IIR Filter Design

EE4015 Digital Signal Processing

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IIR Digital Filter Design Methods

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    - $H_a(s) \rightarrow h(t) \xrightarrow{sampling} h[n] \rightarrow H(z)$
  - **Bilinear Transformation**
    - $H_a(s) \xrightarrow{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} H(z)$
- **Digital-to-Digital Transformations**
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    - $H(z) \xrightarrow{z^{-1}=\frac{z^{-1}-a}{1-az^{-1}}} H(z)$
  - **Pole-zero placement method**

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_{M-1} z^{M-1} + b_M z^M}{a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{N-1} + b_N z^N}$$

Analog-to-Digital Filter Transformation

- Some of the IIR filters design methods are to apply a transformation to an **existing analog filter** to obtain the digital filter transfer functions, such as

$$H_a(s) = \frac{1}{s-a} \xleftrightarrow{Transformation} H(z) = \frac{1}{1-bz^{-1}}$$

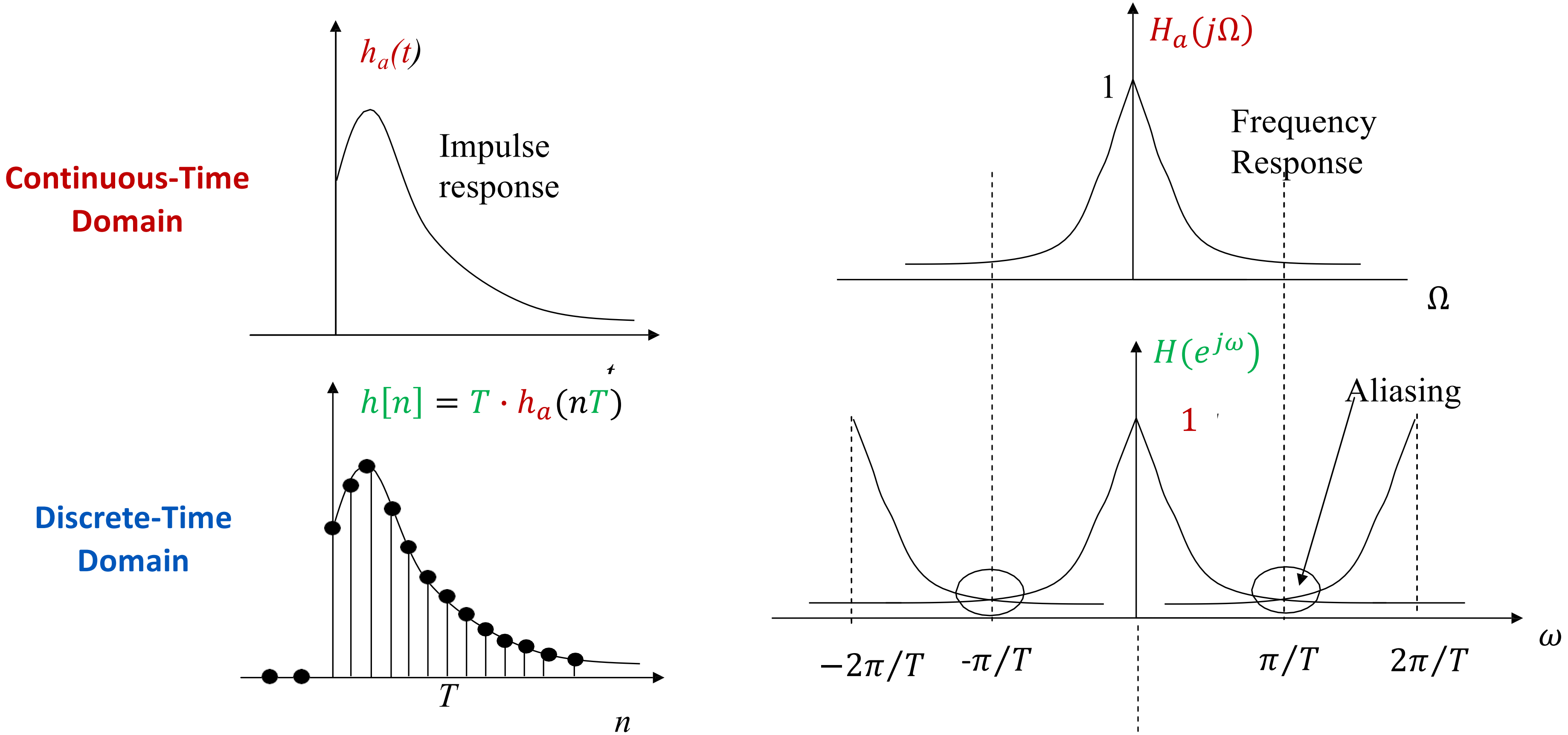
- Their common feature is that a **stable analog filter will transform to a stable discrete-time system** with transfer function $H(z)$.
- Left half of s-plane maps into inside of unit circle in z-plane

Impulse Invariance Method

Impulse Invariance Method

- This method starts from an analog filter's transfer function $H_a(s)$ to obtain its impulse response $h_a(t)$ by inverse Laplace Transform.
- The objective of the design is to realize an IIR filter with an impulse response $h[n]$ which satisfies :
  - $h[n] = T \cdot h_a(nT)$ where T is the sampling period $(T = 1/F_s)$
- Finally, use z-Transform to obtain $H(z)$ with filter coefficients from the sampled impulse response $h[n]$.
- The main feature of this method is that the impulse response $h[n]$ of the resulting digital filter is a sampled version of the impulse response $h_a(t)$ of the analog filter.

Sampling of the Impulse Response



- The problem of Impulse Invariance is sensitivity to the choice of T. Too large T will **create aliasing problem** to distorting the frequency response of the IIR digital Filter.

Spectrum of the Sampled Impulse Response

- With the use of convolution property, $H(e^{j\omega})$ is

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_a\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

- Where the analog and digital frequencies

$$\omega = \Omega T$$

- The impulse response of the resultant IIR filter is similar to that of the analog filter
- Aliasing due to the overlapping of $\left\{H_a\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)\right\}$ which are not bandlimited.

Impact of the Sampling Frequency

- **The sampling frequency affects the frequency response of the impulse invariant discrete filter.**
- A sufficient high sampling frequency is necessary for the frequency response to be close to that of the equivalent analog filter.
- Due to aliasing, therefore, the frequency response of the digital filter will **not be identical to that of the analog filter**.

