EE4015 Digital Signal Processing
2022 Mid-Term Exam Solution

Solution of Question 1

1(a) The difference equation is

$$y[n] = 3x[n] + x[n-1] - 3x[n-2]$$

No. The system is not memoryless.
It is because the output at time n depends on input at n , n -1, and n -2.

[3 marks]

1(b)

This system is causal as its output $y[n]$ depends only the present input $x[n]$ and previous inputs of $x[n-1]$ and $x[n-2]$. It does not depend on future value.

[3 marks]

1(c)

For bounded input $|x[n]| < B_x$, the output

$$|y[n]| = |3x[n] + x[n-1] - 3x[n-2]| < 3B_x + B_x + 3B_x = 7B_x < \infty$$

Thus, the system is BIBO stable.

[3 marks]

1(d)
According to convolution formula, we can determine the system impulse response $h[n]$ as:

$$h[n] = 3\delta[n] + \delta[n-1] - 3\delta[n-2]$$

[3 marks]

1(e)

This is a FIR system because the impulse response $h[n]$ only has three non-zero samples.

$$h[n] = \{3,1,-3\}$$

[4 marks]

1(f)

Yes, it is a linear-phase system because its impulse response fulfils the odd negative-symmetric condition.

[4 marks]

1(g)

$$h[n] = 3\delta[n] + \delta[n-1] - 3\delta[n-2] = \{3, 1, -3\}$$

$$x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] = \{1,2,3\}$$

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \{3,7,8,-3,-9\} \\ &= 3\delta[n] + 7\delta[n-1] + 8\delta[n-2] - 3\delta[n-3] - 9\delta[n-4] \end{aligned}$$

[6 marks]

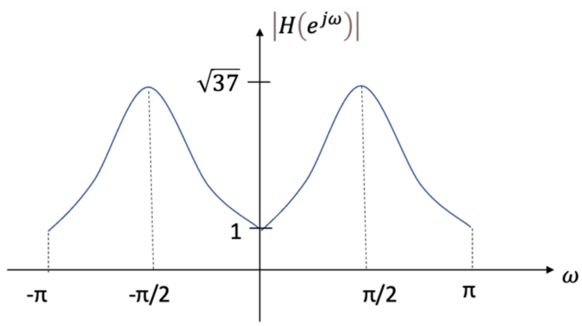
1(h)

$$\left|H(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} h[n]e^{-j\omega}\right| = \left|3 + e^{-j\omega} - 3e^{-j2\omega}\right|$$

$$= \left|e^{-j\omega}(3e^{j\omega} + 1 - 3e^{-j\omega}) = \left|e^{-j\omega}(1 + 3e^{j\omega} - 3e^{-j\omega})\right|\right|$$

$$= \left|e^{-j\omega}(1 + j6\sin\omega)\right| = \sqrt{1 + 36\sin^2\omega}$$

[2 marks]



This is a bandpass filter with center frequency at $\frac{\pi}{2}$

[2 marks]

Solution of Question 2

2(a) The frequencies existing in the continuous-time signal $x(t)$ are:

$$f_1 = \frac{\Omega_1}{2\pi} = \frac{30\pi}{2\pi} = 15\ Hz$$
$$f_2 = \frac{\Omega_2}{2\pi} = \frac{80\pi}{2\pi} = 40Hz$$

The maximum frequency of the signal is $40Hz$, then Nyquist frequency is 40 Hz.

[2 marks]

According to the sampling theorem, the Nyquist rate is two times of the Nyquist frequency

The Nyquist rate is 80 Hz

[2 marks]

2(b)

$$x(t) = 2\cos(30\pi t) + 4\cos(80\pi t)$$

$$= 2 \cdot \frac{1}{2}(e^{j30\pi} + e^{-j30\pi}) + 4 \cdot \frac{1}{2}(e^{j80\pi} + e^{-j80\pi})$$

$$= (e^{j30\pi} + e^{-j30\pi}) + 2(e^{j80\pi} + e^{-j80\pi})$$

[3 marks]

We know that $CTFT\{e^{j\Omega}\} = \delta(\Omega)$

$$X(j\Omega) = 2\pi\delta(\Omega - 30\pi) + 2\pi\delta(\Omega + 30\pi) + 4\pi\delta(\Omega - 80\pi) + 4\pi\delta(\Omega + 80\pi)$$

