1\right)} = \frac{s^3}{(s+1)(s^2+s+1)}$$

Lowpass to Bandpass Transformations (1)

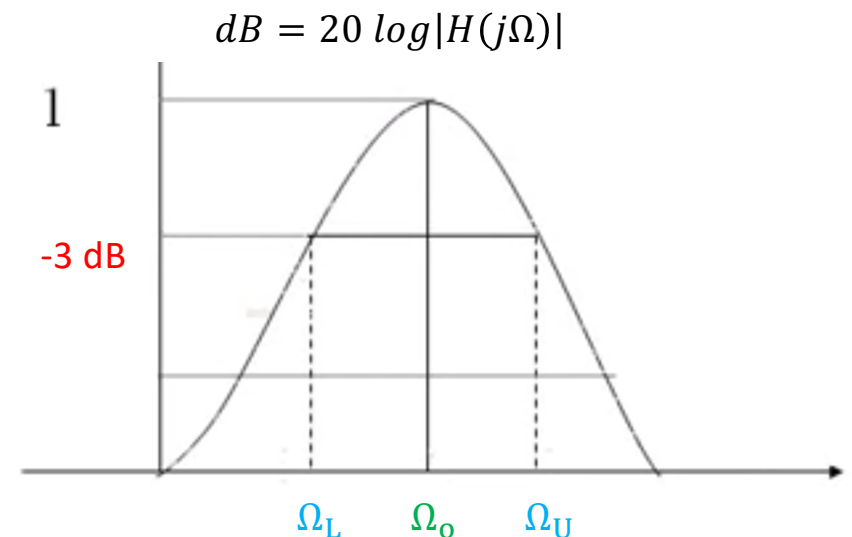
- By definition, a bandpass filter **rejects both low and high frequency** components and passes a certain band of frequencies some where between them
- Thus, the properties of bandpass filter frequency response $|H_{BP}(j\Omega)|$ are
 - $|H_{BP}(j\Omega)| = 0$ at both $\Omega = 0$ and $\Omega = \infty$
 - $|H_{BP}(j\Omega)| = 1$ for a frequency band centered at Ω_o , where Ω_o is the mid frequency of the filter.



Lowpass to Bandpass Transformations (2)

- To convert lowpass filter $H_a(s)$ with unity cutoff frequency into a **Bandpass filter** $H_{BP}(s)$ with **lower cutoff frequency** Ω_L and the **upper cutoff frequency** Ω_U , we replace