Derive Transfer Function $H(z)$ (1)

- To derive the IIR filter transfer function $H(z)$ from $H_a(s)$, we first obtain the **partial fraction expansion**:

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - c_k}$$

  where $\{c_k\}$ are the poles on the left half of the s-plane

- The inverse Laplace transform of $H_a(s)$ is given as:

$$h_a(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{c_k t}, & t \geq 0 \\ 0, & t < 0 \end{cases} = \left(\sum_{k=1}^{N} A_k e^{c_k t} \right) u(t)$$

Derive Transfer Function $H(z)$ (2)

- Sample and scale $h_a(t)$ with period T to obtain $h[n]$:

$$h[n] = \sum_{k=1}^{N} T A_k e^{c_k T n} u[n]$$

- The z-transform of $h[n]$ is

$$H(z) = \sum_{k=1}^{N} \frac{T A_k}{1 - e^{c_k T} z^{-1}}$$

Relationship between $H_a(s)$ and $H(z)$

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - c_k} \quad \Rightarrow \quad H(z) = \sum_{k=1}^{N} \frac{TA_k}{1 - e^{c_k T} z^{-1}}$$

- It can be seen that a pole of $s = c_k$ in the s-plane transforms to a pole at $z = e^{c_k T}$ in the z-plane:

$$\boxed{z = e^{sT}}$$

- Expressing $s = \sigma + j\Omega$

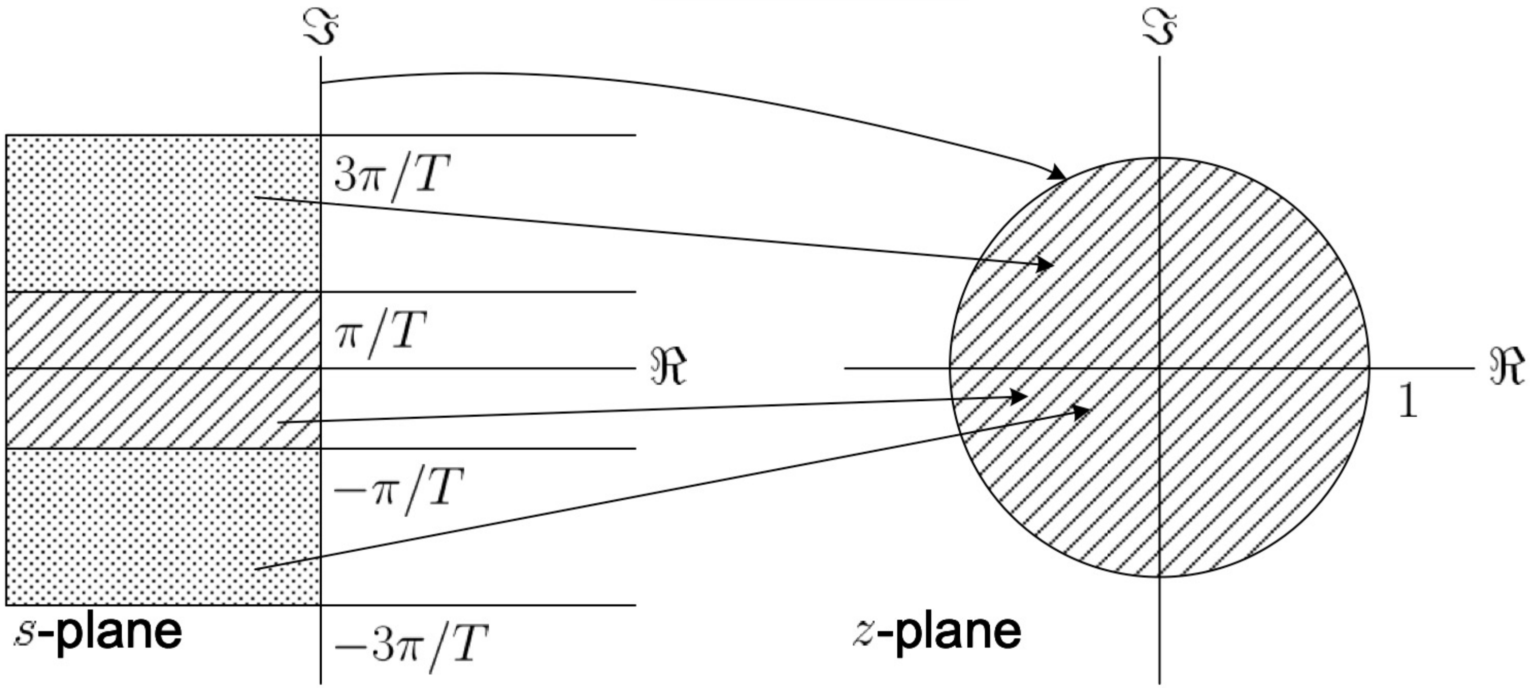
$$z = e^{\sigma T} \cdot e^{j\Omega T} = e^{\sigma T} \cdot e^{j(\Omega + 2\pi k/T)T}$$

where k is any integer, indicating a many-to-one mapping

- Each infinite horizontal strip of with $2\pi/T$ maps into the entire z-plane

Mapping between s and z in Impulse Invariance Method

$$z = e^{sT}$$



Stability of Impulse Invariance Method

- $\sigma = 0$ maps to $|z| = 1$, that is, $j\Omega$ axis in the s-plane transforms to the unit circle in the z-plane
- $\sigma < 0$ maps to $|z| < 1$, left half of the s-plane maps into the inside of the unit circle in the z-plane. Then, **stable** $H_a(s)$ produces **stable** $H(z)$
- $\sigma > 0$ maps to $|z| > 1$, right half of the s-plane maps into the outside of the unit circle in the z-plane

Procedure of Impulse Invariance Method

Given the magnitude square response specifications of $H(e^{j\omega})$ in terms of ω_p, ω_s, R_p and A_s, the design procedure for $H(z)$ can be summarized as :

1. Select a value for the sampling interval T and then compute the passband and stopband frequencies for the analog lowpass filter according to $\Omega_p = \omega_p/T$ and $\Omega_s = \omega_s/T$
2. Design the analog Butterworth filter with transfer function $H_a(s)$ according to Ω_p, Ω_s, R_p and A_s
3. Perform partial fraction expansion on $H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s-c_k}$
4. Obtain $H(z)$ by

$$H(z) = \sum_{k=1}^{N} \frac{T A_k}{1 - e^{c_k T} z^{-1}}$$

Impulse Invariance Method Example 1

$$H(s) = \frac{1}{(s+1)(s+3)}$$

- Using partial fraction this can be written as

$$H(s) = \frac{0.5}{s+1} - \frac{0.5}{s+3}$$

$$\downarrow \quad \frac{A}{s-c} \rightarrow \frac{AT}{1 - e^{cT} z^{-1}}$$

$$H(z) = \frac{0.5T}{1 - e^{-T} z^{-1}} - \frac{0.5T}{1 - e^{-3T} z^{-1}} = \frac{0.5T(e^{-T} - e^{-3T}) z^{-1}}{1 - (e^{-T} - e^{-3T}) z^{-1} + e^{-4T} z^{-2}}$$

Impulse Invariance Method Example 2

- Determine the transfer function $H(z)$ of a digital lowpass filter whose magnitude requirements are $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $R_p = 8\ dB$ and $A_s = 16\ dB$.
- Use the Butterworth lowpass filter and Impulse Invariance Method with sampling interval $T = 0.1$ in the design.

Example 2 Solution

- With $T = 0.1$, the analog frequency parameters can be determined as

$$\Omega_p = \frac{\omega_p}{T} = 4\pi \quad \text{and} \quad \Omega_s = \frac{\omega_s}{T} = 6\pi$$

- Based on the example 2 in analog filter design results with $\Omega_c = 10$, we know the 3rd order Butterworth filter can meets the magnitude requirements and its transfer function is

$$H_a(s) = \frac{1}{(s^2 + s + 1)(s + 1)}\bigg|_{s=\frac{s}{\Omega_c}} = \frac{1000}{(s^2 + 10s + 100)(s + 10)}$$

- Performing partial fraction expansion on $H_a(s)$, we have

$$H_a(s) = \frac{10}{s + 10} + \frac{-5 - j2.8868}{s + 5 - j8.6603} + \frac{-5 + j2.8868}{s + 5 + j8.6603}$$

- Using the mapping of $H(z)=\sum_{k=1}^{N}\frac{TA_k}{1-e^{c_kT}z^{-1}}$, the transfer function of IIR filter is