[3 marks]

2(c) $T = \frac{1}{50}$

$$x(t) = 2\cos(30\pi t) + 4\cos(80\pi t)$$

$$x[n] = x(nT) = 2\cos\left(30\pi \cdot n\frac{1}{50}\right) + 4\cos\left(80\pi \cdot n\frac{1}{50}\right)$$
$$= 2\cos\left(\pi\left(\tfrac{3}{5}\right)n\right) + 4\cos\left(\pi\left(\tfrac{8}{5}\right)n\right)$$
$$= 2\cos\left(2\pi\left(\frac{3}{10}\right)n\right) + 4\cos\left(2\pi\left(\frac{8}{10}\right)n\right)$$
$$= 2\cos\left(2\pi\left(\frac{3}{10}\right)n\right) + 4\cos\left(2\pi\left(1 - \frac{1}{5}\right)n\right)$$
$$= 2\cos\left(2\pi\left(\frac{3}{10}\right)n\right) + 4\cos\left(2\pi n - 2\pi\left(\frac{1}{5}\right)n\right)$$
$$= 2\cos\left(2\pi\left(\frac{3}{10}\right)n\right) + 4\left[\cos(2\pi n)\cos(-2\pi\left(\frac{1}{5}\right)n) - \sin(2\pi n)\sin(-2\pi\left(\frac{1}{5}\right)n)\right]$$
$$= 2\cos\left(2\pi\left(\frac{3}{10}\right)n\right) + 4\cos\left(-2\pi\left(\frac{1}{5}\right)n\right)$$

$$= 2\cos\left(2\pi\left(\frac{3}{10}\right)n\right) + 4\cos\left(2\pi\left(\frac{1}{5}\right)n\right)$$

[6 marks]

3(d) It is because only frequency components at 15 Hz and 10 Hz are present in the sampled signal, the CT signal we can recover is

$$y(t) = \cos(30\pi t) + 4\cos(20\pi t)$$

Which is obviously different from the original signal $x(t)$. The 10Hz sinusoidal ($\cos(20\pi t)$) is due to the sampling frequency of 50Hz is lower than the Nyquist rate of 80Hz. Thus, aliasing effect is occurred from the sinusoidal of $4\cos(80\pi t)$ with frequency of 40Hz.

[4 marks]

Solution of Question 3

3(a)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 0.24z^{-1}}{1 - 0.36z^{-2}}$$

$$=> \ Y(z)[1 - 0.36z^{-2}] = X(z)[1 - 0.24z^{-1}]$$

$$=> \ y[n] - 0.36y[n-2] = x[n] - 0.24x[n-1]$$

$$=> \ y[n] = x[n] - 0.24x[n-1] + 0.36y[n-2]$$

[4 marks]

3(b)

$$H(z) = \frac{1 - 0.24z^{-1}}{1 - 0.36z^{-2}} = \frac{z(z - 0.24)}{(z - 0.6)(z + 0.6)}$$

The poles are located at $z = 0.6$ and $z = -0.6$
The zeros are located at $z = 0$ and $z = 0.24$

[4 marks]

As this system is causal, the ROC is $|z| > 0.6$, which include the unit circle in the z-plane. Therefore, the system is stable.

[2 marks]

3(c) As all the poles and zeros of the transfer $H(z)$ are inside the unit circle, the poles and zeros of the inverse system $G(z) = 1/H(z)$ are zeros and poles of the $H(z)$. Therefore, they are also inside the unit circle for causal system. Then, the inverse system stable and exist.

[3 marks]

3(d) The partial fraction expansion for $H(z)$ is

$$H(z) = \frac{1 - 0.24z^{-1}}{1 - 0.36z^{-2}} = \frac{0.3}{1 - 0.6z^{-1}} + \frac{0.7}{1 + 0.6z^{-1}}$$

Since the system is causal, the ROC is $|z| > 0.6$.

