Stability and Hopf Bifurcation of a General Delayed Recurrent Neural Network

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***Abstract*—In this paper, stability and bifurcation of a general recurrent neural network with multiple time delays is considered, where all the variables of the network can be regarded as bifurcation parameters. It is found that Hopf bifurcation occurs when these parameters pass through some critical values where the conditions for local asymptotical stability of the equilibrium are not satisfied. By analyzing the characteristic equation and using the frequency domain method, the existence of Hopf bifurcation is proved. The stability of bifurcating periodic solutions is determined by the harmonic balance approach, Nyquist criterion, and graphic Hopf bifurcation theorem. Moreover, a critical condition is derived under which the stability is not guaranteed, thus a necessary and sufficient condition for ensuring the local asymptotical stability is well understood, and from which the essential dynamics of the delayed neural network are revealed. Finally, numerical results are given to verify the theoretical analysis, and some interesting phenomena are observed and reported.**

***Index Terms*—Hopf bifurcation, frequency domain approach, harmonic balance, recurrent neural network, stability.**

I. Introduction

IN THE LAST three decades, neural networks, particularly the Hopfield neural network (HNN) [1] and Cohen–Grossberg neural network (CGNN) [2], have received increasing attention due to their wide and important applications in such areas as signal processing, image processing, pattern recognition, and optimizations. Some applications of neural networks require the knowledge of dynamical behaviors of the neural networks, such as the uniqueness and asymptotical stability of an equilibrium point of a specific neural network. Therefore, the problem of stability analysis for neural networks has been a focal topic of research in this field.

In practice, due to the finite speeds of the switching and transmitting signals, time delays exist in various neural networks, and therefore, should be taken into consideration [3]–[6], [37], [38]. It is well known that time delays may result in oscillatory behaviors or network instability (periodic oscillation and chaos) [10]–[18], [30], [31], [34], hence the study of delayed neural networks is very important. In fact, stability problems of delayed neural networks have been intensively studied [7], [8], [28], [38], where however, most derived conditions are sufficient conditions for the asymptotical stability and are generally too conservative.

Stability of equilibrium point in neural networks is widely investigated in associative memory, pattern recognition, and optimization, which can be considered as a single storage or memory pattern, or an optimum object. Owing to the limited information stored in equilibrium point, there have been great interests in using periodic solutions for associative memory and pattern recognition. It is known that periodic solutions can restore various complex patterns unlike most existing patterns based on the stable equilibrium point. When the dynamics of neural network pass the Hopf bifurcation, various local periodic solutions arise from the equilibrium point of neural networks. In order to realize a memory system, the authors examined the neural network dynamics phenomenon as bifurcations of attractors in [33]. A noisy self-organizing neural network with bifurcation dynamics for combinatorial optimization is investigated in [32]. Hopf bifurcation cannot only provide a guide to design stable neural networks, but also paves the way for the application of periodic solutions. Therefore, in order to reveal the dynamics of artificial neural networks, it is very urgent and significant to study the bifurcation analysis of the neural network model.

In [17], Olien and Bélair investigated the following system with two delays:

$$\dot{x}_1(t) = -x_1(t) + a_{11}f(x_1(t-\tau_1)) + a_{12}f(x_2(t-\tau_2))$$
$$\dot{x}_2(t) = -x_2(t) + a_{21}f(x_1(t-\tau_1)) + a_{22}f(x_2(t-\tau_2))$$

for which several cases were discussed, such as $\tau_1 = \tau_2, a_{11} = a_{22} = 0$, etc.

In [11], Yu and Cao extended the aforementioned model and studied the following delayed network model:

$$\dot{x}_1(t) = -a_1x_1 + b_{11}f_1(x_1(t-\tau)) + b_{12}f_2(x_2(t-\tau))$$
$$\dot{x}_2(t) = -a_2x_2 + b_{21}f_1(x_1(t-\tau)) + b_{22}f_2(x_2(t-\tau))$$

where $a_i(i=1,2)$ are positive constants, $x_1(t)$ and $x_2(t)$ denote the activations of two neurons, τ denotes the synaptic transmission delay, $b_{ij}(1 \leq i,j \leq 2)$ are the synaptic weights, $f_i(i=1,2)$ are the activations function, and $f_i : R \longrightarrow R$ are C^3 smooth functions with $f_i(0)=0$.

In [18], Campbell *et al.* studied a neural network model with multiple delays

$$C_j\dot{u}_j(t) = -\frac{1}{R_j} + F_j(u_j(t-\sigma)) + G_j(u_{j-1}(t-\tau)), \quad j=1,2,\ldots,n$$

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where $C_j > 0$ and $R_j > 0$ represent the capacitance and resistance of each neuron, respectively, and F_j and G_j are nonlinear functions representing, respectively, the feedback from neuron j to itself and the connection from j to $j-1$, for which only the case of $n = 4$ was discussed.

In [13], Song *et al.* considered a simplified BAM neural network model as follows:

$$\begin{cases} \dot{x}_1(t) = -\mu_1 x_1(t) + c_{21} f_1(y_1(t-\tau_2)) + c_{31} f_1(y_2(t-\tau_2)) \\ \dot{y}_1(t) = -\mu_2 y_1(t) + c_{12} f_2(x_1(t-\tau_1)) \\ \dot{y}_2(t) = -\mu_3 y_2(t) + c_{13} g_3(x_1(t-\tau_1)). \end{cases}$$

In [10], Yu and Cao studied a more complex BAM neural network model

$$\begin{cases} \dot{x}_1(t) = -\mu_1 x_1(t) + c_{11} f_{11}(y_1(t-\tau_3)) + c_{12} f_{12}(y_2(t-\tau_3)) \\ \dot{x}_2(t) = -\mu_2 x_2(t) + c_{21} f_{21}(y_1(t-\tau_4)) + c_{22} f_{22}(y_2(t-\tau_4)) \\ \dot{y}_1(t) = -\mu_3 y_1(t) + d_{11} g_{11}(x_1(t-\tau_1)) + d_{12} g_{12}(x_2(t-\tau_2)) \\ \dot{y}_2(t) = -\mu_4 y_2(t) + d_{21} g_{21}(x_1(t-\tau_1)) + d_{22} g_{22}(x_2(t-\tau_2)). \end{cases}$$

From this introduction, it is easy to see that bifurcation analysis about delayed systems under investigation today are mainly lower dimensional systems with a few time delays. The dimensions of the delayed systems studied in [9]–[18], [29] are no more than four, and actually with no more than two delays as bifurcation parameters. Though in [4], an n-dimensional delayed model was investigated, the model is very simple with only one delay. It is well known that neural networks are very complex and large scale nonlinear systems, but neural network models under study today have been dramatically simplified [10]–[18], [29]. In this paper, a general model with multiple delays is investigated, aiming at giving a clearer understanding of the transition from stability to bifurcation and explaining some phenomena that otherwise cannot be interpreted by the existing results.