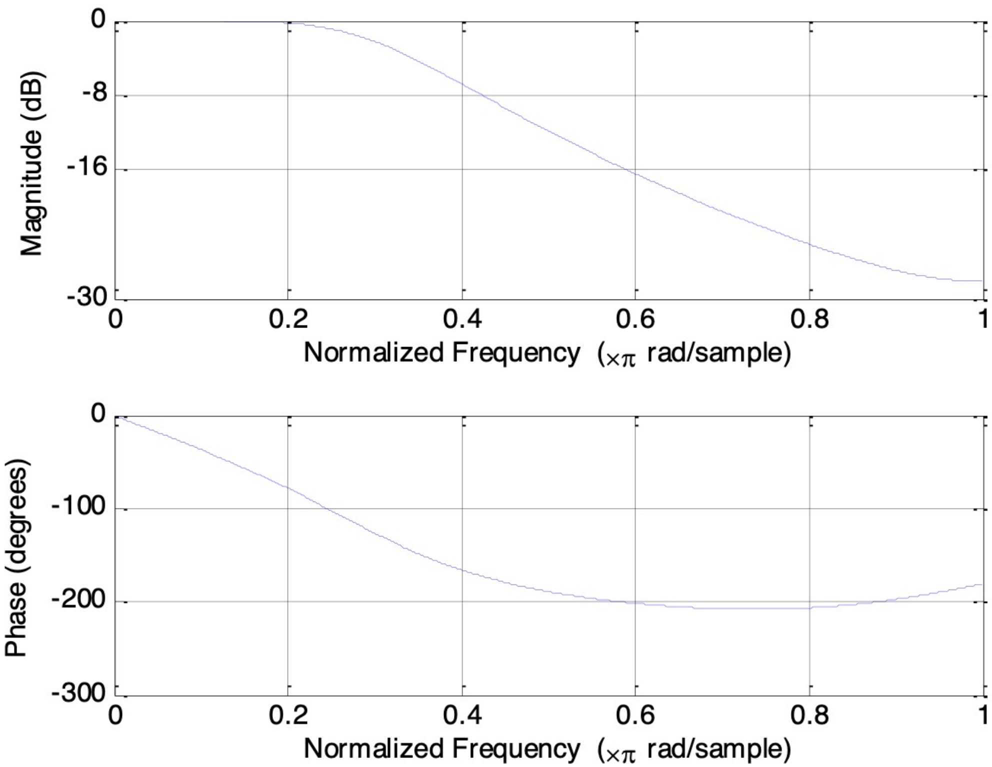
$$H(z)=\frac{0.1\cdot 10}{1-e^{-10\cdot 0.1}z^{-1}}+\frac{0.1\cdot(-5-j2.8868)}{1-e^{(-5+j8.6603)\cdot 0.1}z^{-1}}+\frac{0.1\cdot(-5+j2.8868)}{1-e^{(-5-j8.6603)\cdot 0.1}z^{-1}}$$

$$=\frac{1}{1-0.3679z^{-1}}+\frac{-0.5-j0.2887}{1-(0.3929+j0.4620)z^{-1}}+\frac{-0.5+j0.2887}{1-(0.3929-j0.4620)z^{-1}}$$

$$=\frac{1}{1-0.3679z^{-1}}+\frac{-1+0.6597z^{-1}}{1-0.7859z^{-1}+0.3679z^{-2}}$$

$$=\frac{0.2417z^{-1}+0.1262z^{-2}}{1-1.1538z^{-1}+0.6570z^{-2}-0.1354z^{-3}}$$

Magnitude and phase responses based on impulse invariance Method

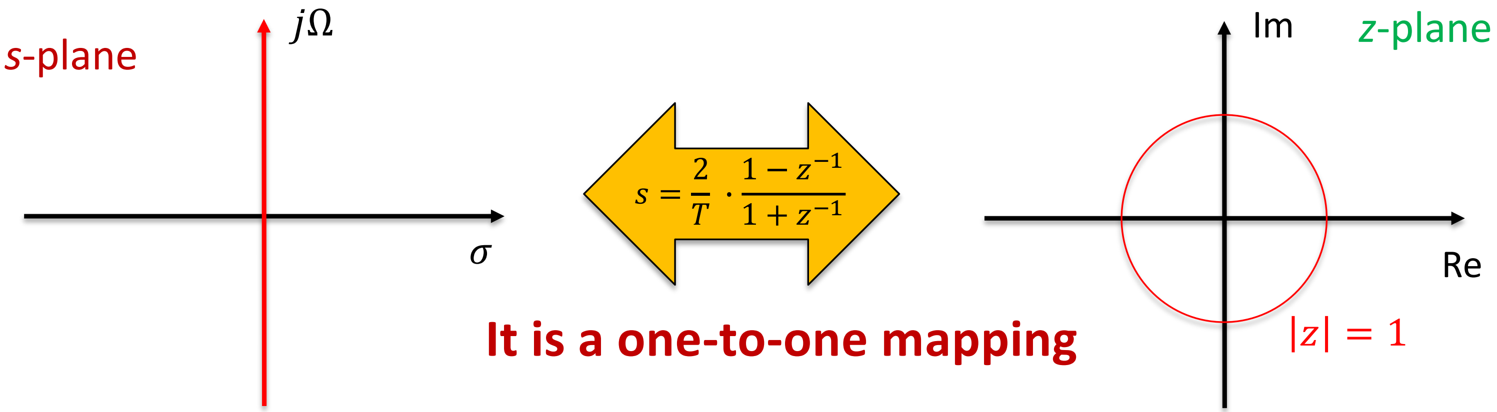


Bilinear Transformation Method

Bilinear Transformation Method

- Map continuous-time systems to discrete-time system by relating s and z

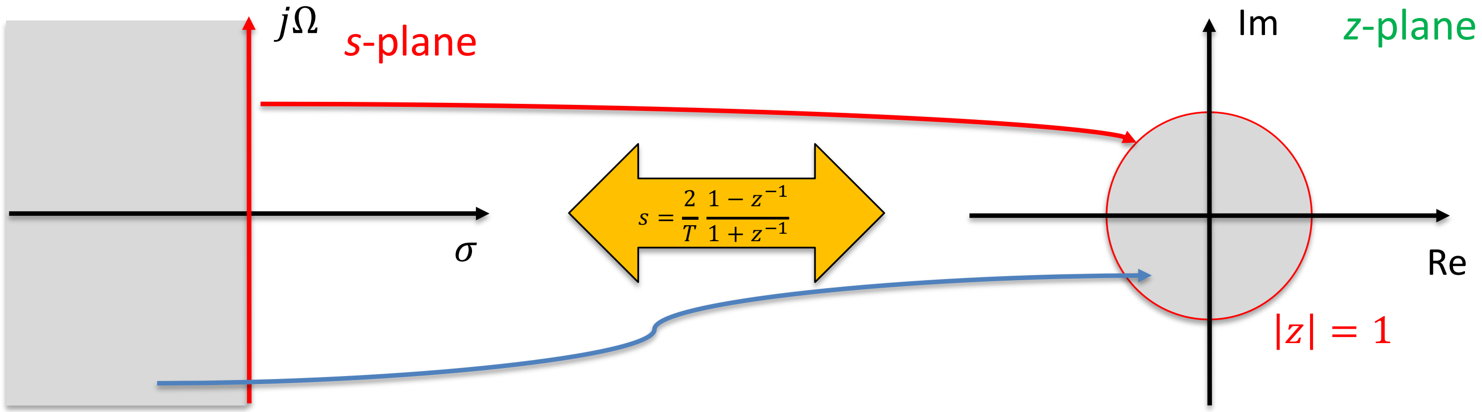
$$z = e^{sT} = \frac{e^{\frac{sT}{2}}}{e^{\frac{-sT}{2}}} = \frac{1 + \frac{sT}{2} + \left(\frac{sT}{2}\right)^2 \cdot \frac{1}{2} + \cdots}{1 - \frac{sT}{2} + \left(\frac{sT}{2}\right)^2 \cdot \frac{1}{2} + \cdots} \Rightarrow z \approx \frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}} \Rightarrow \boxed{s = \frac{2}{T} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}}$$