$$s \rightarrow \frac{s^2 + \Omega_L \Omega_U}{s(\Omega_U - \Omega_L)}$$



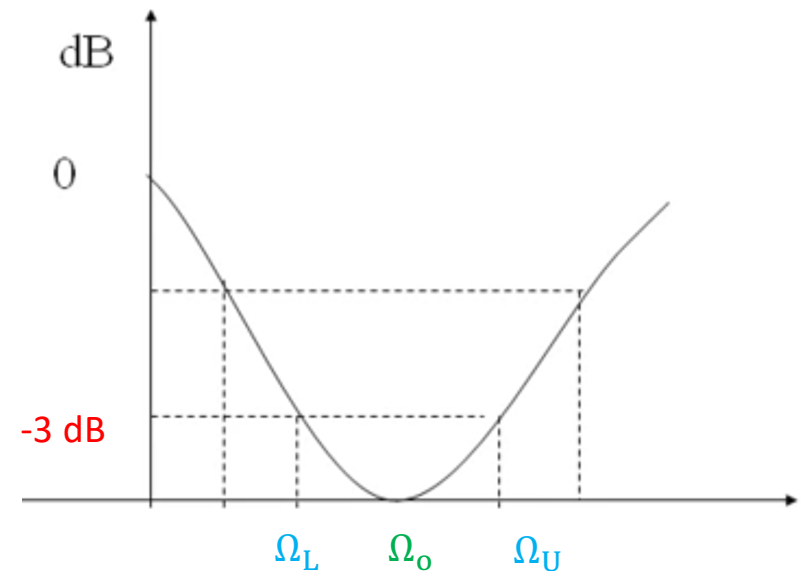
$$B = \Omega_U - \Omega_L$$

Bandwidth of
the bandpass filter

Lowpass to Bandstop Transformation

- Similarly to convert a lowpass filter $H_a(s)$ with unity cutoff frequency into a **bandstop filter** $H_{BS}(s)$ with **lower cutoff frequency** Ω_L and **upper cutoff frequency** Ω_U , we replace:

$$s \rightarrow \frac{s(\Omega_U - \Omega_L)}{s^2 + \Omega_L\Omega_U}$$



Another Transformation

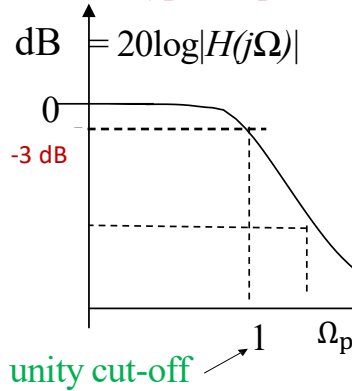
- A lowpass to bandpass transformation can also be performed by

$$s \rightarrow \frac{s^2 + \Omega_o^2}{B s}$$

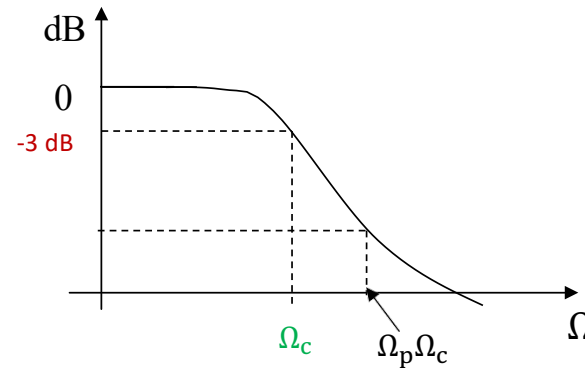
- B – Bandwidth of the band-pass filter ($\Omega_U - \Omega_L$)
- Ω_o – Center frequency