Hopf bifurcation of dynamical systems is investigated in [9], [10], [22], and [35] by using the normal form method and center manifold theorem. Then, a new method of multiple scales versus center manifold is used for order reduction of retarded nonlinear systems [36]. However, these approaches are still difficult to analyze Hopf bifurcation of a general system with multiple delays. To the best of our knowledge, the Hopf bifurcation of a general neural network model with multiple time delays has not been investigated elsewhere. In this paper, we try to find out some critical conditions under which the stability is not guaranteed and the Hopf bifurcation occurs when the parameters of the network model pass through some critical values. The results obtained in this paper give an explicit view of the dynamics of a general delayed neural network based on Hopf bifurcation analysis from the harmonic balance approach, Nyquist criterion, and graphic Hopf bifurcation theorem [20], [21], rather than the normal form method and center manifold theorem [22].

The rest of this paper is organized as follows. In Section II, local asymptotical stability analysis is established by analyzing the characteristic equation and using Nyquist criterion. Then applying harmonic balance approach and graphic Hopf bifurcation theorem, existence and stability of bifurcating periodic solutions of a general neural network are investigated in Section III. In Section IV, simulation examples are constructed to verify the theoretical analysis in this paper. Finally, the conclusions are drawn.

II. Local Asymptotical Stability Analysis

Consider the following delayed recurrent neural network model:

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t-\tau)) + E \quad (1)$$

namely

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j)) + E_i, \qquad i = 1, 2, \ldots, n \quad (2)$$

where n denotes the number of neurons in the network, $\tau_j (j = 1, 2, \ldots, n)$ are the time delays, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in R^n$ is the state vector associated with the neurons, $E = (E_1, E_2, \ldots, E_n)^T \in R^n$ is the external input vector, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)))^T \in R^n$ and $f(x(t-\tau)) = (f_1(x_1(t-\tau_1)), f_2(x_2(t-\tau_2)), \ldots, f_n(x_n(t-\tau_n)))^T \in R^n$ correspond to the activation functions and delayed activation functions of neurons, respectively, $C = diag(c_1, c_2, \ldots, c_n) > 0, A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively, and the initial conditions are given by $\phi_i(t) \in \mathcal{C}([-r, 0], R)$, where $r = \max_{1\le i\le n}\{\tau_i\}$, with $\mathcal{C}([-r, 0], R)$ denoting the set of all continuous functions from $[-r, 0]$ to R.

Assume that model (1) has an equilibrium $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ for a given E. Without loss of generality, assume the equilibrium point x^*. Using the transformation

$$y(t) = x(t) - x^* \quad y(t-\tau) = x(t-\tau) - x^*$$

model (1) can be transformed into the following form:

$$\dot{y}(t) = -Cy(t) + Ag(y(t)) + Bg(y(t-\tau)) \quad (3)$$

namely

$$\dot{y}_i(t) = -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} g_j(y_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(y_j(t-\tau_j)), \qquad i = 1, 2, \ldots, n \quad (4)$$

where $g(y(t)) = (g_1(y_1(t)), g_2(y_2(t)), \ldots, g_n(y_n(t)))^T \in R^n$ with $g_i(y_i(t)) = f_i(y_i(t) + x_i^*) - f_i(x_i^*)$ and $g(0) = 0$.

Next, the stability and Hopf bifurcation of delayed system (3) are discussed.

By introducing a "state-feedback control" u, one obtains a linear system with a nonlinear feedback, which is equivalent to (3), as follows:

$$\begin{cases} \dot{y}(t) = -Cy(t) + u(z(t)) \\ z(t) = -y(t) \\ u(z(t)) = Ag(-z(t)) + Bg(-z(t-\tau)) \end{cases} \quad (5)$$

where $z(t) = (z_1(t), z_2(t), \ldots, z_n(t))^T \in R^n$ and $z(t-\tau) = (z_1(t-\tau_1), z_2(t-\tau_2), \ldots, z_n(t-\tau_n))^T \in R^n$.

In order to study the Hopf bifurcation of delayed neural networks, one can parameterize the feedback system (5) as

$$\begin{cases} \dot{y}(t) = -C(\mu)y(t) + u(z(t);\mu) \\ z(t) = -y(t) \\ u(z(t);\mu) = A(\mu)g(-z(t);\mu) + B(\mu)g(-z(t-\tau(\mu));\mu) \end{cases} \tag{6}$$

where μ is the bifurcation parameter.

Next, taking a Laplace transform $\mathcal{L}(\cdot)$ on (6) yields

$$\mathcal{L}(y) = [sI + C(\mu)]^{-1}\mathcal{L}(u(z;\mu))$$

and

$$\begin{aligned} \mathcal{L}(z) &= -\mathcal{L}(y) = -[sI + C(\mu)]^{-1}\mathcal{L}(u(z;\mu)) \\ &= -G(s;\mu)\mathcal{L}(u(z;\mu)) \end{aligned} \tag{7}$$

where

$$G(s;\mu) = [sI + C(\mu)]^{-1} \tag{8}$$

is the standard transfer matrix of the linear part of the system. Throughout this paper, I is the n-dimensional identity matrix.

It follows from (7) that one may only deal with $z(t)$ in the frequency domain, without directly considering $y(t)$. In so doing, first observe that if y^* is an equilibrium solution of the first equation of (6), then

$$z^* = -G(0;\mu)u(z^*;\mu). \tag{9}$$

Let $u(z(t);\mu) = A(\mu)\widetilde{u}(z(t);\mu) + B\widetilde{u}(z(t-\tau(\mu));\mu)$, where $\widetilde{u}(z(t);\mu) = g(-z(t);\mu)$.

Clearly, $y = 0$ is the equilibrium of the linearized feedback system. If one linearizes the feedback system about the equilibrium z^*, then the Jacobian of $\widetilde{u}$ is given by

$$J(\mu) = \left(\frac{\partial \widetilde{u}}{\partial z}\right)\bigg|_{z=0} \tag{10}$$

where $J(\mu) = (J_{ij})_{n\times n}, J_{ij} = (\partial\widetilde{u}_i/\partial z_j)|_{y=0}(i,j = 1,2,\ldots,n)$. The Jacobian of the nonlinear feedback u is then given by $\widetilde{J}(s;\mu) = A(\mu)J(\mu) + B(\mu)J(\mu)e^{-s\tau(\mu)}$, where $e^{-s\tau(\mu)} = diag(e^{-s\tau_1(\mu)}, e^{-s\tau_2(\mu)}, \ldots, e^{-s\tau_n(\mu)})$. The closed-loop transfer matrix of the linearized feedback system (6) is

$$H(s;\mu) = -[I + G(s;\mu)\widetilde{J}(s;\mu)]^{-1}G(s;\mu). \tag{11}$$

To this end, stability analysis is established based on the Nyquist stability criterion [27]. Let $\Re\{\cdot\}$ be the real part of the complex constant.