It is a one-to-one mapping

Bilinear Transformation Property 1

- Left half of s-plane (σ<0) maps to interior of unit circle in the z-plane



$$z = \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}} = \frac{1+\frac{T}{2}\sigma + j\frac{T}{2}\Omega}{1-\frac{T}{2}\sigma - j\frac{T}{2}\Omega}$$

$$if \begin{matrix} \sigma < 0, & |z| < 1 \\ \sigma > 0, & |z| > 1 \end{matrix}$$

- Stable Continuous-Time Systems ⇔ Stable Discrete-time Systems

Bilinear Transformation Property 2

- $j\Omega$ axis maps to unit circle via a nonlinear warping

$$z=\frac{1+\frac{sT}{2}}{1-\frac{sT}{2}}\xrightarrow{s=j\Omega} z=\frac{1+j\frac{T}{2}\Omega}{1-j\frac{T}{2}\Omega}\xrightarrow{|z|=1} e^{j\omega}=\frac{1+j\frac{T}{2}\Omega}{1-j\frac{T}{2}\Omega}$$

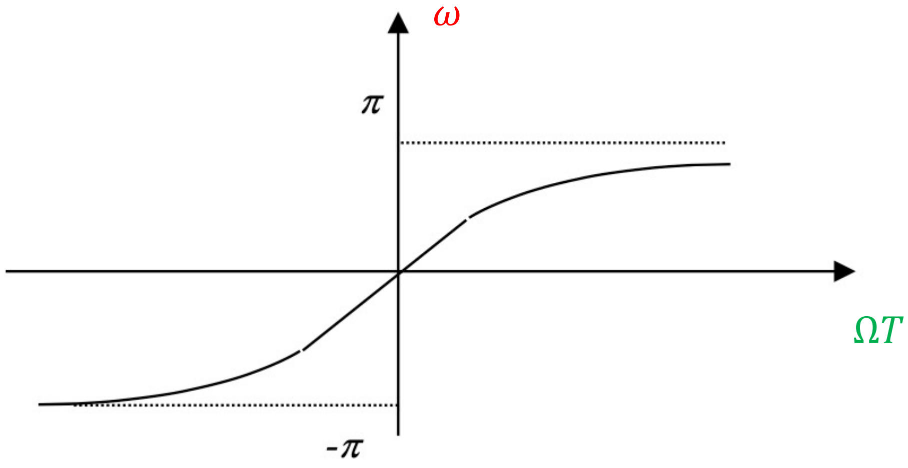
=> $$j\Omega=\frac{2}{T}\frac{1-e^{-j\omega}}{1+e^{-j\omega}}=\frac{2}{T}\frac{2je^{-\frac{j\omega}{2}}\sin\frac{\omega}{2}}{2je^{-\frac{j\omega}{2}}\cos\frac{\omega}{2}}=j\frac{2}{T}\tan\frac{\omega}{2}$$

Then, we have

$$\Omega=\frac{2}{T}\tan\frac{\omega}{2}$$

and

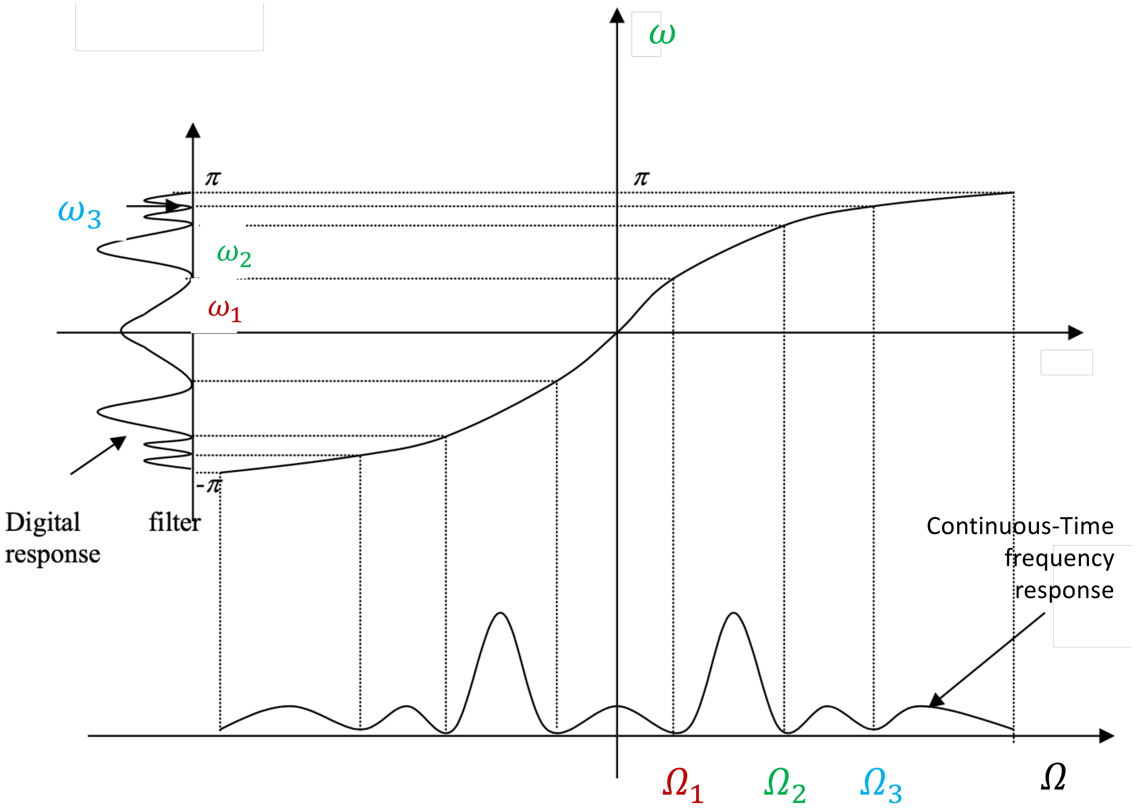
$$\omega=2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$



We see that a nonlinear relation exists between Ω and ω.
This effect is called ‘Warping’.

Pro and Con of Frequency Warping

- **No aliasing of the frequency characteristic** can occur in the transformation of an analog filter to a discrete filter
- We must however **check carefully just how the various characteristic frequencies** of the continuous characteristic frequencies of the discrete filter.
- In designing a digital filter by this method, we must **first prewarp** the given filter specifications to find the continuous filter to which we are going to apply the bilinear transformation.



