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Second-order leader-following consensus of nonlinear multi-agent systems via pinning control

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ABSTRACT

Without assuming that the interaction diagraph is strongly connected or contains a directed spanning tree, this paper studies the second-order leader-following consensus problem of nonlinear multi-agent systems with general network topologies. Based on graph theory, matrix theory, and LaSalle's invariance principle, a pinning control algorithm is proposed to achieve leader-following consensus in a network of agents with nonlinear second-order dynamics. Furthermore, a pinning consensus protocol is developed for coupled double-integrators with a constant reference velocity. In particular, this paper addresses what kind of agents and how many agents should be pinned, and establishes some sufficient conditions to guarantee that all agents asymptotically follow the virtual leader. Numerical simulations are given to verify the theoretical analysis.

1. Introduction

Since the pioneering works of Reynolds [1] and Vicsek et al. [2], the consensus problem of multi-agent systems has attracted researchers from a wide range of disciplines, such as biology, physics, robotics, computer science, social science and control engineering, due to its applications in the formation control of unmanned air vehicles, the cooperative control of mobile robots, the design of distributed sensor networks, and so on. For a cooperative multi-agent system, leaderless consensus means that each agent updates its state based on local information of its neighbors such that all agents eventually reach an agreement on a common value, while leader-following consensus means that there exists a virtual leader which specifies an objective for all agents to follow.

In the past few years, the first-order consensus problem of multi-agent systems has been intensively studied. Olfati-Saber and Murray [3] presented a systematic framework to analyze the first-order consensus algorithms, and showed that the consensus problem can be solved if the diagraph is strongly connected. Ren and Beard [4] further proved that first-order consensus can be achieved if the union of the dynamically changing interaction graphs has a directed spanning tree frequently enough as the system evolves. Sun et al. [5] discussed the first-order average consensus problem of dynamic agents with multiple time-varying communication delays. Lu et al. [6] studied the first-order consensus problem over directed networks with arbitrary finite communication delays and nonlinear couplings. Recently, the second-order consensus problem of multi-agent systems has received increasing attention [7–13]. Unlike the first-order consensus, Ren and Atkins [7] showed that the existence of a directed spanning tree is a necessary rather than a sufficient condition to reach second-order consensus, i.e., second-order consensus may not be achievable even if the interaction topology has a directed spanning tree. Therefore, the extension of consensus algorithms from first-order to second-order is non-trivial [7], and the second-order consensus problem is more complicated and challenging than the first-order case.

Broadly speaking, the first-order consensus problem of multi-agent systems can be treated as a special case of the synchronization problem of complex dynamical networks, which has been extensively studied in the past few decades [14–16]. Usually, a complex network consists of a set of nonlinear oscillators with first-order dynamics. Nevertheless, in reality some oscillators, e.g., harmonic oscillators [17,18] and pendulums [19–21], are governed by second-order dynamics with the position and velocity terms. Hence, it is necessary to study the consensus problem of a multi-agent system composed of second-order oscillators, in which the dynamics of each agent is not only determined by the interactions

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among agents, but also by its own dynamics, i.e., self-dynamics. Up to date, few consensus protocols consider the intrinsic dynamics of each individual agent in a multi-agent system with second-order dynamics. It is noted that the second-order consensus problem of coupled linear oscillators was addressed in [17,18]. However, all physical systems are nonlinear in nature [21]. Recently, Yu et al. [22] proposed a nonlinear multi-agent system by introducing complex dynamics into each agent, and studied the leaderless second-order consensus with directed topology. Thus, it is high time to deeply investigate the second-order consensus problem of multi-agent systems with nonlinear dynamics.

For a multi-agent system with fixed interaction topology, most leaderless consensus algorithms require the existence of a directed spanning tree. Otherwise, one can design some appropriate controllers to force all agents to reach on the state of the virtual leader. Considering that it is practically impossible to apply control actions to all agents in a large-scale multi-agent system, some authors developed consensus algorithms based on pinning control, which means that only a small fraction of agents have access to the reference state. Chen et al. [23], Liu et al. [24] and Ren [25] studied first-order consensus of multi-agent systems via pinning control. Ren [10] considered second-order consensus of coupled double-integrators with partial access to a time-varying reference state.

Due to the fact that many pinning control schemes for complex networks have been developed in the past few years, it would be beneficial to apply these pinning control techniques to study the consensus problem of multi-agent systems. It is well-known that one of the most difficult problems in the pinning control of complex networks is how to choose a set of pinned nodes. For undirected complex networks, Wang and Chen [26] pointed out that it is better to specifically pin the most-highly connected nodes. Li et al. [27] showed that there is no significant difference between specifically and randomly pinning schemes for random networks. Yu et al. [28] found that the nodes with low degrees should be pinned first when the coupling strength is small. Based on the concept of V-stability [29], Xiang and Chen [30] proposed a renormalization approach to select pinned nodes. However, it is quite challenging to select pinned nodes for a directed complex network. Chen et al. [31] pointed out that it is possible to pin a complex network with strong coupling strength by a single controller. Using minimum number of controllers, Lu et al. [32] studied the pinning stabilization of linearly coupled stochastic neural networks. Xiong et al. [33] developed an algorithm to choose pinned nodes by computing the determinants of principle minors. Lu et al. [34] proposed an approach to select pinned nodes by finding the strongly connected components which have no edges with heads in and tails out. Song and Cao [35] showed that the nodes whose out-degrees are bigger than their in-degrees should be chosen as pinned candidates.

The main purpose of this paper is to study the pinning-controlled second-order leader-following consensus of nonlinear multi-agent systems. The contributions of the paper are four-fold. First, the intrinsic dynamics of each individual agent in a multi-agent system is considered, and the interaction topology of the multi-agent system is general, which can neither be strongly connected nor have a directed spanning tree. Second, a consensus algorithm based on pinning control for a nonlinear multi-agent system with second-order dynamics is proposed. Also, a pinning consensus protocol is developed for coupled double-integrators. Third, this paper addresses two challenging problems in the pinning control of multi-agent systems: what kind of agents should be pinned and how many agents are needed to be pinned such that all agents reach consensus? Some sufficient conditions are established to ensure that the states of all agents globally asymptotically approach the state of the virtual leader. Fourth, comparing with the second-order leaderless consensus in [22], this paper further studies how to achieve second-order leader-following consensus of nonlinear multi-agent systems via pinning control when the interaction digraph does not have a directed spanning tree.

The remainder of the paper is organized as follows. Section 2 provides some necessary preliminaries. Section 3 presents pinning-controlled algorithms for nonlinear multi-agent systems and coupled double-integrators, respectively, and then develops a pinned-agent selection scheme to achieve second-order leader-following consensus. Numerical examples are given in Section 4 to verify the theoretical analysis. Finally, some concluding remarks and future trends are stated in Section 5.

2. Preliminaries

2.1. Notations

Some mathematical notations are used throughout this paper. Let $I_N \in \mathbb{R}^{N\times N}$ be an N-dimensional identity matrix, $1_N = (1, \ldots, 1)^T \in \mathbb{R}^N$ be a vector of all ones, A^T and A^{-1} be the transpose and the inverse of matrix A, respectively. For a matrix $M \in \mathbb{R}^{N\times N}$, denote the ith row and the ith column of M as the ith row–column pair [26], and let M_l represent its minor matrix by removing arbitrary $l\,(1 \le l < N)$ row–column pairs of M, $\|M\|$ and $\lambda_{\max}(M)$ be the Euclidean norm and maximal eigenvalue of matrix M, respectively. Write $M > 0\,(M < 0)$ if M is positive (negative) definite. For two real symmetric matrices X and Y, $X > Y\,(X < Y)$ means that $X - Y > 0\,(X - Y < 0)$. The symbol $\otimes$ denotes the Kronecker product.