Lemma 1 [21]: Let $G \in R$ have $p(G)$ poles in $\Re(s) > 0$ and let $\mathcal{D}$ be a simple closed contour consisting of an interval $[-i\omega, i\omega]$ on the imaginary axis together with a semicircle in $\Re(s) > 0$, large enough to contain all the poles in $\Re(s) > 0$. Suppose $\mathcal{D}$ is indented if necessary to exclude any poles on the imaginary axis. Let Γ_G be the image of $\mathcal{D}$ under G as $\mathcal{D}$ is traversed clockwise. Then, H, defined in (11), has no poles in $\Re(s) \geq 0$ if Γ_G encircles $-1/\widetilde{J}(s)$ $p(G)$ times counterclockwise, and $|1 + G(s)\widetilde{J}(s)| \nrightarrow 0$ as $|s| \to \infty$.

Next, a theorem is given to ensure the local asymptotical stability of the nonlinear feedback system (6).

Theorem 1 [21]: Let $G(s)\widetilde{J}(s) \in R^{n\times n}$ have characteristic functions $\lambda_1(s), \lambda_2(s), \ldots, \lambda_q(s)$, with a total of $p(G\widetilde{J})$ poles counted according to multiplicity. Let the jth characteristic locus Γ_{λ_j} encircle -1 for a total of n_j times counterclockwise. Then, the closed-loop system (5) is stable if $\sum_{j=1}^{q} n_j = p(G\widetilde{J})$. In this case, the recurrent neural network (3) is locally asymptotically stable.

Applying the generalized Nyquist stability criterion, the following result can be established.

Theorem 2 [20]: If an eigenvalue of the corresponding Jacobian of the nonlinear system, in the time domain, assumes a purely imaginary value $i\omega_0$ at a particular value $\mu = \mu_0$, then the corresponding eigenvalue of the constant matrix $[G(i\omega_0;\mu_0)\widetilde{J}(i\omega_0;\mu_0)]$ in the frequency domain must assume the value $-1 + i0$ at $\mu = \mu_0$.

Set

$$h(\lambda, s;\mu) = |\lambda I - G(s;\mu)\widetilde{J}(s;\mu)|. \tag{12}$$

To apply Theorem 2, let $\widehat{\lambda} = \widehat{\lambda}(i\omega;\mu)$ be the eigenvalue of $G(i\omega;\mu)\widetilde{J}(i\omega;\mu)$ that satisfies $\widehat{\lambda}(i\omega_0;\mu_0) = -1 + i0$. Then

$$h(-1, s;\mu) = |-I - (sI + C)^{-1}(AJ + BJe^{-s\tau})|. \tag{13}$$

Let $h(-1, s;\mu) = 0$. Then

$$|(sI + C) + (AJ + BJe^{-s\tau})| = 0. \tag{14}$$

It is easy to see that (14) is equivalent to the characteristic equation of (3).

III. Existence and Stability of Bifurcating Periodic Solutions

Based on Theorems 1 and 2 and the results in [23]–[26], we now derive some formulas for the existence and the stability of bifurcating periodic solutions.

Suppose that a second-order harmonic balance approximation for the solution has the following form:

$$z(t) = z^* + \Re\left\{\sum_{k=0}^{2} Z_k e^{ik\omega t}\right\} \tag{15}$$

where z^* is the equilibrium point.

Let $G(i\omega) = G(s;\mu)|_{s=i\omega}$. Then, equating the input and the output of (6) gives

$$Z_k = -G(i\omega)F_k, \qquad k = 0, 1, 2 \tag{16}$$

where F_k is the Fourier coefficients of $u(z(t);\mu)$. Equation (16) is known as the second-order harmonic balance equation [20], [24], [25], [26]. By expanding the trial expression for z, which contains the time delays, one has

$$z(t - \tau(\mu)) = z^* + \Re\left\{\sum_{k=0}^{2} e^{-ik\omega\tau(\mu)} Z_k e^{ik\omega t}\right\}. \tag{17}$$