Procedure of Bilinear Method

Given the magnitude square response specifications of $H_{LP}(e^{j\omega})$ in terms of ω_p, ω_s, R_p and A_s, the design procedure for $H(z)$ can be summarized as :

1. Select a value for the sampling interval T and then compute the passband and stopband frequencies for the analog lowpass filter according to $\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2}$ and $\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2}$
2. Design the analog filter with transfer function $H_a(s)$ according to Ω_p, Ω_s, R_p and A_s
3. Obtain $H(z)$ from $H_a(s)$ by the bilinear transformation

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

Bilinear Transformation Example 1

$$H_{LP}(s) = \frac{2s}{s^2 + 6s + 8}$$

- Using the bilinear transformation with $T = 1$ to transform to a digital filter with transfer function $H(z)$

$$H(z) = \frac{2s}{s^2 + 6s + 8}\bigg|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{2 \cdot 2\frac{1 - z^{-1}}{1 + z^{-1}}}{\left(2\frac{1 - z^{-1}}{1 + z^{-1}}\right)^2 + 6 \cdot 2\frac{1 - z^{-1}}{1 + z^{-1}} + 8}$$

$$= \frac{4 - 4z^{-2}}{15 + 14z^{-1} + 9z^{-2}} = \frac{0.2667 - 40.2667z^{-2}}{1 + 0.933z^{-1} + 0.6z^{-2}}$$

Bilinear Transformation Example 2

- Determine the transfer function $H_{LP}(z)$ of a digital lowpass filter with
  - Sampling frequency F_s : 8kHz
  - Passband edge frequency f_p at 1.6kHz with 8dB maximum passband ripple (R_p),
  - Stopband edge frequency f_s at 2.4kHz with a minimum 16dB stopband attenuation (A_s).
- Use the Butterworth lowpass filter and Bilinear Transformation method with sampling interval $T = 0.1$ in the design.

Example 2 Solution

- Determine the continuous-time critical frequencies Ω_p and Ω_s by frequency warping of the Bilinear Transformation method with $F_s = 8000$ Hz.
- The discrete-time critical frequencies are given by

$$\omega_p = \frac{f_p}{F_s} \cdot 2\pi = \frac{1600}{8000} \cdot 2\pi = 0.4\pi \qquad \omega_s = \frac{f_s}{F_s} \cdot 2\pi = \frac{2400}{8000} \cdot 2\pi = 0.6\pi$$

- Apply frequency pre-warping to obtain the continuous-time critical frequencies with sampling interval $T = 0.1$:

$$\Omega_p = \frac{2}{T} \tan \frac{\omega_p}{2} = \frac{2}{0.1} \tan\left(\frac{0.4\pi}{2}\right) = 14.5309\, rad\ s^{-1}$$

$$\Omega_s = \frac{2}{T} \tan \frac{\omega_s}{2} = \frac{2}{0.1} \tan\left(\frac{0.6\pi}{2}\right) = 27.5276\, rad\ s^{-1}$$

Example 2 Solution

- Using the Butterworth order estimation formula, we have

$$N = \left\lceil \frac{\log_{10}\left[\left(10^{R_p/10} - 1\right)/\left(10^{A_s/10} - 1\right)\right]}{2\log_{10}\left(\Omega_p/\Omega_s\right)} \right\rceil = \left\lceil \frac{\log_{10}\left[\left(10^{8/10} - 1\right)/\left(10^{16/10} - 1\right)\right]}{2\log_{10}(14.5309/27.5276)} \right\rceil = \lceil 1.56 \rceil = 2$$

- With $N = 2$, we can obtain the range of analog cutoff frequency Ω_c as

$$\Omega_c \in \left[\frac{\Omega_p}{\left(10^{\frac{R_p}{10}} - 1\right)^{1/(2N)}}, \frac{\Omega_s}{\left(10^{\frac{A_s}{10}} - 1\right)^{1/(2N)}}\right] = \left[\frac{14.5309}{\left(10^{\frac{8}{10}} - 1\right)^{1/(2\cdot 2)}}, \frac{27.5276}{\left(10^{\frac{16}{10}} - 1\right)^{1/(2\cdot 2)}}\right] = [9.5725, 11.0289]$$

- For simplicity, $\Omega_c = 10$ is employed

Python Code

```
import numpy as np
Rp=8
As=16
Wp= 14.5309
Ws= 27.5276

N=(np.log((pow(10,Rp/10)-1)/(pow(10,As/10)-1)))/(2*np.log(Wp/Ws))

N
1.5566977828103328
```

$$N = \left\lceil \frac{\log_{10}\left[\left(10^{8/10} - 1\right)/\left(10^{16/10} - 1\right)\right]}{2\log_{10}(14.5309/27.5276)} \right\rceil = \lceil 1.56 \rceil = 2$$

```
import numpy as np
Rp=8
As=16
Wp= 14.5309
Ws= 27.5276
N=2
Wc1 = Wp/(pow((pow(10,Rp/10)-1),(1/(2*N))))
Wc2 = Ws/(pow((pow(10,As/10)-1),(1/(2*N))))

Wc1
9.572549171535371
Wc2
11.028855136049335
```

$$\Omega_c \in \left[\frac{14.5309}{\left(10^{\frac{8}{10}} - 1\right)^{1/(2\cdot 2)}}, \frac{27.5276}{\left(10^{\frac{16}{10}} - 1\right)^{1/(2\cdot 2)}}\right] = [9.5725, 11.0289]$$

Normalized Butterworth Filter Transfer Functions

For $\Omega_c = 1$, we have the transfer functions of the Butterworth Filters in different order *N*.

N	Butterworth Transfer Functions
1	$\frac{1}{s+1}$
2	$\frac{1}{s^2+\sqrt{2}s+1}$
3	$\frac{1}{(s^2+s+1)(s+1)}$
4	$\frac{1}{(s^2+0.76536s+1)(s^2+1.84776s+1)}$
5	$\frac{1}{(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)}$

https://en.wikipedia.org/wiki/Butterworth_filter#Normalized_Butterworth_polynomials

- From the Table of the Normalized Butterworth Filter Transfer Functions, we know that the 2nd Order Butterworth Lowpass filter with cutoff frequency $\Omega_c = 1$ is

$$H_{LP}(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

- Then, we map the cutoff frequency of this transfer function to $\Omega_c = 10$ as

$$H_{LP}(s) = \frac{1}{s^2 + \sqrt{2}s + 1}\bigg|_{s=\frac{s}{\Omega_c}} = \frac{1}{\left(\frac{s}{10}\right)^2 + \sqrt{2}\left(\frac{s}{10}\right) + 1} = \frac{100}{s^2 + 10\sqrt{2}s + 100}$$