2.2. Graph theory

Information exchange among agents in a multi-agent system can be modeled by an interaction graph. Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ be a weighted diagraph with a node set $\mathcal{V} = \{1, \ldots, N\}$, an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = (a_{ij})_{N\times N}$ with nonnegative elements [3]. A directed edge denoted by (j, i) means that node i has access to node j, i.e., node i can receive information from node j. The elements of the adjacency matrix $\mathcal{A}$ are defined such that $a_{ij} > 0 \Leftrightarrow (j, i) \in \mathcal{E}$, while $a_{ij} = 0 \Leftrightarrow (j, i) \notin \mathcal{E}$ [4,7]. We assume that $a_{ii} = 0$ for all $i \in \mathcal{V}$. The neighbor set of node i is defined by $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$, and the in-degree and out-degree of node i are, respectively, defined as [35]

$$\deg_{\text{in}}(i) = \sum_{j=1, j\neq i}^{N} a_{ij}, \qquad \deg_{\text{out}}(i) = \sum_{j=1, j\neq i}^{N} a_{ji}. \tag{1}$$

A diagraph is called balanced if $\deg_{\text{in}}(i) = \deg_{\text{out}}(i)$, $\forall i \in \mathcal{V}$ [3].

The Laplacian matrix $L = (l_{ij})_{N\times N}$ associated with the adjacency matrix $\mathcal{A}$ is defined as [7,10,11,25]

$$\begin{aligned} l_{ij} &= -a_{ij}, \quad i \neq j, \\ l_{ii} &= -\sum_{j=1, j\neq i}^{N} l_{ij}, \quad i, j = 1, \ldots, N, \end{aligned} \tag{2}$$

which ensures that $\sum_{j=1}^{N} l_{ij} = 0$. Generally speaking, the Laplacian matrix of a diagraph is asymmetric.

A directed path from node j to node i is a sequence of edges $(j, i_1), (i_1, i_2), \ldots, (i_l, i)$ in the directed graph $\mathcal{G}$ with distinct nodes i_k, $k = 1, \ldots, l$ [22]. A directed graph is strongly connected if for any two distinct nodes j and i, there exists a directed path from node j to node i. A directed graph has a directed spanning tree if there exists at least one node called root which has a directed path to all the other nodes [7].

2.3. Supporting lemmas

The following lemmas are needed to derive our main results.

Lemma 1 (*Schur complement [36]*). *The following linear matrix inequality (LMI)*

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0,$$

where $Q(x) = Q^T(x)$, $R(x) = R^T(x)$, is equivalent to either of the following conditions:

(i) $Q(x) > 0$, $R(x) - S^T(x)Q^{-1}(x)S(x) > 0$;
(ii) $R(x) > 0$, $Q(x) - S(x)R^{-1}(x)S^T(x) > 0$.

Lemma 2 (*[37]*). *Let A and B be m by m Hermitian matrices, $\alpha_1 \geq \cdots \geq \alpha_m$, $\beta_1 \geq \cdots \geq \beta_m$, $\gamma_1 \geq \cdots \geq \gamma_m$ be eigenvalues of matrices A, B, and A + B, respectively. Then one has $\alpha_i + \beta_m \leq \gamma_i \leq \alpha_i + \beta_1$, $i = 1, \ldots, m$.*

Lemma 3 (*[35]*). *For a symmetric matrix $M \in \mathbb{R}^{N\times N}$ and a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_l, \underbrace{0, \ldots, 0}_{N-l})$ with $d_i > 0, i = 1, \ldots, l\,(1 \leq l < N)$, let $M - D = \begin{bmatrix} A - \widetilde{D} & B \\ B^T & M_l \end{bmatrix}$ where M_l is the minor matrix of M by removing its first l row–column pairs, A and B are matrices with appropriate dimensions, $\widetilde{D} = \mathrm{diag}(d_1, \ldots, d_l)$. If $d_i > \lambda_{\max}(A - BM_l^{-1}B^T)$, $i = 1, \ldots, l$, $M - D < 0$ is equivalent to $M_l < 0$.*

Lemma 4 (*[38]*). *For matrices A, B, C with appropriate dimensions, one has*

(1) $(A \otimes B)^T = A^T \otimes B^T$;
(2) $(A + B) \otimes C = A \otimes C + B \otimes C$.

3. Main results

This section presents pinning control algorithms for nonlinear multi-agent systems and coupled double-integrators to achieve second-order leader-following consensus. In particular, a pinned-agent selection scheme is proposed to determine what kind of agents and how many agents should be pinned.

3.1. Problem description

Within the general framework of multi-agent systems, we consider a nonlinear multi-agent system composed of N coupled autonomous agents with second-order dynamics [22]:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= f(t, x_i(t), v_i(t)) + u_i(t), \qquad i = 1, \ldots, N, \end{aligned} \tag{3}$$

where $x_i = (x_{i1}, \ldots, x_{in})^T$ and $v_i = (v_{i1}, \ldots, v_{in})^T$ are the position and velocity states of agent i, respectively, $f(t, x_i, v_i) = (f_1(t, x_i, v_i), \ldots, f_n(t, x_i, v_i))^T \in \mathbb{R}^n$ is a nonlinear vector-valued continuous function to describe the self-dynamics of agent i, u_i is the control input for agent i. When $f \equiv 0$, the multi-agent system (3) has double-integrator dynamics [7–12].

The virtual leader for multi-agent system (3) is an isolated agent described by

$$\begin{aligned} \dot{x}_r(t) &= v_r(t) \\ \dot{v}_r(t) &= f(t, x_r(t), v_r(t)), \end{aligned} \tag{4}$$

where $x_r \in \mathbb{R}^n$ and $v_r \in \mathbb{R}^n$ are the position and velocity states of the virtual leader, respectively.

Definition 1. The multi-agent system (3) is said to achieve second-order leader-following consensus if its solution satisfies $\lim_{t\to\infty} \|x_i(t) - x_r(t)\| = 0$, $\lim_{t\to\infty} \|v_i(t) - v_r(t)\| = 0$, $i = 1, \ldots, N$, for any initial condition.

3.2. Pinning-controlled second-order leader-following consensus of nonlinear multi-agent systems

Before moving on, we need the following assumption.

Assumption 1. For the nonlinear function f in (3), there exist two constant matrices $W = (w_{ij})_{n\times n}$ and $M = (m_{ij})_{n\times n}$, in which $w_{ij} \geq 0$, $m_{ij} \geq 0$, such that

$$|f_i(t, x, v) - f_i(t, y, z)| \leq \sum_{j=1}^{n} (w_{ij}|x_j - y_j| + m_{ij}|v_j - z_j|),$$
$$i = 1, \ldots, n, \ \forall x, v, y, z \in \mathbb{R}^n. \tag{5}$$

When $f \neq 0$, the reference velocity, i.e., the velocity of the virtual leader (4), is time-varying. Now we start to design a consensus algorithm based on pinning control such that all agents in nonlinear multi-agent system (3) can asymptotically follow the virtual leader (4). Without loss of generality, rearrange the order of all agents and let the first $l\,(1 \leq l < N)$ agents in multi-agent system (3) be controlled. We consider the following pinning control algorithm to implement second-order leader-following consensus of multi-agent system (3):

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= f(t, x_i(t), v_i(t)) + c \sum_{j\in\mathcal{N}_i} a_{ij}[(x_j(t) - x_i(t)) \\ &\quad + (v_j(t) - v_i(t))] - d_i[(x_i(t) - x_r(t)) + (v_i(t) - v_r(t))], \\ &\quad i = 1, \ldots, N, \end{aligned} \tag{6}$$

where $c > 0$, a_{ij} is the (i, j)-th entry of the adjacency matrix $\mathcal{A} \in \mathbb{R}^{N\times N}$, $\mathcal{N}_i$ is the neighbor set of agent i, the local feedback gains satisfy $d_i > 0$, $i = 1, \ldots, l$ and $d_i = 0$, $i = l + 1, \ldots, N$.