Summary: Analog to Analog Transformation

Butterworth Prototype Response

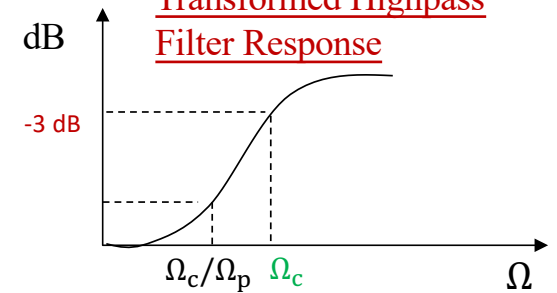


Transformed Lowpass Filter Response



$$s \rightarrow \frac{s}{\Omega_c}$$

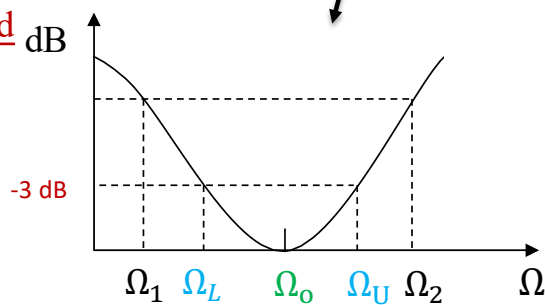
Transformed Highpass Filter Response



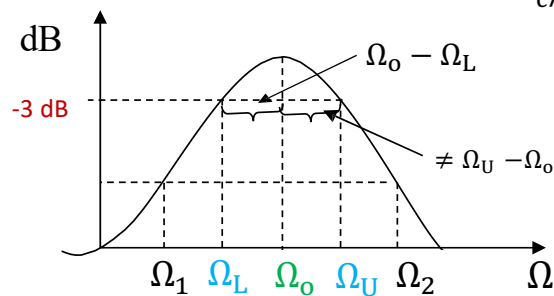
$$s \rightarrow \frac{\Omega_c}{s}$$

$$s \rightarrow \frac{s(\Omega_U - \Omega_L)}{s^2 + \Omega_L \Omega_U}$$

Transformed Bandstop Filter Response



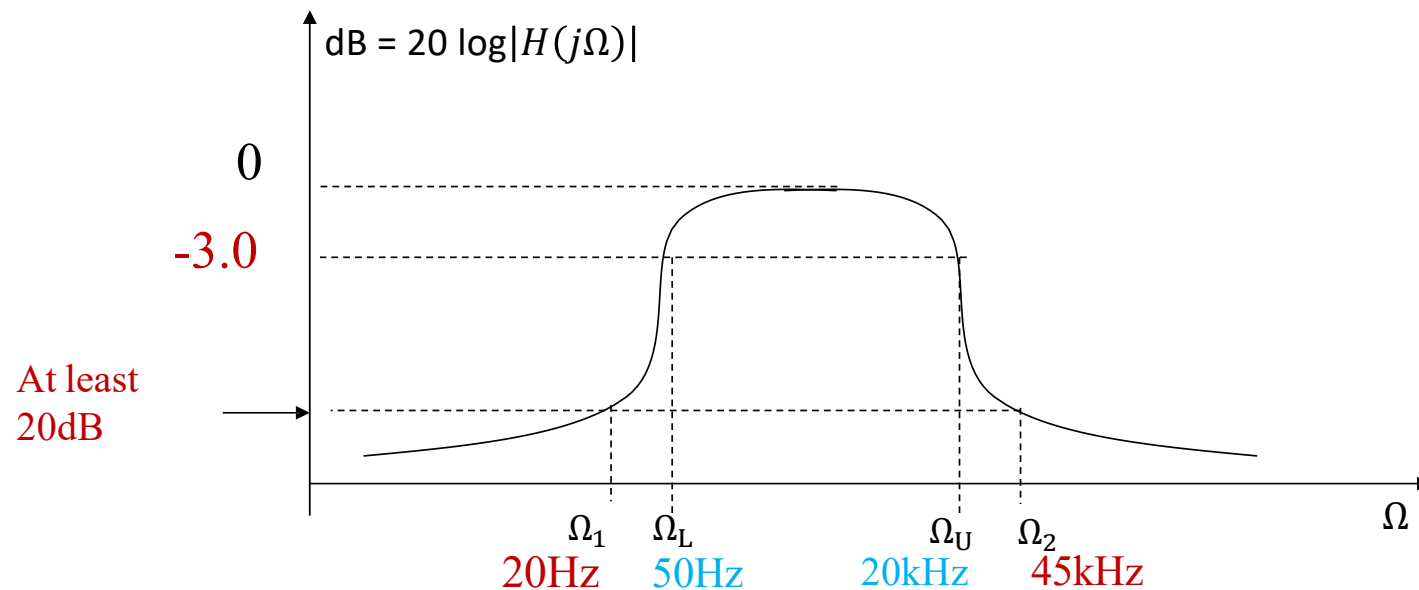
$$s \rightarrow \frac{s^2 + \Omega_L \Omega_U}{s(\Omega_U - \Omega_L)}$$



Transformed Bandpass Filter Response

Analog Bandpass Filter Design Example

- Design an **analog bandpass filter** with the following characteristics
 - A monotonic frequency response.
 - **-3.0 dB** upper and lower cutoff frequency of **50Hz** and **20kHz**
 - A stopband attenuation of at least **20dB** at **20Hz** and **45 kHz**