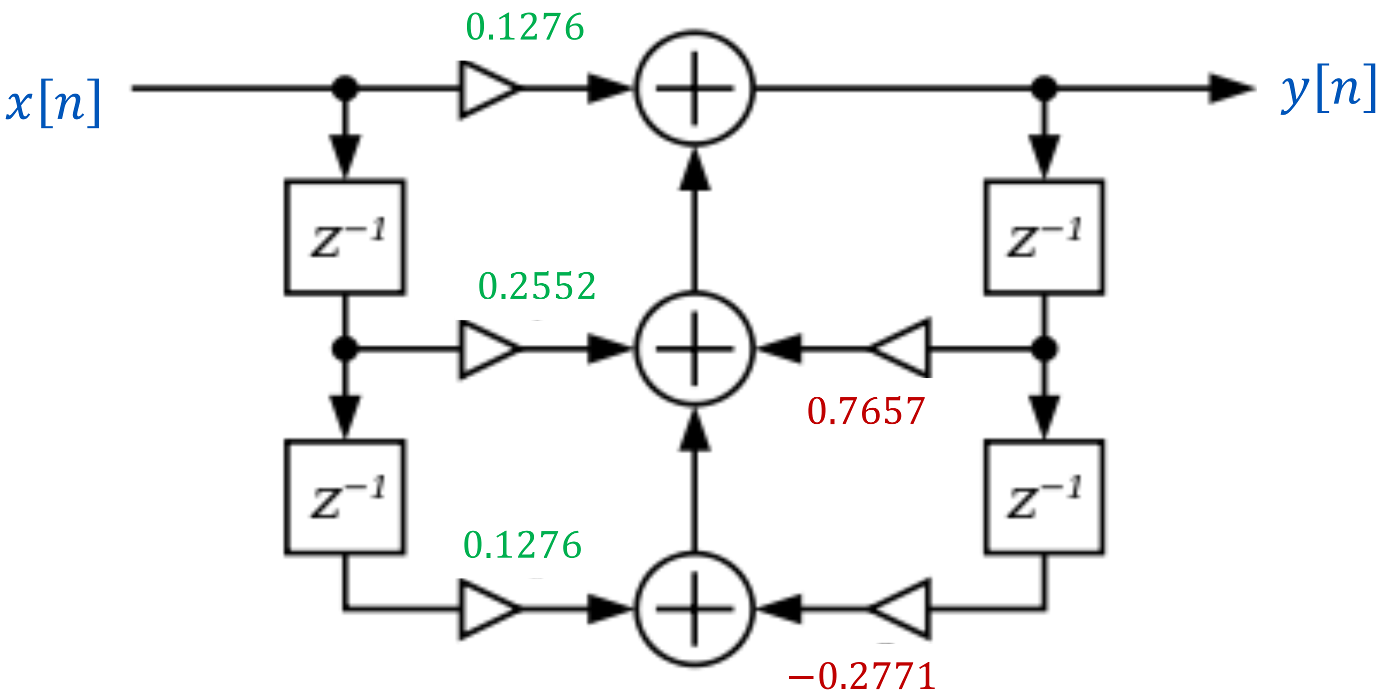
- Finally, we apply the Bilinear Transformation with $T = 0.1$ to obtain $H_{LP}(z)$ as

$$H_{LP}(z) = H_{LP}(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{100}{\left(\frac{2}{0.1}\cdot\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 10\sqrt{2}\cdot\frac{2}{0.1}\cdot\frac{1-z^{-1}}{1+z^{-1}} + 100}$$

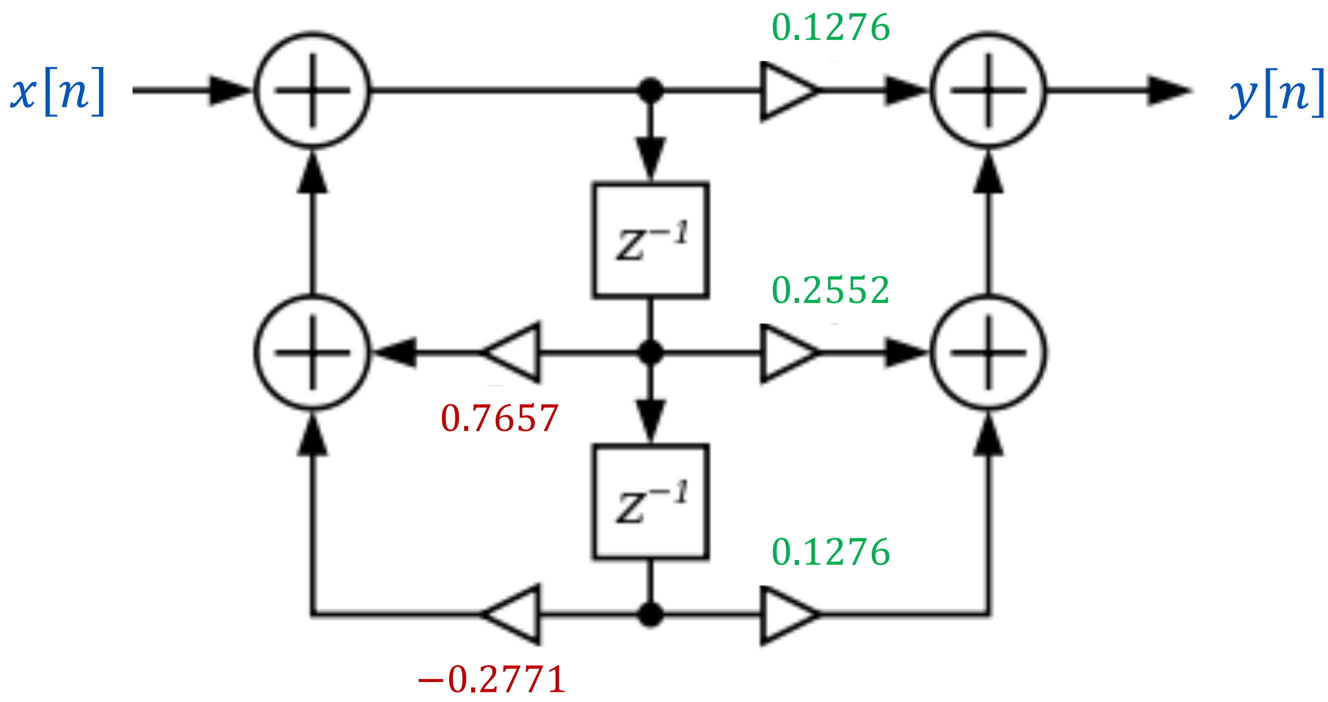
$$= \frac{1 + 2z^{-1} + z^{-2}}{7.8284 - 6z^{-1} + 2.1716z^{-2}}$$

$$= \frac{0.1276 + 0.2552z^{-1} + 0.1276z^{-2}}{1 - 0.7657z^{-1} + 0.2771z^{-2}}$$

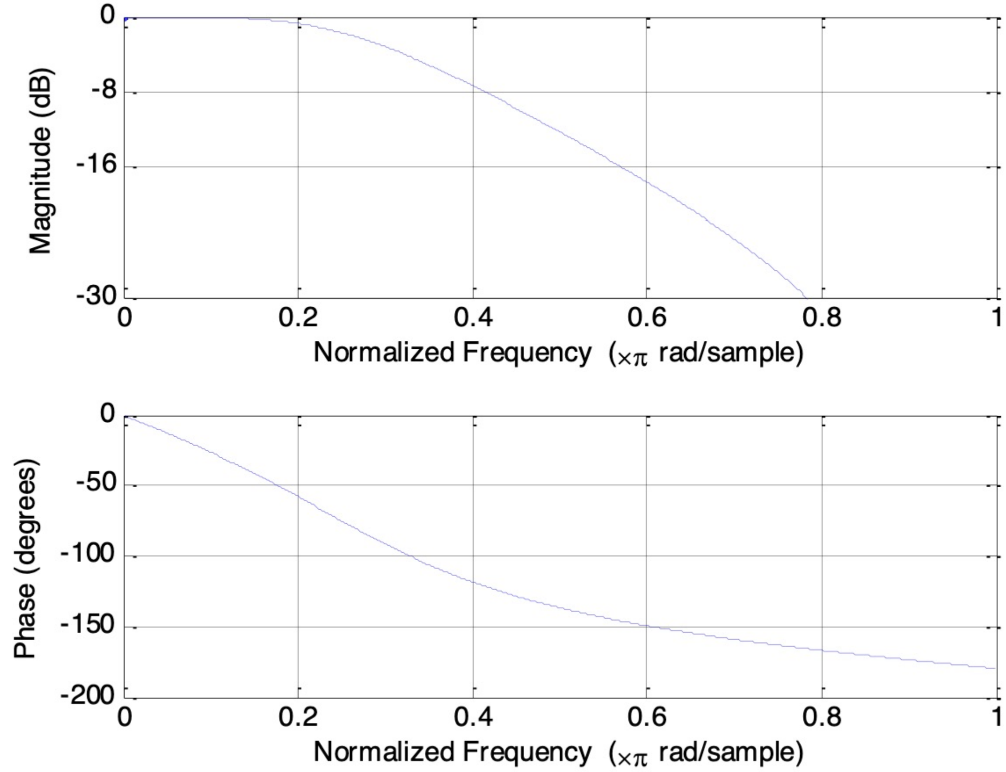
Direct Form Filter Structure Implementation