By Assumption 1, define

$$\rho = \max\{p, q + 1\}, \tag{7}$$

where $p = \frac{1}{2} \max_{1\leq j\leq n} \sum_{k=1}^{n} (w_{jk}^{2\varepsilon} + m_{jk}^{2\varepsilon} + 2w_{kj}^{2(1-\varepsilon)})$ and $q = \frac{1}{2} \max_{1\leq j\leq n} \sum_{k=1}^{n} (m_{jk}^{2\varepsilon} + w_{jk}^{2\varepsilon} + 2m_{kj}^{2(1-\varepsilon)})$ with $\varepsilon \in [0, 1]$.

Let

$$G = -L, \tag{8}$$

where L is the Laplacian matrix defined in (2).

Construct the following symmetric matrix

$$H = c\frac{G + G^T}{2} + \rho I_N. \tag{9}$$

Let $D = \mathrm{diag}(d_1, \ldots, d_l, \underbrace{0, \ldots, 0}_{N-l})$ where $d_i > 0, i = 1, \ldots, l$, are pinning feedback gains in (6). Using matrix decomposition, we have

$$H - D = \begin{bmatrix} A - \widetilde{D} & B \\ B^T & H_l \end{bmatrix}, \tag{10}$$

where $H_l = (c\frac{G+G^T}{2} + \rho I_N)_l$ is the minor matrix of H by removing its first $l\,(1 \leq l < N)$ row–column pairs, A and B are matrices with appropriate dimensions, $\widetilde{D} = \mathrm{diag}(d_1, \ldots, d_l)$.

Theorem 1. *Suppose that Assumption 1 holds and let $G = -L$. The second-order leader-following consensus in the pinning-controlled multi-agent system (6) is achieved if the following two conditions are satisfied:*

$$\lambda_{\max}\left(\left(\frac{G + G^T}{2}\right)_l\right) < -\frac{\rho}{c}, \tag{11}$$

and

$$d_i > \lambda_{\max}(A - BH_l^{-1}B^T), \quad i = 1, \ldots, l, \tag{12}$$

where A, B, H_l are defined in (10), $d_i > 0, i = 1, \ldots, l$, are pinning feedback gains, $(\frac{G+G^T}{2})_l$ is the minor matrix of $\frac{G+G^T}{2}$ by removing its first l row–column pairs, ρ is defined in (7).

Proof. Let $\widehat{x}_i(t) = x_i(t) - x_r(t), \widehat{v}_i(t) = v_i(t) - v_r(t), i = 1, \ldots, N$. From (4) and (6), we have

$$\begin{aligned}\dot{\widehat{x}}_i(t) &= \hat{v}_i(t)\\ \dot{\widehat{v}}_i(t) &= f(t, x_i(t), v_i(t)) - f(t, x_r(t), v_r(t))\\ &\quad - c\sum_{j=1}^{N} l_{ij}[\widehat{x}_j(t) + \widehat{v}_j(t)] - d_i\,[\widehat{x}_i(t) + \widehat{v}_i(t)],\\ &\quad i = 1, \ldots, N,\end{aligned} \tag{13}$$

where l_{ij} is the (i, j)-th entry of the Laplacian matrix L defined by (2), $d_i > 0, i = 1, \ldots, l$ and $d_i = 0, i = l + 1, \ldots, N$.

Let

$$\widehat{x}(t) = (\widehat{x}_1^T(t), \ldots, \widehat{x}_N^T(t))^T, \qquad \widehat{v}(t) = (\widehat{v}_1^T(t), \ldots, \widehat{v}_N^T(t))^T,$$
$$F(t, x(t), v(t)) = (f^T(t, x_1(t), v_1(t)), \ldots, f^T(t, x_N(t), v_N(t)))^T.$$

Rewrite (13) in the matrix form as

$$\begin{aligned}\dot{\widehat{x}}(t) &= \widehat{v}(t)\\ \dot{\widehat{v}}(t) &= F(t, x(t), v(t)) - 1_N \otimes f(t, x_r(t), v_r(t))\\ &\quad - ((cL + D) \otimes I_n)(\widehat{x}(t) + \widehat{v}(t)),\end{aligned} \tag{14}$$

where $D = \mathrm{diag}(d_1, \ldots, d_l, \underbrace{0, \ldots, 0}_{N-l})$.

Let $\widehat{y}(t) = (\widehat{x}^T(t), \widehat{v}^T(t))^T$. Construct the following Lyapunov functional candidate

$$V(t) = \frac{1}{2}\widehat{y}^T(t)(\Omega \otimes I_n)\widehat{y}(t), \tag{15}$$

where $\Omega = \begin{bmatrix} c(L+L^T)+2D & I_N\\ I_N & I_N\end{bmatrix}$.

Now we show that the Lyapunov functional candidate (15) is valid, which means that $\Omega > 0$. It is obvious that Ω is symmetric. In terms of Lemma 1, we know that $\Omega > 0$ is equivalent to $c(L + L^T) + 2D - I_N > 0$, which indicates that $c\frac{L+L^T}{2} + D - \frac{I_N}{2} > 0$. Considering $G = -L$, we just need to show that $c\frac{G+G^T}{2} + \frac{I_N}{2} - D < 0$. It follows from condition (11) that $c\lambda_{\max}((\frac{G+G^T}{2})_l) + \rho < 0$. Then by Lemma 2, we know that $\lambda_{\max}((c\frac{G+G^T}{2} + \rho I_N)_l) \le c\lambda_{\max}((\frac{G+G^T}{2})_l) + \rho < 0$. So we have $H_l = (c\frac{G+G^T}{2} + \rho I_N)_l < 0$, where H_l is defined in (10). Then, when pinning feedback gains satisfy condition (12), in view of (9), (10) and Lemma 3, we know that $c\frac{G+G^T}{2} + \rho I_N - D = H - D < 0$. With the definition of ρ in (7), we have $\rho \ge 1$. Thus, $c\frac{G+G^T}{2} + \frac{I_N}{2} - D < c\frac{G+G^T}{2} + \rho I_N - D < 0$, which implies that $c(L + L^T) + 2D - I_N > 0$ since $G = -L$. Then by Lemma 1, we can conclude that $\Omega > 0$, which means that Lyapunov functional candidate (15) satisfies $V(t) \ge 0$ and $V(t) = 0$ if and only if $\widehat{y}(t) = 0$.

Rewrite (15) as

$$\begin{aligned}V(t) &= \frac{1}{2}\widehat{x}^T(t)[(c(L + L^T) + 2D) \otimes I_n]\widehat{x}(t)\\ &\quad + \widehat{x}^T(t)\widehat{v}(t) + \frac{1}{2}\widehat{v}^T(t)\widehat{v}(t).\end{aligned} \tag{16}$$

In view of Lemma 4, taking the time derivative of $V(t)$ along the trajectory (14) yields

$$\begin{aligned}\dot{V}(t) &= \widehat{x}^T(t)[(c(L + L^T) + 2D) \otimes I_n]\widehat{v}(t) + \widehat{v}^T(t)\widehat{v}(t)\\ &\quad + \widehat{x}^T(t)[F(t, x(t), v(t)) - 1_N \otimes f(t, x_r(t), v_r(t))\\ &\quad - ((cL + D) \otimes I_n)(\widehat{x} + \widehat{v})] + \widehat{v}^T(t)[F(t, x(t), v(t))\\ &\quad - 1_N \otimes f(t, x_r(t), v_r(t)) - ((cL + D) \otimes I_n)(\widehat{x} + \widehat{v})]\\ &= \widehat{x}^T(t)\,[(-cL - D) \otimes I_n]\widehat{x}(t) + \widehat{v}^T(t)\\ &\quad \times [(I_N - cL - D) \otimes I_n]\,\widehat{v}(t)\\ &\quad + \sum_{i=1}^{N}\widehat{x}_i^T(f(t, x_i, v_i) - f(t, x_r, v_r))\\ &\quad + \sum_{i=1}^{N}\widehat{v}_i^T(f(t, x_i, v_i) - f(t, x_r, v_r)).\end{aligned} \tag{17}$$