Example Solution

- The monotonic requirement can be satisfied with a **Butterworth filter**
 - $\Omega_1 = 2\pi(20) = 125.663 \text{ rad / sec}$
 - $\Omega_2 = 2\pi(45,000) = 2.82743 \times 10^5 \text{ rad / sec}$
 - $\Omega_L = 2\pi(50) = 314.159 \text{ rad / sec}$
 - $\Omega_U = 2\pi(20,000) = 1.25663 \times 10^5 \text{ rad / sec}$
- Lowpass \rightarrow Bandpass Transformation is

$$s \rightarrow \frac{s^2 + \Omega_L \Omega_U}{s(\Omega_U - \Omega_L)}$$

- For bandpass filter to satisfy the stopband attenuation requirement at Ω_1 , we must have equality within the transformation. i.e.

$$j\Omega_c = \frac{(j\Omega_1)^2 + \Omega_L\Omega_U}{j\Omega_1(\Omega_U - \Omega_L)} \Rightarrow \Omega_c = \frac{\Omega_1^2 + \Omega_L\Omega_U}{\Omega_1(\Omega_U - \Omega_L)} = 2.5053$$

- Substitute values for Ω_1 , Ω_L , and Ω_U .
- Similarly solving Ω_1 , to satisfy stopband attenuation requirement at Ω_2

$$j\Omega_c = \frac{(j\Omega_2)^2 + \Omega_L\Omega_U}{j\Omega_2(\Omega_U - \Omega_L)} \Rightarrow \Omega_c = \frac{\Omega_2^2 + \Omega_L\Omega_U}{\Omega_2(\Omega_U - \Omega_L)} = 2.2545$$

- We choose $\Omega_c = \min\{|2.5053|, |2.2545|\} = \underline{2.2545}$
- First design a lowpass filter $H(s)$ with Ω_c and then apply transformation (LP to BP) using Ω_U and Ω_L to obtain BP filter.

- The lowpass filter order N can be calculated as

$$N = \log_{10} \left[\frac{10^{\frac{3.0}{20}} - 1}{10^{\frac{20}{20}} - 1} \right] / 2 \log_{10} \left[\frac{1}{2.2545} \right] = 2.829$$

- Therefore, $N = 3$ should be chosen. Then, the transfer function of the 3rd LP filter is

$$H_{LP}(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

- LP \rightarrow BP

$$s \rightarrow \frac{s^2 + \Omega_L \Omega_U}{s(\Omega_U - \Omega_L)} = \frac{s^2 + 3.94784 \times 10^7}{s(1.25349 \times 10^5)}$$

- The BP transfer function

$$H_{BP}(s) = \frac{1}{\left[\frac{s^2 + 3.94784 \times 10^7}{s(1.25349 \times 10^5)} \right]^3 + 2 \left[\frac{s^2 + 3.94784 \times 10^7}{s(1.25349 \times 10^5)} \right]^2 + 2 \left[\frac{s^2 + 3.94784 \times 10^7}{s(1.25349 \times 10^5)} \right] + 1}$$

Python: Analog Bandpass Filter Design

```
from scipy import signal
import numpy as np
import matplotlib.pyplot as plt

b, a = signal.butter(3, [4000*2*np.pi, 5000*2*np.pi], btype='band', analog=True)
w, h = signal.freqs(b, a)

print("b = ", b)
print("a = ", a)

# plt.plot(w, 20 * np.log10(abs(h)))
plt.plot(w, abs(h))
plt.title('3rd Butterworth Bandpass Filter Frequency Response')
plt.xlabel('Frequency [radians / second]')
plt.ylabel('Magnitude [dB]')
plt.margins(0, 0.1)
plt.grid(which='both', axis='both')
plt.show()

b = [2.48050213e+11 0.00000000e+00 0.00000000e+00 0.00000000e+00]
a = [1.00000000e+00 1.25663706e+04 2.44766189e+09 2.00920673e+13
     1.93259637e+18 7.83410393e+21 4.92231267e+26]
```