Canonical Form Filter Structure Implementation



Magnitude and phase responses based on bilinear transformation



Bilinear Transformation Example 3

- Determine the transfer function $H_{HP}(z)$ of a digital highpass filter whose magnitude requirements are $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $R_p = 8\ dB$ and $A_s = 16\ dB$.
- Use the normalized Butterworth lowpass filter and analog to analog transformation to the analog highpass transfer function.
- Finally, use Bilinear Transformation method with sampling interval $T = 0.1$ to obtain the $H_{HP}(z)$.

Example 3 Solution

- Based on the results of the Example 2, we know that
  - The order of the filter is 2
  - The analog cutoff frequency $\Omega_c = 10$
  - From the Table of the Normalized Butterworth Filter Transfer Functions, we know that the 2nd Order Butterworth Lowpass filter with cutoff frequency $\Omega_c = 1$ is

$$H_{LP}(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

- Now, we can apply Analog-to-Analog Transformation to obtain the highpass transfer function $H_{HP}(s)$ as

$$H_{HP}(s) = H_{LP}(s)\Big|_{s=\frac{\Omega_c}{s}} = \frac{1}{\left(\frac{10}{s}\right)^2 + \sqrt{2}\left(\frac{10}{s}\right) + 1} = \frac{s^2}{s^2 + 10\sqrt{2}s + 100}$$

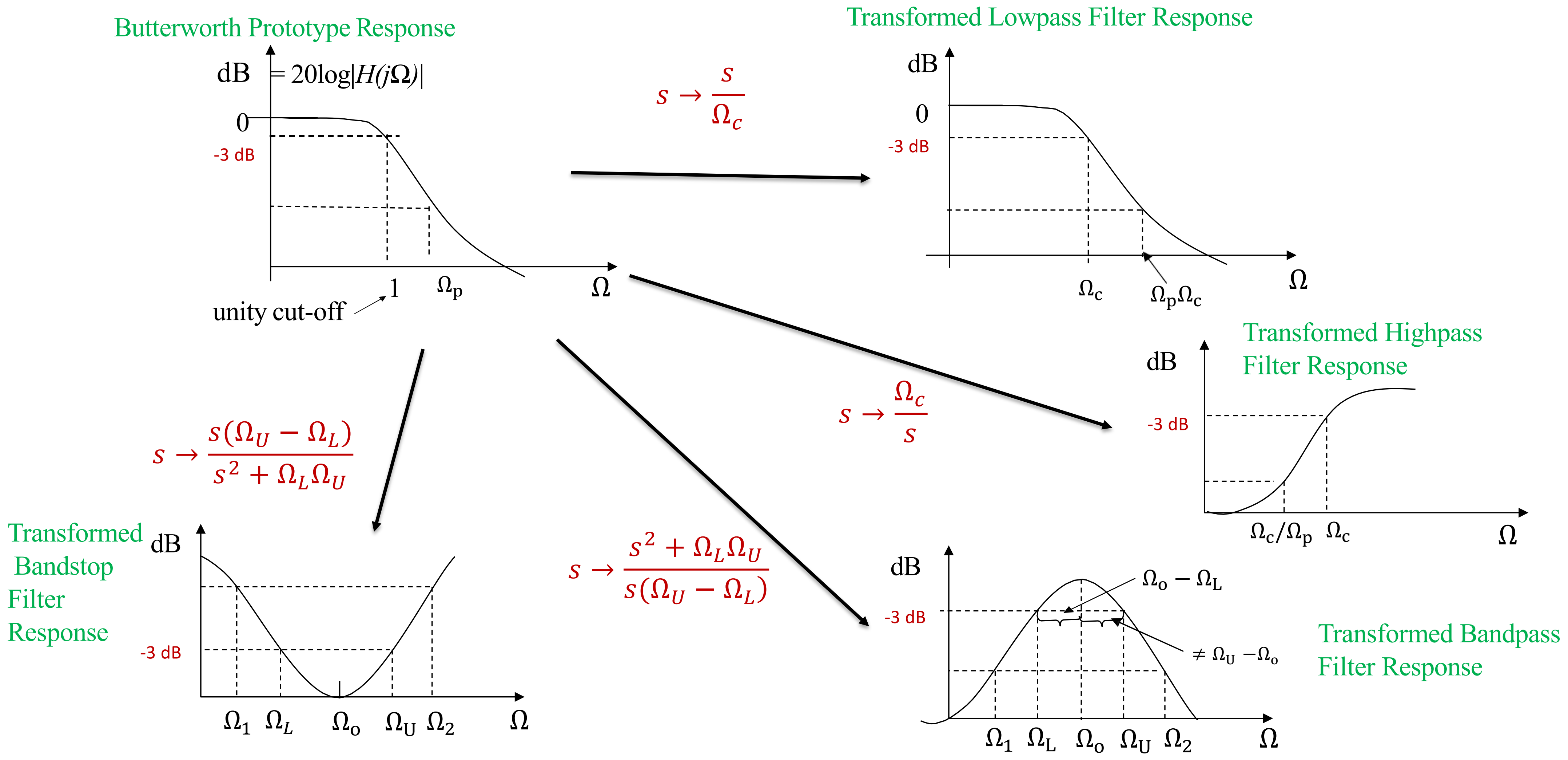
- Finally, we apply the Bilinear Transformation with $T = 0.1$ to obtain $H_{HP}(z)$ as

$$H_{HP}(z) = H_{HP}(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}z}} = \frac{s^2}{s^2 + 10\sqrt{2}s + 100}\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}z}}$$

$$= \frac{\left(\frac{2}{0.1}\cdot\frac{1-z^{-1}}{1+z^{-1}}\right)^2}{\left(\frac{2}{0.1}\cdot\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 10\sqrt{2}\cdot\frac{2}{0.1}\cdot\frac{1-z^{-1}}{1+z^{-1}} + 100} = \frac{1 - 2z^{-1} + z^{-2}}{1.9571 - 1.5z^{-1} + 0.5429z^{-2}}$$

$$= \frac{0.5110 - 1.0219z^{-1} + 0.5110z^{-2}}{1 + 0.7664z^{-1} + 0.2774z^{-2}}$$

Analog to Analog Transformation



Digital Frequency Band Transformations

Digital Frequency Band Transformation

- The operations are like that of the bilinear transformation but now the mapping is performed only in the z-plane:

$$z_o^{-1} = T(z^{-1})$$

where z_o and z correspond to the lowpass and resultant filters, respectively, and $T(\ )$ denotes the transformation operator.

- To ensure the transformed filter to be stable and causal, the unit circle of the z_o-plane should map into those of the z-plane, respectively.