By Assumption 1 and the fact that $2\mu|xy| \le \mu^{2\varepsilon}x^2 + \mu^{2(1-\varepsilon)}y^2, \forall\mu \ge 0, x, y \in \mathbb{R}, \varepsilon \in [0, 1]$, we obtain

$$\begin{aligned}\sum_{i=1}^{N}\widehat{x}_i^T(f(t, x_i, v_i) - f(t, x_r, v_r)) &\le \frac{1}{2}\sum_{i=1}^{N}\left[\sum_{j=1}^{n}\sum_{k=1}^{n}(w_{jk}^{2\varepsilon} + m_{jk}^{2\varepsilon}\right.\\ &\quad \left. + w_{kj}^{2(1-\varepsilon)})\widehat{x}_{ij}^2 + \sum_{j=1}^{n}\sum_{k=1}^{n} m_{kj}^{2(1-\varepsilon)}\widehat{v}_{ij}^2\right]\end{aligned} \tag{18}$$

and

$$\begin{aligned}\sum_{i=1}^{N}\widehat{v}_i^T(f(t, x_i, v_i) - f(t, x_r, v_r)) &\le \frac{1}{2}\sum_{i=1}^{N}\left[\sum_{j=1}^{n}\sum_{k=1}^{n}(m_{jk}^{2\varepsilon} + w_{jk}^{2\varepsilon}\right.\\ &\quad \left. + m_{kj}^{2(1-\varepsilon)})\widehat{v}_{ij}^2 + \sum_{j=1}^{n}\sum_{k=1}^{n} w_{kj}^{2(1-\varepsilon)}\widehat{x}_{ij}^2\right].\end{aligned} \tag{19}$$

Considering (7), (17)–(19) and $G = -L$, we have

$$\begin{aligned}\dot{V}(t) &\le \widehat{x}^T(t)\,[(-cL - D) \otimes I_n]\widehat{x}(t) + \widehat{v}^T(t)\\ &\quad \times [(I_N - cL - D) \otimes I_n]\widehat{v}(t)\\ &\quad + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{n}\sum_{k=1}^{n}(w_{jk}^{2\varepsilon} + m_{jk}^{2\varepsilon} + 2w_{kj}^{2(1-\varepsilon)})\widehat{x}_{ij}^2\\ &\quad + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{n}\sum_{k=1}^{n}(m_{jk}^{2\varepsilon} + w_{jk}^{2\varepsilon} + 2m_{kj}^{2(1-\varepsilon)})\widehat{v}_{ij}^2\\ &\le \widehat{x}^T(t)[(pI_N - cL - D) \otimes I_n]\widehat{x}(t) + \widehat{v}^T(t)\\ &\quad \times[((q + 1)I_N - cL - D) \otimes I_n]\widehat{v}(t)\\ &= \widehat{x}^T(t)\left[\left(c\frac{G + G^T}{2} + pI_N - D\right) \otimes I_n\right]\widehat{x}(t)\\ &\quad + \widehat{v}^T(t)\left[\left(c\frac{G + G^T}{2} + (q + 1)\,I_N - D\right) \otimes I_n\right]\widehat{v}(t),\end{aligned} \tag{20}$$

where the parameters p and q are defined in (7).

With conditions (11) and (12), we have shown that $c\frac{G+G^T}{2} + \rho I_N - D < 0$ when we prove that $\Omega > 0$. Recall that $\rho = \max\,\{p, q + 1\}$ according to its definition in (7). Then it is easy to conclude that $c\frac{G+G^T}{2} + pI_N - D < 0$ and $c\frac{G+G^T}{2} + (q+1)I_N - D < 0$. In view of (20), we have $\dot{V}(t) \le 0$ and $\dot{V}(t) = 0$ if and only if $\widehat{x}(t) = 0$ and $\widehat{v}(t) = 0$. So the set $\mathcal{M} = \{(\widehat{x}^T(t), \widehat{v}^T(t))^T | \widehat{x}(t) = \widehat{v}(t) = 0\}$ is the largest invariant set contained in the set $\mathcal{D} = \{(\widehat{x}^T(t), \widehat{v}^T(t))^T | \dot{V}(t) = 0\}$ for system (14). According to LaSalle's invariance principle [21], starting from any initial condition, every solution of system (14) approaches $\mathcal{M}$ as $t \to \infty$, i.e., $\|\widehat{x}(t)\| \to 0$ and $\|\widehat{v}(t)\| \to 0$. Therefore, the states of the pinning-controlled multi-agent system (6) globally asymptotically approach the state of virtual leader (4), i.e., $\|x_i(t) - x_r(t)\| \to 0$ and $\|v_i(t) - v_r(t)\| \to 0, i = 1, \ldots, N$, as $t \to \infty$. So the second-order leader-following consensus in pinning-controlled multi-agent system (6) is achieved. □

Remark 1. For a directed network, we construct a characteristic matrix $H = c\frac{G+G^T}{2} + \rho I_N$ in (9), which can be treated as the extension of the characteristic matrix defined in [28–30] for an undirected complex network.

Remark 2. Based on the low-dimensional pinning condition (11), we know that $\lambda_{\max}((\frac{G+G^T}{2})_l) < 0$ and a lower bound of the parameter c can be given by $c^* = -\rho/\lambda_{\max}((\frac{G+G^T}{2})_l) > 0$, which can be treated as a function of l. We can let $c = c^* + \epsilon_1$, where ϵ_1 is a small positive number, such that the condition (11), i.e., $\lambda_{\max}((\frac{G+G^T}{2})_l) < -\rho/c$, holds.

Remark 3. Although the pinning feedback gains $d_i, i = 1, \dots, l$, in Theorem 1 can be chosen to be large enough, we estimate one of their lower bounds with (12), i.e., $d^* = \lambda_{\max}(A - BH_l^{-1}B^T)$, such that they are not much larger than the values needed in practice.

Remark 4. In [10], Ren proposed several consensus algorithms, including a pinning control algorithm, for multi-agent systems with double-integrator dynamics and time-varying reference velocity. Different from the counterpart in [10], the pinning control algorithm (6) in this paper is developed to achieve leader-following consensus in a multi-agent system composed of nonlinear second-order oscillators, in which the dynamics of each agent is not only determined by interactions among agents, but also by its intrinsic dynamics.

Remark 5. Assuming that the interaction graph consisting of several strongly connected components contains a directed spanning tree, Yu et al. [22] gave some sufficient conditions for reaching second-order leaderless consensus in nonlinear multi-agent system (3) by defining generalized algebraic connectivity. In this paper, we work on the leader-following consensus problem of multi-agent system (3) with general network topology, which may neither be strongly connected nor have a directed spanning tree. By applying local feedback actions to a small fraction of agents under some conditions, we show that the states of pinning-controlled multi-agent system (6) can globally asymptotically approach the state of virtual leader (4).

3.3. Pinning-controlled second-order leader-following consensus of coupled double-integrators

Let $f \equiv 0$ in (3), which implies that two zero matrices can satisfy Assumption 1, i.e., $W = M = 0$. So we have $\rho = 1$ according to (7). When $f \equiv 0$, each agent in multi-agent system (3) reduces to the following double-integrator [7–12]

$$\dot{x}_i(t) = v_i(t), \qquad \dot{v}_i(t) = u_i, \quad i = 1, \dots, N, \tag{21}$$

and the virtual leader (4) is described by

$$\dot{x}_r(t) = v_r(t), \qquad \dot{v}_r(t) = 0, \tag{22}$$

which indicates that the reference velocity is constant.

The following pinning control protocol can be directly derived from algorithm (6) to solve the leader-following consensus problem of double-integrator dynamics (21) with constant reference velocity:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= c\sum_{j\in\mathcal{N}_i} a_{ij}[(x_j(t) - x_i(t)) + (v_j(t) - v_i(t))] \\ &\quad - d_i[(x_i(t) - x_r(t)) + (v_i(t) - v_r(t))], \\ &\quad i = 1, \dots, N, \end{aligned} \tag{23}$$

where the parameters $c, a_{ij}, \mathcal{N}_i, d_i$ are defined in (6).