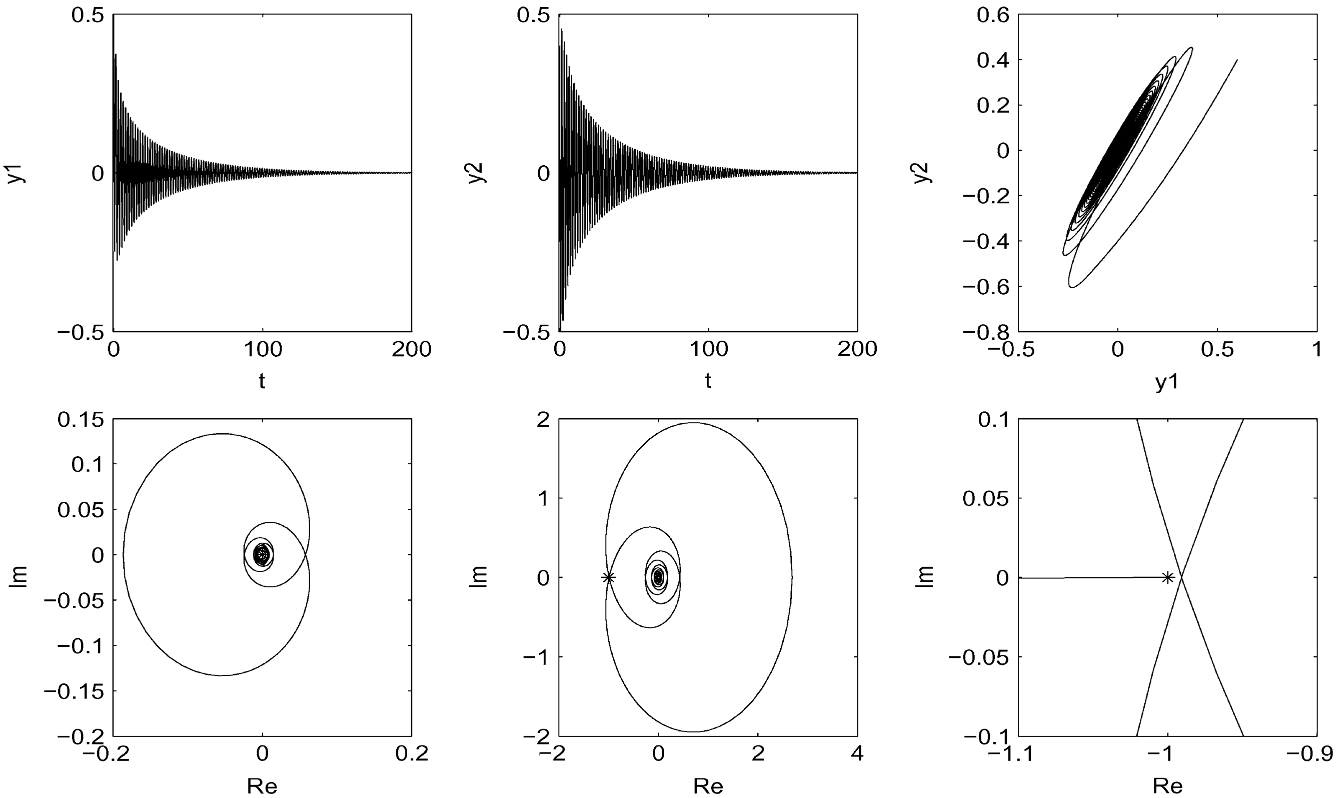

Fig. 1. Waveform, phase, frequency, and magnification of the frequency graph ($\tau_1 = 0.48, \tau_2 = 0.52$).

In order to derive the main results, it is convenient to use the following notations:

$$D_2^1 = A(\widetilde{\mu})\left.\frac{\partial^2 \widetilde{u}(z;\widetilde{\mu})}{\partial z^2}\right|_{z=0} + B(\widetilde{\mu})\left.\frac{\partial^2 \widetilde{u}(z;\widetilde{\mu})}{\partial z^2}\right|_{z=0} \times \left(I \otimes e^{-i\widetilde{\omega}\tau(\widetilde{\mu})}\right) \tag{18}$$

$$D_2^2 = A(\widetilde{\mu})\left.\frac{\partial^2 \widetilde{u}(z;\widetilde{\mu})}{\partial z^2}\right|_{z=0} + B(\widetilde{\mu})\left.\frac{\partial^2 \widetilde{u}(z;\widetilde{\mu})}{\partial z^2}\right|_{z=0} \times \left(e^{i\widetilde{\omega}\tau(\widetilde{\mu})} \otimes e^{-2i\widetilde{\omega}\tau(\widetilde{\mu})}\right) \tag{19}$$

$$D_3 = A(\widetilde{\mu})\left.\frac{\partial^3 \widetilde{u}(z;\widetilde{\mu})}{\partial z^3}\right|_{z=0} + B(\widetilde{\mu})\left.\frac{\partial^3 \widetilde{u}(z;\widetilde{\mu})}{\partial z^3}\right|_{z=0} \times \left(e^{-i\widetilde{\omega}\tau(\widetilde{\mu})} \otimes e^{-i\widetilde{\omega}\tau(\widetilde{\mu})} \otimes e^{i\widetilde{\omega}\tau(\widetilde{\mu})}\right) \tag{20}$$

$$D_2^3 = A(\widetilde{\mu})\left.\frac{\partial^2 \widetilde{u}(z;\widetilde{\mu})}{\partial z^2}\right|_{z=0} + B(\widetilde{\mu})\left.\frac{\partial^2 \widetilde{u}(z;\widetilde{\mu})}{\partial z^2}\right|_{z=0} \times \left(e^{-i\widetilde{\omega}\tau(\widetilde{\mu})} \otimes e^{i\widetilde{\omega}\tau(\widetilde{\mu})}\right) \tag{21}$$

$$D_2^4 = A(\widetilde{\mu})\left.\frac{\partial^2 \widetilde{u}(z;\widetilde{\mu})}{\partial z^2}\right|_{z=0} + B(\widetilde{\mu})\left.\frac{\partial^2 \widetilde{u}(z;\widetilde{\mu})}{\partial z^2}\right|_{z=0} \times \left(e^{-i\widetilde{\omega}\tau(\widetilde{\mu})} \otimes e^{-i\widetilde{\omega}\tau(\widetilde{\mu})}\right) \tag{22}$$

where $\otimes$ is the tensor product operator, $\widetilde{\mu}$ is the fixed value of the parameter μ, and $\widetilde{\omega}$ is the frequency of the intersection between the $\widehat{\lambda}$ locus and the negative real axis closest to the point $(-1 + i0)$.

Combining (15)–(22), one obtains

$$\begin{aligned} Z_0 &= -G(0;\widetilde{\mu})F_0. \\ &= -G(0;\widetilde{\mu})\left[J(0;\widetilde{\mu})Z_0 + \frac{D_2^3}{2!}\left(\frac{1}{2}Z_1 \otimes \bar{Z}_1\right) + \rho_0\right] \end{aligned} \tag{23}$$

$$\begin{aligned} Z_1 &= -G(i\widetilde{\omega};\widetilde{\mu})F_1. \\ &= -G(i\widetilde{\omega};\widetilde{\mu})\left[J(i\widetilde{\omega};\widetilde{\mu})Z_1 + \frac{D_2^1}{2!}(2Z_0 \otimes Z_1) + \frac{D_2^2}{2!} \times (\bar{Z}_1 \otimes Z_2) + \frac{D_3}{3!}\left(\frac{3}{4}Z_1 \otimes Z_1 \otimes \bar{Z}_1\right) + \rho_1\right] \end{aligned} \tag{24}$$

$$\begin{aligned} Z_2 &= -G(2i\widetilde{\omega};\widetilde{\mu})F_2. \\ &= -G(2i\widetilde{\omega};\widetilde{\mu})\left[J(2i\widetilde{\omega};\widetilde{\mu})Z_2 + \frac{D_2^4}{2!}\left(\frac{1}{2}Z_1 \otimes Z_1\right) + \frac{\rho_2}{2}\right] \end{aligned} \tag{25}$$

where $\bar{\cdot}$ denotes the complex conjugate, and ρ_0, ρ_1, ρ_2 are higher order terms. From (23)–(25), one can calculate Z_0 and Z_2 as functions of Z_1, which yields

$$Z_0 = -[I + G(0;\widetilde{\mu})J(0;\widetilde{\mu})]^{-1}G(0;\widetilde{\mu})\frac{D_2^3}{2!}\left(\frac{1}{2}Z_1 \otimes \bar{Z}_1\right) \tag{26}$$

$$Z_2 = -[I + G(2i\widetilde{\omega};\widetilde{\mu})J(2i\widetilde{\omega};\widetilde{\mu})]^{-1}G(2i\widetilde{\omega};\widetilde{\mu})\frac{D_2^4}{2!}\left(\frac{1}{2}Z_1 \otimes Z_1\right). \tag{27}$$