Frequency Band Transformation Operators

Filter Type	Transformation Operator	Design Parameter
Lowpass	$z_o^{-1} = \dfrac{z^{-1} - a}{1 - az^{-1}}$	$a = \dfrac{\sin\left(\dfrac{\omega_{c_o} - \omega_o}{2}\right)}{\sin\left(\dfrac{\omega_{c_o} + \omega_o}{2}\right)}$
Highpass	$z_o^{-1} = -\dfrac{z^{-1} - a}{1 - az^{-1}}$	$a = -\dfrac{\cos\left(\dfrac{\omega_{c_o} + \omega_o}{2}\right)}{\cos\left(\dfrac{\omega_{c_o} - \omega_o}{2}\right)}$
Bandpass	$z_o^{-1} = \dfrac{z^{-2} - \dfrac{2ab}{b+1} z^{-1} + \dfrac{b-1}{b+1}}{\dfrac{b-1}{b+1} z^{-2} - \dfrac{2ab}{b+1} z^{-1} + 1}$	$a = \dfrac{\cos\left(\frac{\omega_{c_2} + \omega_{c_1}}{2}\right)}{\cos\left(\frac{\omega_{c_2} - \omega_{c_2}}{2}\right)}$ $b = \cot\left(\frac{\omega_{c_2} + \omega_{c_1}}{2}\right) \tan\left(\frac{\omega_{c_o}}{2}\right)$
Bandstop	$z_o^{-1} = \dfrac{z^{-2} - \dfrac{2a}{1+b} z^{-1} + \dfrac{1-b}{1+b}}{\dfrac{1-b}{1+b} z^{-2} - \dfrac{2a}{1+b} z^{-1} + 1}$	$a = \dfrac{\cos\left(\frac{\omega_{c_2} + \omega_{c_1}}{2}\right)}{\cos\left(\frac{\omega_{c_2} - \omega_{c_1}}{2}\right)}$ $b = \cot\left(\frac{\omega_{c_2} - \omega_{c_1}}{2}\right) \tan\left(\frac{\omega_{c_o}}{2}\right)$

Frequency Band Transformation Example

- Determine the transfer function $H(z)$ of a digital highpass filter whose magnitude requirements are $\omega_p = 0.6\pi$, $\omega_s = 0.4\pi$, $R_p = 8\ dB$ and $A_s = 16\ dB$.
- Use the Butterworth lowpass filter and bilinear transformation in the design.

Use the Results of the Bilinear Transformation Example 2

- Determine the transfer function $H_{LP}(z)$ of a digital lowpass filter whose magnitude requirements are $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $R_p = 8\ dB$ and $A_s = 16\ dB$.
- Use the Butterworth lowpass filter and Bilinear Transformation method with sampling interval $T = 0.1$ in the design.

Solution

- Using the Example 2 of the Bilinear Transformation, the corresponding lowpass digital filter function $H_{LP}(z)$ is :

$$H_{LP}(z_o) = \frac{0.1276 + 0.2552 z_o^{-1} + 0.1276 z_o^{-2}}{1 - 0.7657 z_o^{-1} + 0.2771 z_o^{-2}}$$

- Assigning the cutoff frequencies as the midpoints between the passband and stopband frequencies, we have

$$\omega_{c_o} = \omega_c = \frac{0.4\pi + 0.6\pi}{2} = 0.5\pi$$

- Using the Table, the corresponding value of *a* is

$$a = -\frac{\cos\left(\frac{\omega_{c_o} + \omega_o}{2}\right)}{\cos\left(\frac{\omega_{c_o} - \omega_o}{2}\right)} = -\frac{\cos(0.5\pi)}{\cos(0)} = 0$$

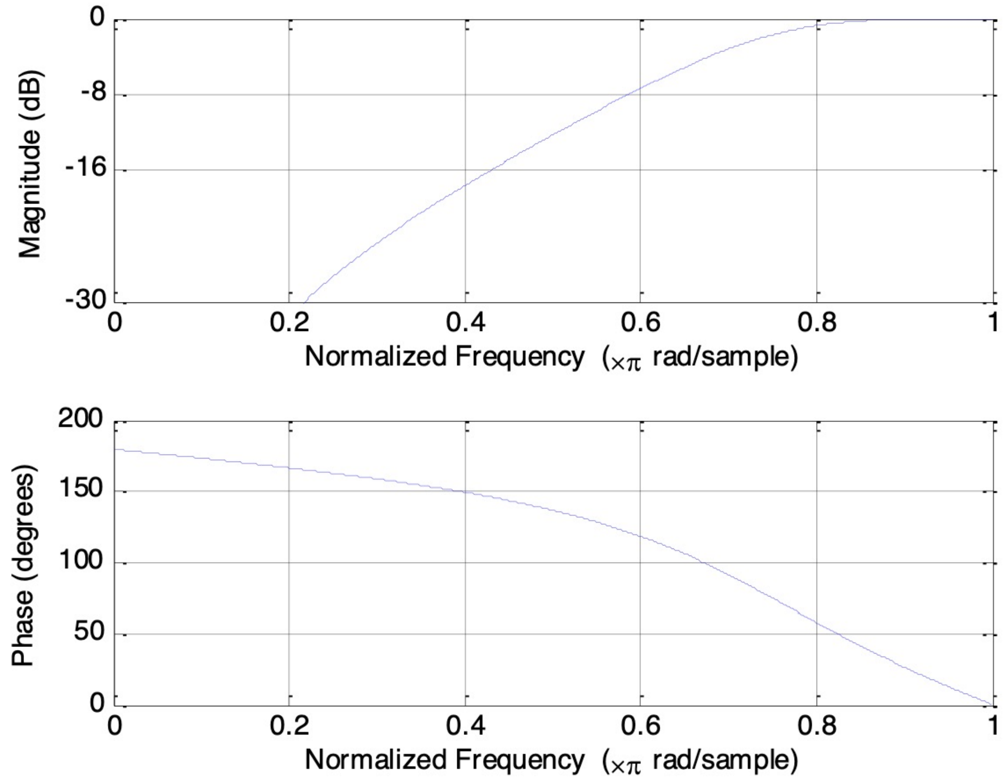
which gives the transformation operator:

$$z_o^{-1} = -\frac{z^{-1} - a}{1 - az^{-1}} = -z^{-1}$$

- As a result, the digital highpass filter transfer function is:

$$H_{HP}(z) = H_{LP}(z_o)\Big|_{z_o^{-1} = -z^{-1}} = \frac{0.1276 - 0.2552z^{-1} + 0.1276z^{-2}}{1 + 0.7657z^{-1} + 0.2771z^{-2}}$$

Magnitude and phase responses based on frequency band transformation



Digital Highpass Butterworth Filter to Remove a Single Tone from a Signal

- Generate a signal made up of 10 Hz and 20 Hz, sampled at 1 kHz.
- Design a digital highpass filter at 15 Hz to remove the 10 Hz tone and apply it to the signal.
- It's recommended to use second-order sections (SOS) format when filtering, to avoid numerical error with transfer function (ba) format)

```
t = np.linspace(0, 1, 1000, False)  # 1 second
sig = np.sin(2*np.pi*10*t) + np.sin(2*np.pi*20*t)

sos = signal.butter(10, 15, 'hp', fs=1000, output='sos')
filtered = signal.sosfilt(sos, sig)

fig, (ax1, ax2) = plt.subplots(2, 1, sharex=True)
ax1.plot(t, sig)
ax1.set_title('10 Hz and 20 Hz sinusoids')
ax1.axis([0, 1, -2, 2])

ax2.plot(t, filtered)
ax2.set_title('After 15 Hz highpass filter')
ax2.axis([0, 1, -2, 2])
ax2.set_xlabel('Time [seconds]')
plt.tight_layout()
plt.show()
```