Recall that $\rho = 1$ when $f \equiv 0$. The symmetric matrix in (9) becomes $H = c\frac{G+G^T}{2} + I_N$. Correspondingly, we can determine matrices A, B and H_l in (10) using matrix decomposition. From Theorem 1 and its proof, we have the following result.

Corollary 1. *Let $G = -L$. If pinning feedback gains $d_i > 0, i = 1, \dots, l$, satisfy $d_i > \lambda_{\max}(A - BH_l^{-1}B^T), i = 1, \dots, l$, and the following condition holds:*

$$\lambda_{\max}\left(\left(\frac{G+G^T}{2}\right)_l\right) < -\frac{1}{c}, \tag{24}$$

the states of pinning-controlled coupled double-integrators (23) globally asymptotically agree on the state of virtual leader (22), i.e., $x_i(t) \to x_r(0) + v_r(0)t$ and $v_i(t) \to v_r(0), i = 1, \dots, N$, as $t \to \infty$.

Remark 6. In [10], Ren provided a systematic framework to solve the consensus problem of multi-agent systems with double-integrator dynamics, and proposed a pinning control algorithm with bounded control input and time-varying reference velocity. When the reference velocity is constant, i.e., $\dot{v}_r = 0$, the consensus algorithm based on pinning control in [10] for double-integrator (21) is formulated as

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= \frac{1}{\kappa_i}\left(\sum_{j\in\mathcal{N}_i} a_{ij}\dot{v}_j\right) - \frac{1}{\kappa_i}K_{ri} \\ &\quad \times \tanh\left[\sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j) + a_{i(N+1)}(x_i - x_r)\right] \\ &\quad - \frac{1}{\kappa_i}K_{vi}\tanh\left[\sum_{j\in\mathcal{N}_i} a_{ij}(v_i - v_j) + a_{i(N+1)}(v_i - v_r)\right], \\ &\quad i = 1, \dots, N, \end{aligned} \tag{25}$$

where $a_{i(N+1)} > 0$ if agent i has access to x_r and v_r, $\kappa_i = \sum_{j=1}^{N+1} a_{ij}$, K_{ri} and K_{vi} are $n \times n$ positive definite diagonal matrices. Note that the control input of agent i in pinning control algorithm (25) involves the accelerations of its neighbors, i.e., $\dot{v}_j, j \in \mathcal{N}_i$, while our algorithm (23) only uses the position and velocity states, which might bring some convenience for practical implementations.

3.4. Pinned-agent selection scheme for multi-agent systems with second-order dynamics and directed topologies

Up to this point, a question arises naturally: how to choose a set of pinned agents such that pinning conditions (11) or (24) are satisfied? Actually, this paper uses pinning control techniques for complex networks to study the consensus problem of multi-agent systems. Thus the pinned-node selection scheme for complex networks can be adopted to determine what kind of agents should be pinned and how many agents are needed to be pinned to achieve a second-order leader-following consensus.

Proposition 1 ([35,34]). *For a diagraph $\mathcal{G}$, let $\mathcal{V}$ and $\mathcal{D}$ denote the node set of $\mathcal{G}$ and the pinned-node set, respectively. All nodes in $\mathcal{V} \setminus \mathcal{D}$ should have access to the pinned-node set $\mathcal{D}$, i.e., for any node $i \in \mathcal{V} \setminus \mathcal{D}$, one can always find node $j \in \mathcal{D}$ such that there exists a directed path from node j to node i.*

Remark 7. According to Proposition 1, two useful conclusions can be drawn: (1) The agents with zero in-degrees must be pinned because their states are not influenced by any other agent. (2) The virtual leader and the followers should form a directed spanning tree, in which the virtual leader is the only root, such that the states of all agents can be directly or indirectly affected by the state of the virtual leader.

Consider a complex network with the coupling matrix G. If the network is undirected and connected, the pinned nodes can be specifically or randomly chosen. The authors [26,28] performed simulation-based analysis to study the variation of $\lambda_{\max}(G_l)$ versus l, i.e., the number of pinned nodes, and found that $\lambda_{\max}(G_l)$ decreases by increasing l. For a directed network, how to select a set of pinned nodes is still a challenging problem. In [35], Song and Cao pointed out that the nodes whose out-degrees are bigger than their in-degrees should be chosen as pinned candidates, which results in $\lambda_{\max}((\frac{G+G^T}{2})_l) \le 0$. By rearranging the order of network nodes based on both in-degrees and out-degrees, Song and Cao [35] found that $\lambda_{\max}((\frac{G+G^T}{2})_l)$ decreases with the increase of l. Hence, it is feasible to choose l pinned nodes such that $\lambda_{\max}((\frac{G+G^T}{2})_l) < 0$. It is obvious that condition (11) holds if the parameter c is large enough. However, it is literally impossible to have a very large c due to the high control cost and the difficulty of real implementations. Thus, we should find a relatively lower c for practical use. Note that $c^* = -\rho/\lambda_{\max}((\frac{G+G^T}{2})_l)$ decreases with the increase of l because $-\lambda_{\max}((\frac{G+G^T}{2})_l)$ becomes larger when more nodes are pinned. So we can obtain an acceptable c^* by increasing l and then let $c = c^* + \epsilon_1$, where $\epsilon_1 > 0$ is a small number, such that condition (11) holds.

Furthermore, the pinning feedback gains $d_i, i = 1, \ldots, l$ in algorithms (6) and (23) are not allowed to be very large either. Thus, we estimate their lower bounds, i.e., $d_i > d^* = \lambda_{\max}(A - BH_l^{-1}B^T)$, in Remark 3. To achieve better pinning performance, we need to find a good balance among the number of pinned nodes l, the parameters c and $d_i, i = 1, \ldots, l$ such that l is as small as possible, c and d_i are acceptable for real applications. Motivated by the pinned-node selection scheme in [35], we give the following procedure to select a set of pinned-agents and to design the parameters c and $d_i = 1, \ldots, l$:

(1) Define a degree-difference vector $\deg_{\mathrm{dif}}(i) = \deg_{\mathrm{out}}(i) - \deg_{\mathrm{in}}(i), i = 1, \ldots, N$.
(2) Pick all agents with zero in-degrees as pinned agents and rearrange the remaining agents in descending order according to their degree-differences.
(3) Find the minimum number of agents l_0 which satisfies $\lambda_{\max}((\frac{G+G^T}{2})_{l_0-1}) \ge 0$ and $\lambda_{\max}((\frac{G+G^T}{2})_{l_0}) < 0$. Let $l = l_0$.
(4) Compute the lower bound $c^* = -\rho/\lambda_{\max}((\frac{G+G^T}{2})_l)$ with Remark 2. If c^* is not satisfactory, continuously add more agents to the pinned-agent set based on the degree-difference vector until we have a relatively lower c^*.
(5) Let $c = c^* + \epsilon_1$ where ϵ_1 is a small positive number. Calculate d^* with matrix decomposition (10) and Remark 3. If d^* is not good enough for practical use, add more agents to the pinned-agent set, compute c^* and repeat step 5 until we find a proper d^*.
(6) Let $d_i = d^* + \epsilon_2, i = 1, \ldots, l$, where ϵ_2 is a small positive number.

Remark 8. For most typical networks such as random networks, scale-free networks and small-world networks, we will have no difficulty in finding a proper c by choosing a small fraction of network nodes. However, we must mention that there may exist a special type of network which has quite a small $|\lambda_{\max}((\frac{G+G^T}{2})_l)|$ no matter how we increase l, which means that pinning control is not very effective for this kind of network and we need to pin most network nodes, even all nodes.

4. Numerical results

In this section, two numerical examples are given to verify the effectiveness of the proposed pinning control techniques for multi-agent systems to achieve second-order leader-following consensus.