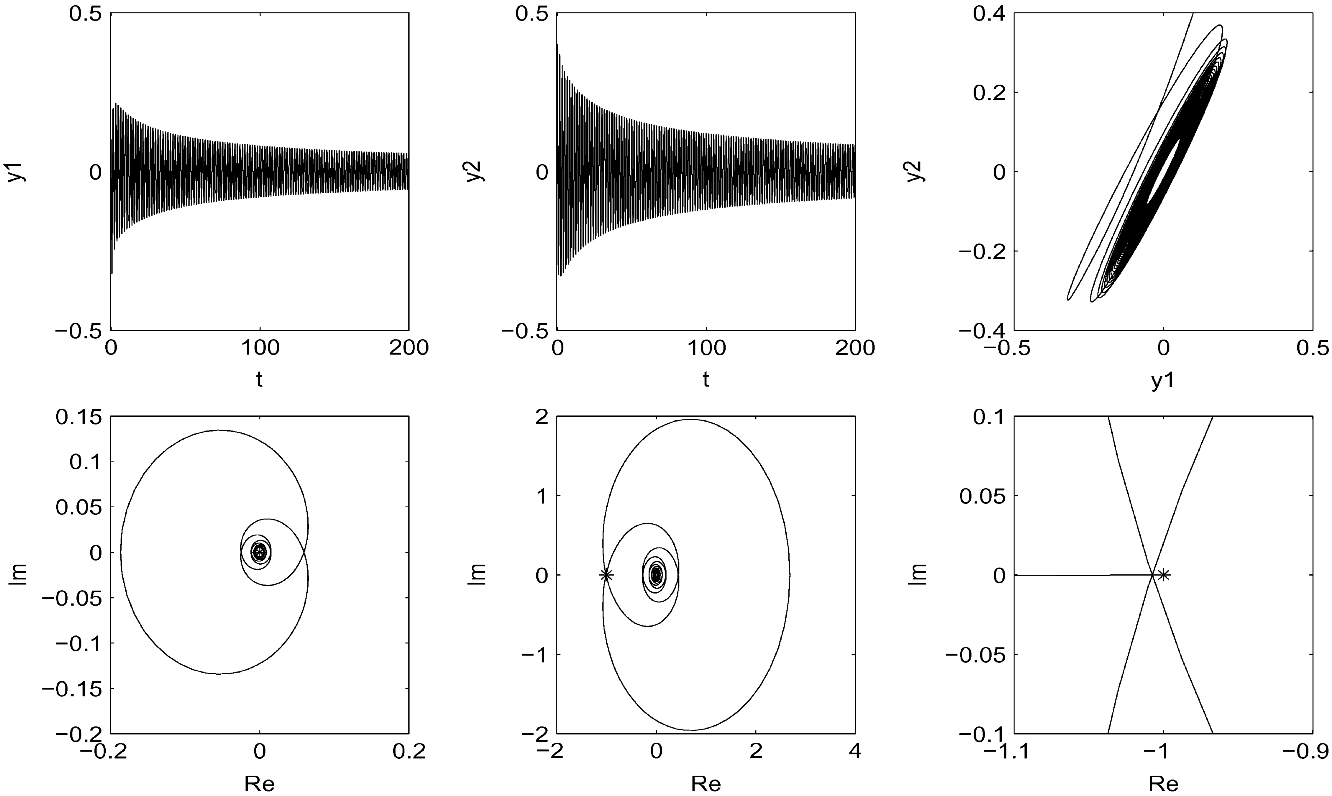

Fig. 2. Waveform, phase, frequency, and magnification of the frequency graph ($\tau_1 = 0.50, \tau_2 = 0.53$).

The complex number $\xi_1(\widetilde{\omega})$, which is used to calculate the amplitude of the emerging periodic solution, is given by

$$\xi_1(\widetilde{\omega}) = \frac{-w^T[G(i\widetilde{\omega};\widetilde{\mu})]p_1}{w^T v} \tag{28}$$

where

$$p_1 = \left[D_2^1(V_{02}\otimes v) + \frac{1}{2}D_2^2(\bar{v}\otimes V_{22}) + \frac{1}{8}D_3(v\otimes v\otimes \bar{v})\right] \tag{29}$$

$$V_{02} = -\frac{1}{4}[I + G(0;\widetilde{\mu})J(0;\widetilde{\mu})]^{-1}G(0;\widetilde{\mu})D_2^3(v\otimes\bar{v}) \tag{30}$$

$$V_{22} = -\frac{1}{4}[I + G(2i\widetilde{\omega};\widetilde{\mu})J(2i\widetilde{\omega};\widetilde{\mu})]^{-1}G(2i\widetilde{\omega};\widetilde{\mu})D_2^4(v\otimes v) \tag{31}$$

and w^T and v are the left and the right eigenvectors of $[G(i\widetilde{\omega};\widetilde{\mu})]J(i\widetilde{\omega};\widetilde{\mu})$, respectively, associated with the value $\widehat{\lambda}(i\widetilde{\omega};\widetilde{\mu})$ that is the closest eigenvalue to the critical point $(-1+i0)$. Clearly, V_{02} and V_{22} are given by (30) and (31) after the replacements $Y_1 = v\theta, Y_0 = V_{02}\theta^2$ and $Y_2 = V_{02}\theta^2$, where θ is a measure of the amplitude of the periodic solution [24]–[26]. For more details about the harmonic balance approach, the reader is referred to [20].

Now, the following Hopf bifurcation theorem formulated in the frequency domain can be stated [20]:

Theorem 3 (The Graphical Hopf Bifurcation Theorem): Suppose that when ω varies, the vector $\xi_1(\widetilde{\omega}) \neq 0$, where $\xi_1(\widetilde{\omega})$ is defined in (28), and that the half-line L_1, starting from $-1+i0$ and pointing to the direction parallel to that of $\xi_1(\widetilde{\omega})$, first intersects the locus of the eigenvalue $\widehat{\lambda}(i\omega;\widetilde{\mu})$ at the point

$$\widehat{P} = \widehat{\lambda}(\widehat{\omega};\widetilde{\mu}) = -1 + \xi_1(\widetilde{\omega})\theta^2 \tag{32}$$

at which $\omega = \widehat{\omega}$ and the constant $\theta = \theta(\widehat{\omega}) \geq 0$. Suppose, furthermore, that the previous intersection is transversal, namely

$$\begin{vmatrix} \Re\{\xi_1(\widehat{\omega})\} & \Im\{\xi_1(\widehat{\omega})\} \\ \Re\left\{\frac{d}{d\omega}\widehat{\lambda}(\omega;\widetilde{\mu})\big|_{\omega=\widehat{\omega}}\right\} & \Im\left\{\frac{d}{d\omega}\widehat{\lambda}(\omega;\widetilde{\mu})\big|_{\omega=\widehat{\omega}}\right\} \end{vmatrix} \neq 0. \tag{33}$$

Then, the following conclusions hold.