Performing inverse z-transform with $|z| > 0.6$, we get

$$h[n] = 0.3(0.6)^n u[n] + 0.7(-0.6)^n u[n]$$

[6 marks]

3(e)

$$x[n] = 0.24^n u[n-1] = 0.24(0.24)^{n-1} u[n-1]$$

With the use of time shifting property, we obtain:

$$X(z) = \frac{0.24z^{-1}}{1 - 0.24z^{-1}}, \qquad |z| > 0.24$$

[4 marks]

Hence

$$Y(z) = H(z)X(z) = \frac{1-0.24z^{-1}}{(1-0.6z^{-1})(1+0.6z^{-1})} \cdot \frac{0.24z^{-1}}{1-0.24z^{-1}}$$

$$= \frac{0.24z^{-1}}{(1-0.6z^{-1})(1+0.6z^{-1})}$$

$$= \frac{0.2}{1-0.6z^{-1}} - \frac{0.2}{1+0.6z^{-1}}, \qquad |z| > 0.6$$

[4 marks]

Taking the inverse z-transform, we get:

$$y[n] = 0.2(0.6)^n u[n] - 0.2(-0.6)^n u[n]$$

[2 marks]

3(f)

Since the ROC of $|z| > 0.6$ includes the unit circle, the DTFT exists. The DTFT has the form of:

$$H(e^{j\omega}) = \frac{1-0.24e^{-j\omega}}{1-0.36e^{-j2\omega}}$$

[2 marks]

The squared magnitude is:

$$H(e^{j\omega})H^*(e^{j\omega}) = \frac{1-0.24e^{-j\omega}}{1-0.36e^{-j2\omega}} \cdot \frac{1-0.24e^{j\omega}}{1-0.36e^{j2\omega}} = \frac{1.0576-0.48\cos(\omega)}{1.1296-0.72\cos(2\omega)}$$

Hence, the magnitude is:

$$|H(e^{j\omega})| = \sqrt{\frac{1.0576-0.48\cos(\omega)}{1.1296-0.72\cos(2\omega)}}$$

[2 marks]

On the other hand,

$$H(e^{j\omega}) = \frac{1-0.24e^{-j\omega}}{1-0.36e^{-j2\omega}} \cdot \left(\frac{1-0.36e^{j2\omega}}{1-0.36e^{j2\omega}}\right)$$

The numerator is:

$$1-0.36e^{j2\omega} - 0.24e^{-j\omega} + 0.0864e^{j\omega}$$

$$= 1-0.36[\cos(2\omega)+j\sin(2\omega)] - 0.24[\cos(\omega)-j\sin(\omega)] + 0.0864[\cos(\omega)+j\sin(\omega)]$$

$$= 1-0.1536\cos(\omega) - 0.36\cos(2\omega) + j[0.3264\sin(\omega) - 0.36\sin(2\omega)]$$

Hence

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{0.3264\sin(\omega)-0.36\sin(2\omega)}{1-0.1536\cos(\omega)-0.36\cos(2\omega)}\right)$$

[2 marks]

Solution of Question 4

4.(a)
For $2 \geq n \geq 0$:

$$h[n] = \begin{cases} 0.5, & n = 0 \\ 1, & n = 1 \\ 0.5, & n = 2 \end{cases}$$

As $h[n]$ is positive symmetric, the system is linear phase.

[6 marks]

4.(b)
According to convolution, we get:

$$y[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2]$$

$$=> \; y[n] = 0.5u[n-1] + u[n-2] + 0.5u[n-3]$$

$$=> y[n] = 0.5\delta[n-1] + 1.5\delta[n-2] + 2u[n-3]$$

[6 marks]

4.(c)
For $2 \geq k \geq 0$:

$$H[k] = \sum_{n=0}^{2} h[n]W_3^{kn} = 0.5 + W_3^k + 0.5W_3^{2k} = 0.5 + e^{-j\frac{2\pi}{3}k} + 0.5e^{-j\frac{4\pi}{3}k}$$

Hence

$H[0] = 2$
$H[1] = -\frac{1}{4} - j\frac{\sqrt{3}}{4}$
$H[2] = -\frac{1}{4} + j\frac{\sqrt{3}}{4}$

As a result,

$$H[k] = \begin{cases} 2, & k = 0 \\ -\frac{1}{4} - j\frac{\sqrt{3}}{4}, & k = 1 \\ -\frac{1}{4} + j\frac{\sqrt{3}}{4}, & k = 2 \\ 0, & otherwise \end{cases}$$

[8 marks]