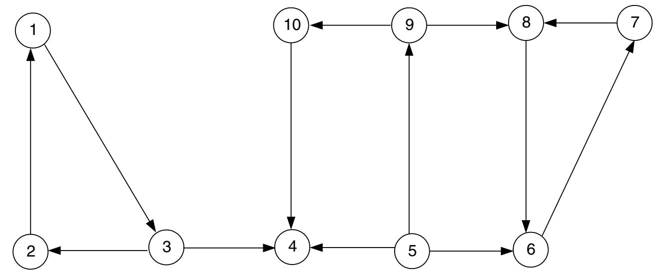

Fig. 1. Diagraph of ten interacting agents.

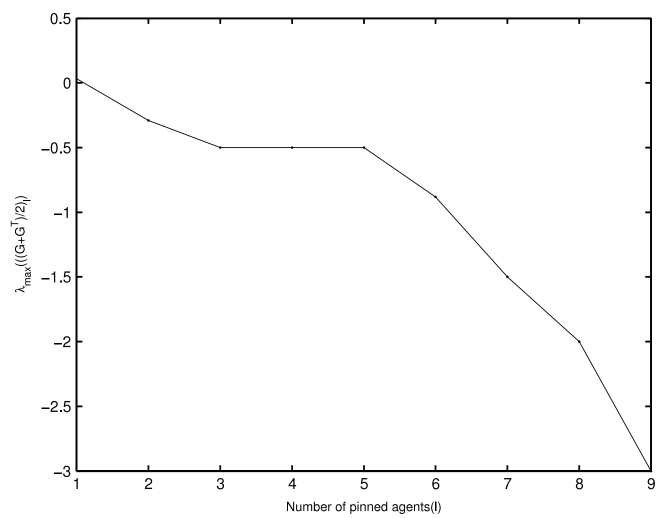

Fig. 2. Variation of $\lambda_{\max}((\frac{G+G^T}{2})_l)$ versus the number of pinned agents.

4.1. Second-order leader-following consensus of a multi-agent system with ten agents

In view of (3), we consider a multi-agent system consisting of ten agents described by

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= f(t, x_i, v_i) + u_i, \quad i = 1, \ldots, 10, \end{aligned} \tag{26}$$

where $x_i, v_i \in \mathbb{R}^3$ are the position and velocity states of agent i, respectively.

The interaction diagraph of multi-agent system (26) is shown in Fig. 1. It is obvious that the interaction diagraph in Fig. 1 does not have a directed spanning tree. Now we examine what agents should be pinned. From Fig. 1, note that agent 5 has zero in-degree and the out-degrees of agents 3 and 9 are bigger than their in-degrees. According to Proposition 1, agent 5 should be pinned first because its state is not affected by others, agent 3 and agent 9 can be chosen as pinned candidates. Furthermore, if we pin agent 3 and agent 5, the virtual leader and followers form a directed spanning tree. Thus, we can choose agents 3 and 5 as pinned agents, and determine that the minimum number of pinned agents is two.

Based on the pinned-agent selection scheme in Section 3.4, we rearrange ten agents and the new order is 5, 3, 9, 1, 2, 7, 10, 6, 8, 4. Letting the number of pinned agents l range from 1 to 9, we illustrate the variation of $\lambda_{\max}((\frac{G+G^T}{2})_l)$ versus l in Fig. 2, from which we see that $\lambda_{\max}((\frac{G+G^T}{2})_l)$ decreases with the increase of l. Table 1 provides some combinations of l and $\lambda_{\max}((\frac{G+G^T}{2})_l)$. Note that $\lambda_{\max}((\frac{G+G^T}{2})_1) = 0.0322 > 0$ and $\lambda_{\max}((\frac{G+G^T}{2})_2) = -0.2906$. This again confirms that at least we need to pin two agents.

With pinning control algorithms (6) and (23), we investigate second-order leader-following consensus of multi-agent system

Table 1
Combinations of l and $\lambda_{max}((\frac{G+G^T}{2})_l)$.

l	1	2	3	4	5	6
$\lambda_{\max}((\frac{G+G^T}{2})_l)$	0.0322	−0.2906	−0.5000	−0.5000	−0.5000	−0.8820

Table 2
Combinations of l and c^* for case 1.

l	2	3	4	5	6
c^*	113.6614	66.0600	66.0600	66.0600	37.4490

(26) with time-varying and constant reference velocities, i.e., $f \neq 0$ and $f \equiv 0$, respectively.

Case 1: *Consensus of multi-agent system with time-varying reference velocity*

By the model of the time-delayed Chua's oscillator in [39], we consider the following nonlinear function f for multi-agent system (26):

$$f(t, x_i(t), v_i(t)) = \begin{pmatrix} 0 \\ 0 \\ -\beta\epsilon\sin(\omega x_{i1}) \end{pmatrix} + \begin{pmatrix} \alpha(v_{i2} - v_{i1} - h(v_{i1})) \\ v_{i1} - v_{i2} + v_{i3} \\ -\beta v_{i2} - \gamma v_{i3} \end{pmatrix}, \tag{27}$$

where $\alpha = 10$, $\beta = 19.53$, $\gamma = 0.1636$, $\epsilon = 0.2$, $\omega = 0.5$, and $h(v_{i1})$ is a piecewise-linear function given by $h(v_{i1}) = bv_{i1} + \frac{a-b}{2}(|v_{i1} + 1| - |v_{i1} - 1|)$ with parameters $a = -1.4325$, $b = -0.7831$.

It is easy to verify that the nonlinear function f in (27) satisfies Assumption 1. By some calculations, we obtain the following two matrices:

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1.953 & 0 & 0 \end{pmatrix},$$
$$M = \begin{pmatrix} 8.663 & 10 & 0 \\ 1 & 1 & 1 \\ 0 & 19.53 & 0.1636 \end{pmatrix}. \tag{28}$$

Choosing $\varepsilon = 0.5$, by (7) and (28) we have $p = 11.2845$, $q = 32.03$, $\rho = 33.03$. According to the pinned-agent selection scheme in Section 3.4, we obtain partial combinations of l and c^* in Table 2. When $l = 2$, agents 3 and 5 are pinned, and we have $\lambda_{\max}((\frac{G+G^T}{2})_l) = -0.2906$, $c^* = 113.6614$ from Tables 1 and 2. Let $c = 114.0$. Based on matrix decomposition (10) and Remark 3, the lower bound of pinning feedback gains is determined to be $d^* = 115.9520$, which is an acceptable value. So we choose $d_1 = d_2 = 116.0$ as pinning feedback gains. With pinning control algorithm (6), the evolutions of positions and velocities of ten agents are shown in Figs. 3 and 4, respectively. We see that the pinning-controlled multi-agent system with nonlinear dynamics (27) reaches second-order leader-following consensus by pinning agent 3 and agent 5.

Case 2: *Consensus of multi-agent system with constant reference velocity*

When $f \equiv 0$, multi-agent system (26) has double-integrator dynamics and the velocity of the virtual leader is constant. Some combinations of l and c^* are shown in Table 3, from which we know that $c^* = 3.4412$ when $l = 2$. Taking $c = 3.5$, we have $d^* = 3.5582$. Let $d_1 = d_2 = 3.6$ be pinning feedback gains. The initial values of reference position and velocity are given by $x_r(0) = (1, 2, 3)^T$ and $v_r(0) = (10, 20, 10)^T$, respectively. With pinning

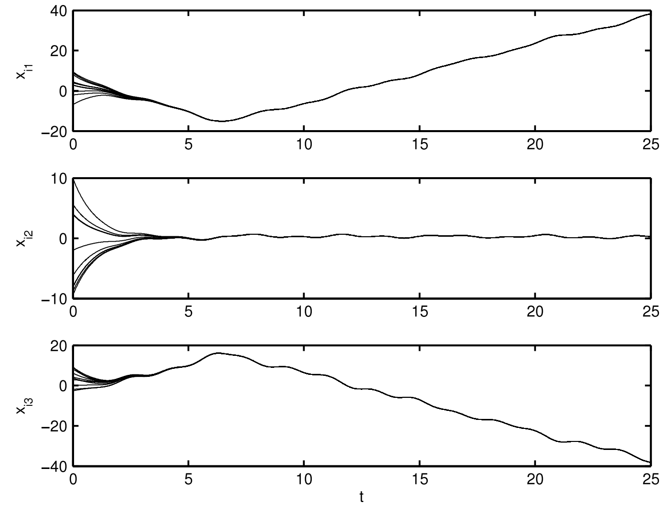

Fig. 3. Positions of ten pinning-controlled agents with time-varying reference velocity.