1) The nonlinear system (6) has a periodic solution $y(t) = y(t;\widehat{y})$. Consequently, there exists a unique limit cycle in the nonlinear (3).
2) If the half-line L_1 first intersects the locus of $\widehat{\lambda}(i\omega)$ when $\widetilde{\mu} > \mu_0(< \mu_0)$, and then the bifurcating periodic solution exists and the Hopf bifurcation is supercritical (subcritical).
3) If the total number of counterclockwise encirclements of the point $P_1 = \widehat{P} + \varepsilon\xi_1(\widetilde{\omega})$, for a small enough $\varepsilon > 0$, is equal to the number of poles of $\lambda(s)$ that have positive real parts, then the limit cycle is stable; otherwise, it is unstable.

It is easy to see that $s = -c_i (i = 1, 2, \ldots, n)$ are the poles of $\lambda(s)$, and the number of poles of $\lambda(s)$ that have positive real parts is zero. Hence, the following Corollary is established.

Corollary 1: Let k be the total number of counterclockwise encirclements of the point $P_1 = \widehat{P} + \varepsilon\xi_1(\widetilde{\omega})$ for a small enough $\varepsilon > 0$, where $\widehat{P}$ is the intersection of the half-line L_1 and the locus $\widehat{\lambda}(i\omega)$. Then, the following conclusions hold.

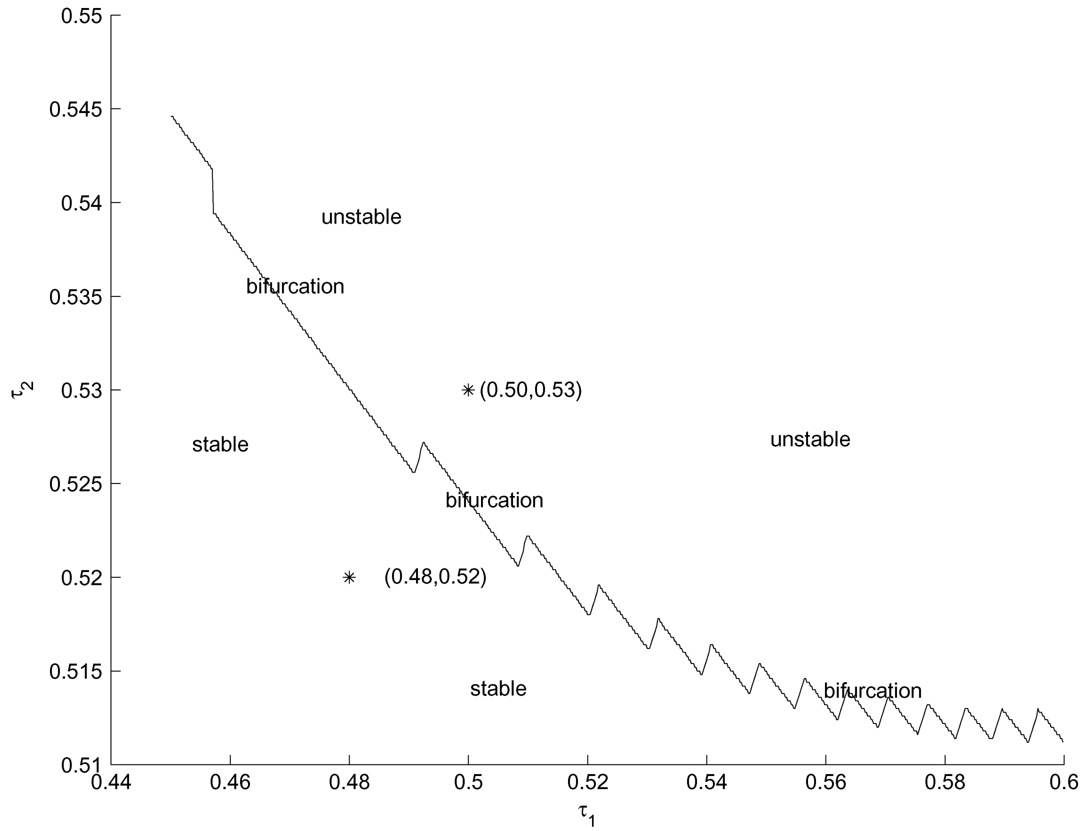

Fig. 3. Two-parameter (τ_1 and τ_2) bifurcation diagram.

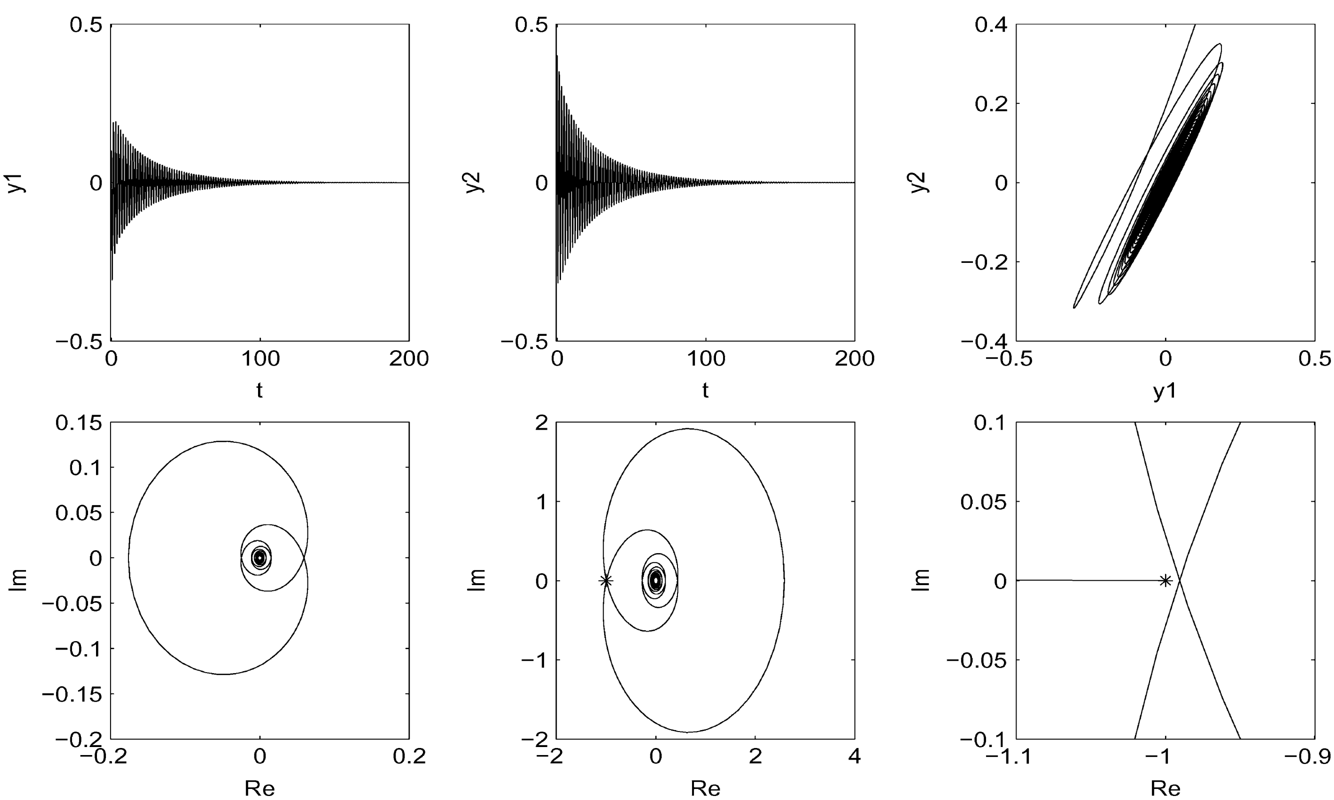

Fig. 4. Waveform, phase, frequency, and magnification of the frequency graph ($c_1 = 1.1, \tau_2 = 0.515$).

1) If $k = 0$, then the bifurcating periodic solutions of system (3) are stable.
2) If $k \neq 0$, then the bifurcating periodic solutions of system (3) are unstable.

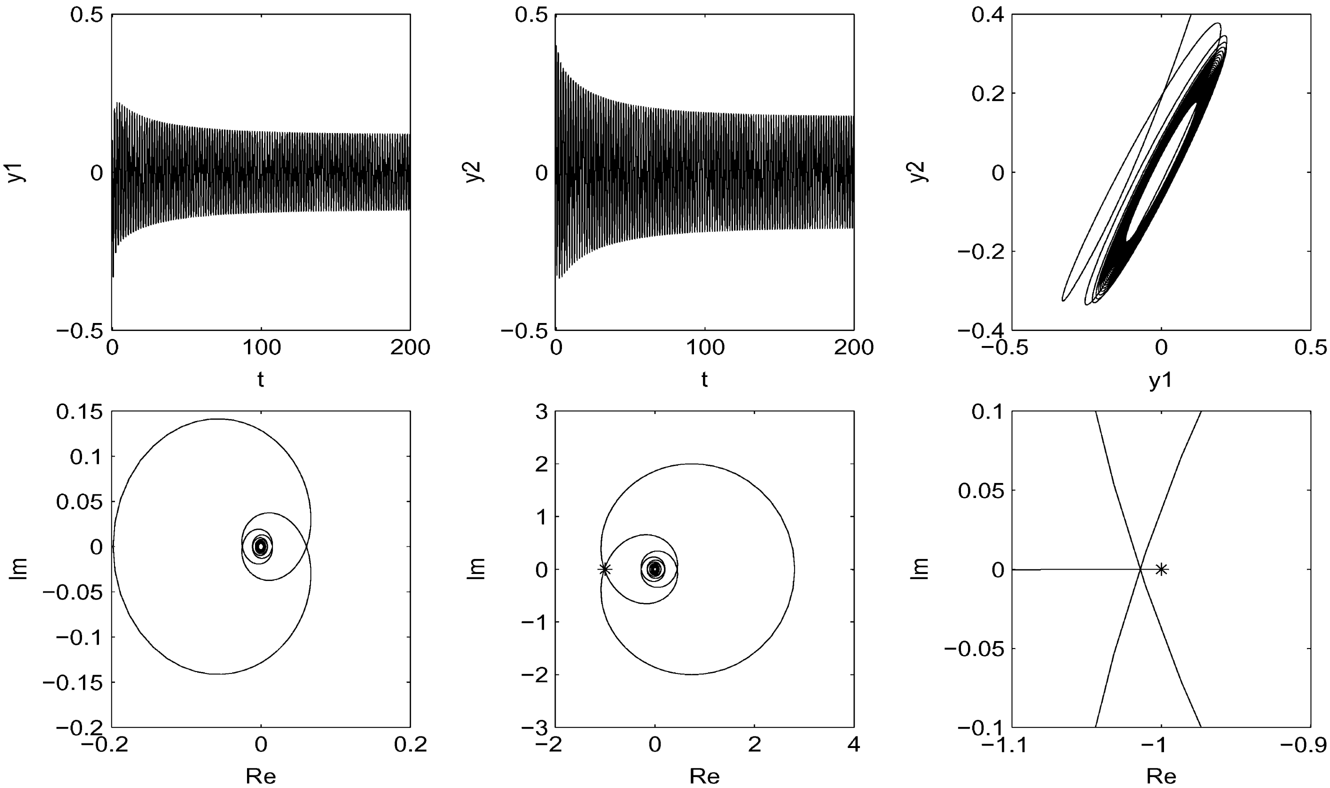

Fig. 5. Waveform, phase, frequency, and magnification of the frequency graph $(c_1 = 0.9, \tau_2 = 0.53)$.

IV. Numerical Examples

In this section, some numerical examples are given to verify the theoretical analysis. Stability of system (3) can be justified by Theorem 1, and thus the stability of the delayed system (1) can be discussed by a transformation. The half-line L_1 and the locus $\widehat{\lambda}(i\omega)$ are shown by the corresponding frequency graphs. If they intersect, a limit cycle exists, or else, no limit cycle exists. Corollary 1 implies that the stability of the bifurcating periodic solution is determined by the total number k of the counterclockwise encirclements of the point $P_1 = \widehat{P} + \varepsilon\xi_1(\widetilde{\omega})$ for a small enough $\varepsilon > 0$. Suppose that the half-line L_1 and the locus $\widehat{\lambda}(i\omega)$ intersect. If $k = 0$, the bifurcating periodic solutions of system (3) is stable; if $k \neq 0$, the bifurcating periodic solutions of system (3) is unstable.