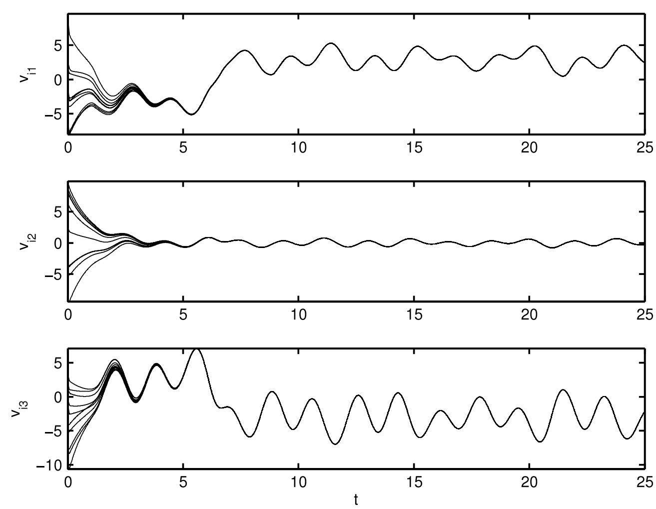

Fig. 4. Velocities of ten pinning-controlled agents with time-varying reference velocity.

Table 3
Combinations of l and c^* for case 2.

l	2	3	4	5	6
c^*	3.4412	2.0000	2.0000	2.0000	1.1338

control algorithm (23), the evolutions of positions and velocities of ten agents are shown in Figs. 5 and 6, respectively. Note that the multi-agent system (26) with double-integrator dynamics reaches second-order leader-following consensus and the velocities of all agents converge to the constant velocity $(10, 20, 10)^T$.

4.2. Second-order leader-following consensus of a multi-agent system with one hundred agents and directed small-world network topology

Consider a multi-agent system composed of one hundred agents described by

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= f(t, x_i, v_i) + u_i, \quad i = 1, \ldots, 100, \end{aligned} \tag{29}$$

where $x_i, v_i \in \mathbb{R}$ are the position and velocity states of agent i.

Table 4
Combinations of l and $\lambda_{\max}((\frac{G+G^T}{2})_l)$.

l	6	7	8	9	10	11	12
$\lambda_{\max}((\frac{G+G^T}{2})_l)$	−0.0045	−0.0350	−0.0724	−0.0808	−0.0842	−0.1544	−0.1719

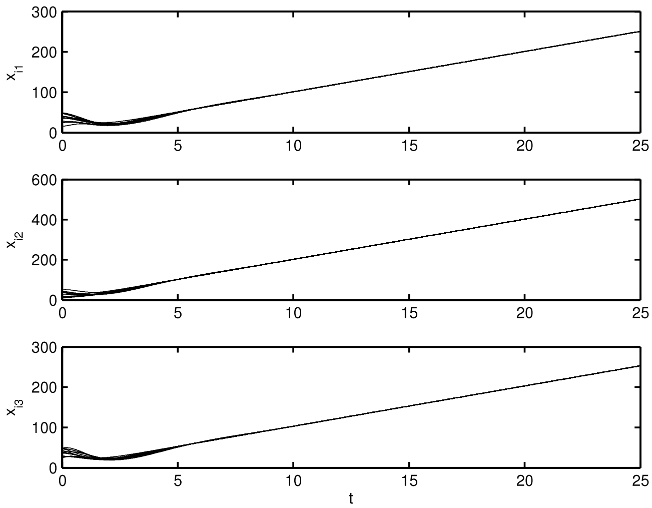

Fig. 5. Positions of ten pinning-controlled agents with constant reference velocity.

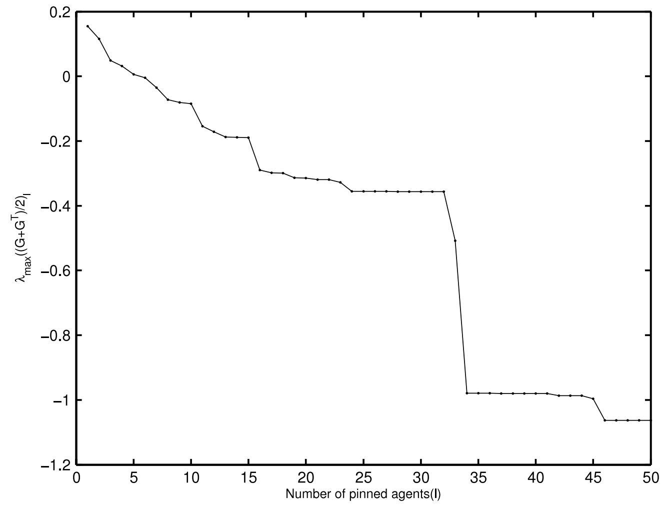

Fig. 7. Variation of $\lambda_{\max}((\frac{G+G^T}{2})_l)$ versus the number of pinned agents.

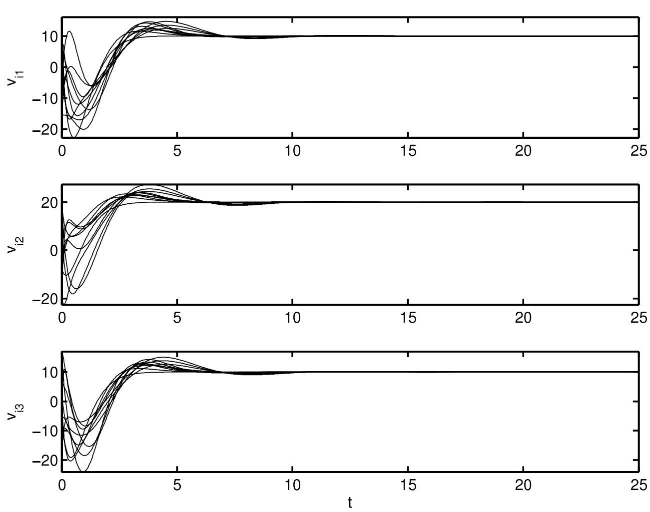

Fig. 6. Velocities of ten pinning-controlled agents with constant reference velocity.

Suppose that the interaction diagraph of multi-agent system (29) is a directed small-world network, which is generated by the algorithms in [40].

Using the pinned-agent selection scheme in Section 3.4, we rearrange the order of one hundred agents and plot the variation of $\lambda_{\max}((\frac{G+G^T}{2})_l)$ versus the number of pinned agents l in Fig. 7, from which it is noted that $\lambda_{\max}((\frac{G+G^T}{2})_l)$ decreases with the increase of l. When $l \le 5$, $\lambda_{\max}((\frac{G+G^T}{2})_l) > 0$ and $\lambda_{\max}((\frac{G+G^T}{2})_6) = -0.0045$. Thus, at least we need to pin the first six rearranged agents. Table 4 provides partial combinations of l and $\lambda_{\max}((\frac{G+G^T}{2})_l)$.

Case 1: *Consensus of multi-agent system with time-varying reference velocity*

Suppose that multi-agent system (29) is composed of one hundred coupled forced pendulums [19,20], in which the dynamics

Table 5
Combinations of l and c^* for case 1.

l	6	7	8	9	10	11	12
c^*	416.6667	53.5714	25.8978	23.2054	22.2684	12.1438	10.9075

of each agent is governed by

$$\dot{x}_i(t) = v_i(t)$$
$$\dot{v}_i(t) = -\sin(x_i) - 0.25 v_i + 1.5\cos(2.5t) + u_i(t), \quad i = 1, \ldots, 100. \tag{30}$$

Let $\varepsilon = 0.5$. In view of Assumption 1, (7) and (30), we obtain $p = 1.625$, $q = 0.875$, $\rho = 1.875$. According to the pinned-agent selection scheme in Section 3.4, we get some combinations of l and c^* in Table 5. When $l = 6$, $c^* = 416.6667$, which is quite large. When $l = 7$, $c^* = 53.5714$; letting $c = 53.6$, we have $d^* = 190.2589$ which is not quite satisfactory. Note that $c^* = 25.8978$ when $l = 8$; taking $c = 26$, we estimate the lower bound of pinning feedback gains to be $d^* = 51.8773$, which is good enough for practical use. Choose $d_i = 52.0$, $i = 1, \ldots, 8$ as pinning feedback gains. With pinning control algorithm (6), the evolutions of the positions and velocities of one hundred agents are shown in Figs. 8 and 9, respectively. Note that the pinning-controlled multi-agent system with one hundred agents can reach second-order leader-following consensus by only pinning eight agents.

Case 2: *Consensus of multi-agent system with constant reference velocity*

When $f \equiv 0$, the multi-agent system (29) has double-integrator dynamics and the velocity of the virtual leader is a constant.

Partial combinations of l and c^* are shown in Table 6. Like the nonlinear case, we pin the first eight rearranged agents. From Table 6, we know that $c^* = 13.8122$ when $l = 8$. Taking $c = 13.9$, we obtain $d^* = 27.7341$. Let $d_i = 28.0$, $i = 1, \ldots, 8$ be pinning feedback gains. The initial values of reference position and velocity are given by $x_r(0) = 1$ and $v_r(0) = 5$, respectively. With pinning control algorithm (23), the evolutions of positions and velocities of

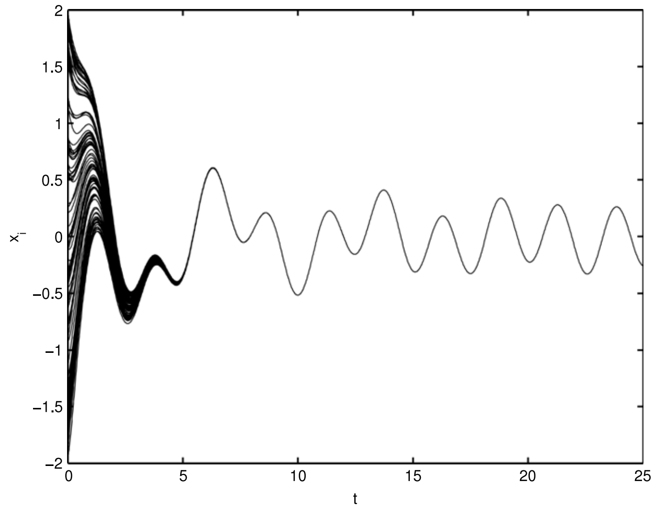

Fig. 8. Positions of one hundred pinning-controlled agents with time-varying reference velocity.

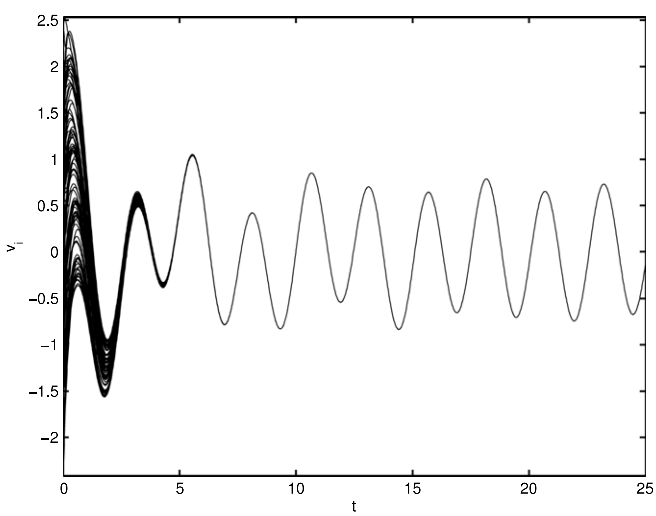

Fig. 9. Velocities of one hundred pinning-controlled agents with time-varying reference velocity.

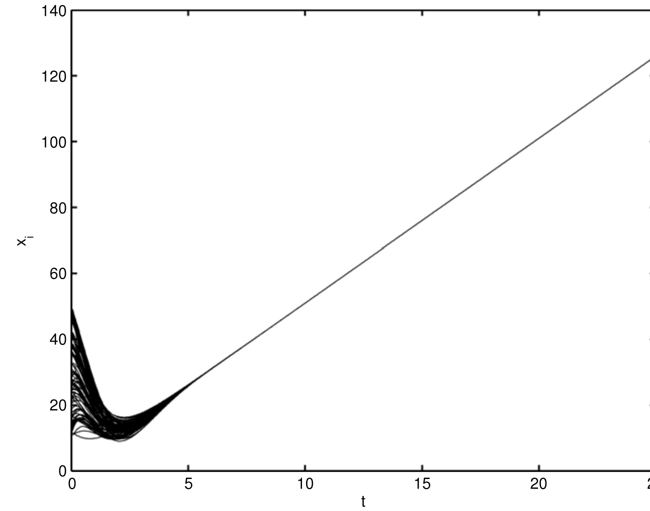

Fig. 10. Positions of one hundred pinning-controlled agents with constant reference velocity.

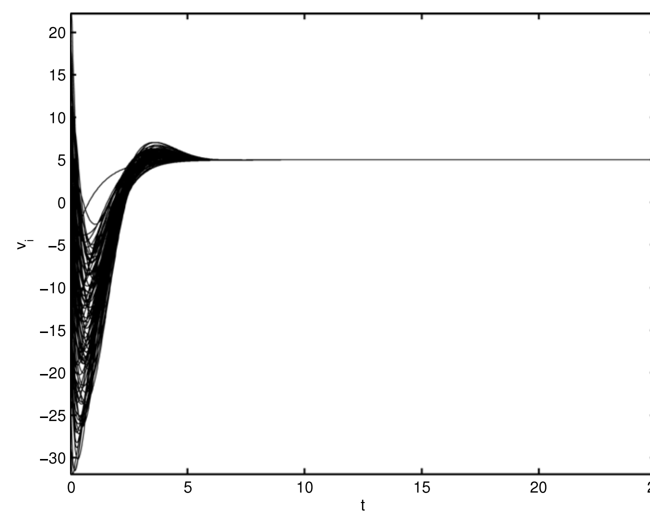

Fig. 11. Velocities of one hundred pinning-controlled agents with constant reference velocity.

Table 6
Combinations of l and c^* for case 2.

l	6	7	8	9	10	11	12
c^*	222.2222	28.5714	13.8122	12.3762	11.8765	6.4767	5.8173

one hundred agents are shown in Figs. 10 and 11, respectively. It is easy to see that the pinning-controlled one hundred agents reach second-order leader-following consensus and the velocities of all agents converge to a constant velocity, i.e., $v_i = 5, i = 1, \ldots, 100$, as $t \to \infty$.

5. Conclusions and future works

In this paper, we have studied the second-order leader-following consensus problem of nonlinear multi-agent systems composed of coupled second-order oscillators. Without assuming that the interaction diagraph contains a directed spanning tree, we have developed leader-following consensus algorithms based on pinning control for second-order multi-agent systems with time-varying and constant reference velocities, respectively. In particular, we have provided a pinned-agent selection scheme to determine what kind of followers and how many followers should be informed by the virtual leader, and have derived some sufficient conditions to guarantee the global asymptotic stability of second-order leader-following consensus. In our future work, we will extend the theoretical analysis to high-order multi-agent systems, and consider the effects of time-delay, nonlinear couplings, and switching interaction graph.

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