Example 1: Consider the following delayed recurrent neural network:

$$\dot{y}(t) = -Cy(t) + Ag(y(t)) + Bg(y(t-\tau)) \tag{34}$$

where

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$
$$g(y) = \begin{pmatrix} -\tanh(y) \\ -\tanh(y) \end{pmatrix}$$

are the same as those discussed in [11]. From [11], $\tau_0 = 0.5183$ is a bifurcation parameter if $\tau_1 = \tau_2$. If $\tau_1 = \tau_2 < \tau_0$, system (34) is locally asymptotically stable; otherwise, if $\tau_1 = \tau_2 > \tau_0$, a periodic solution emerges. Many published papers [10]–[18] have considered the case of only one delay as the bifurcation parameter.

In this paper, the two delays τ_1 and τ_2 are both considered as bifurcation parameters. First, we choose $\tau_1 = 0.48$ and $\tau_2 = 0.52$, respectively. The corresponding waveform, phase, and frequency graph are shown in Fig. 1. The half-line L_1 and locus $\widehat{\lambda}(i\omega)$ do not intersect, so no limit cycle exists. By Theorems 1 and 3, we know that in Fig. 1 its zero solution is asymptotically stable.

Next, we choose $\tau_1 = 0.50$ and $\tau_2 = 0.53$, respectively. The corresponding waveform, phase, and frequency graph are shown in Fig. 2. By Theorem 3, we know that the half-line L_1 intersects the locus $\widehat{\lambda}(i\omega)$, so a limit cycle exists. The total number k of the counterclockwise encirclements of the point $P_1 = \widehat{P} + \varepsilon\xi_1(\widetilde{\omega})$ for a small enough $\varepsilon > 0$ is 0, i.e., $k = 0$, so by Corollary 1, a stable periodic solution exists.

Finally, we show a bifurcation diagram in a local region to verify the theoretical analysis, which is shown in Fig. 3. Here, τ_1 and τ_2 are considered as parameters. In some region of Fig. 3, system (34) is locally asymptotically stable, while in some other region, it is unstable. Hopf bifurcation occurs when τ_1 and τ_2 pass through some critical values where the stability condition of the equilibrium is not satisfied.

Example 2: Consider the following delayed recurrent neural network:

$$\dot{y}(t) = -Cy(t) + Ag(y(t)) + Bg(y(t-\tau)) \tag{35}$$

where

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & 2 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$
$$g(y) = \begin{pmatrix} -\tanh(y) \\ -\tanh(y) \end{pmatrix}$$

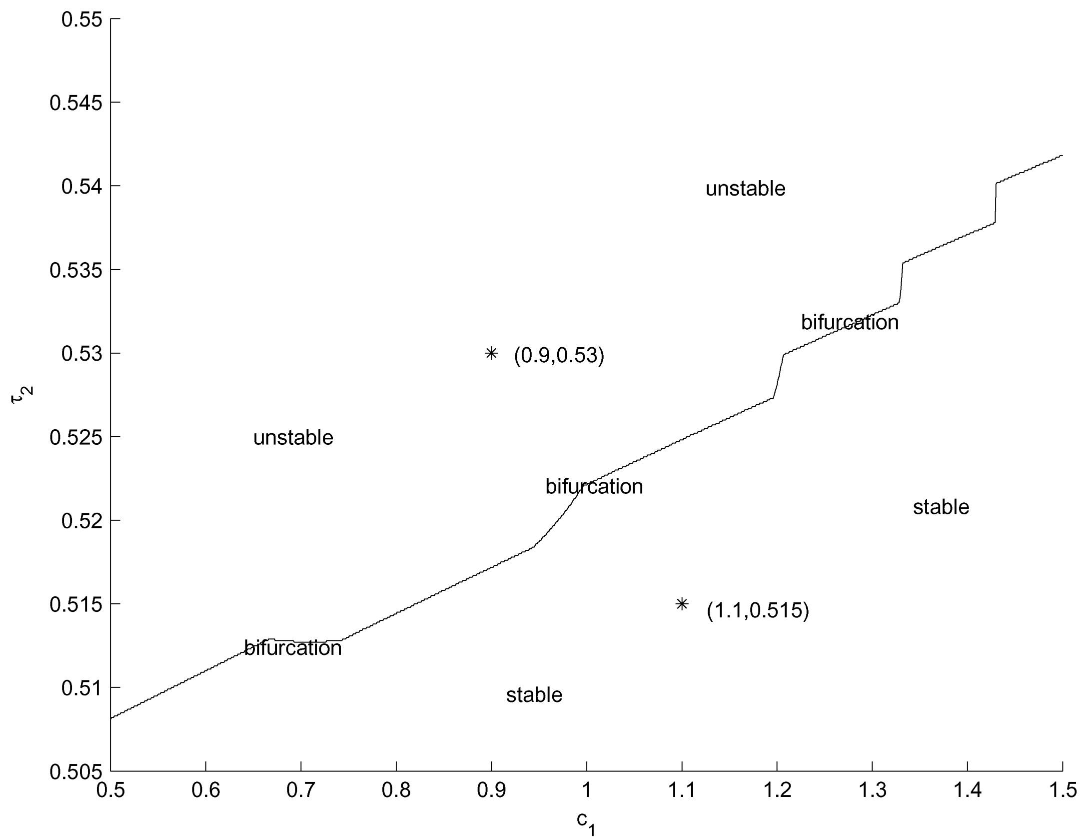

Fig. 6. Two-parameter (c_1 and τ_2) bifurcation diagram.

and $\tau_1 = 0.51$ is fixed. Here, τ_2 and c_1 are both considered as bifurcation parameters. The results can be hardly obtained by the previous analysis. First, we choose $\tau_2 = 0.515$ and $c_1 = 1.1$, respectively. The corresponding waveform, phase, and frequency graph are shown in Fig. 4. The half-line L_1 and locus $\widehat{\lambda}(i\omega)$ do not intersect, so no limit cycle exists. By Theorems 1 and 3, we know that in Fig. 4, its zero solution is asymptotically stable.

Next, we choose $\tau_2 = 0.53$ and $c_1 = 0.9$, respectively. The corresponding waveform, phase, and frequency graph are shown in Fig. 5. By Theorem 3, we know that the half-line L_1 intersects the locus $\widehat{\lambda}(i\omega)$, so a limit cycle exists. The total number k of the counterclockwise encirclements of the point $P_1 = \widehat{P} + \varepsilon\xi_1(\widetilde{\omega})$ for a small enough $\varepsilon > 0$ is 0, i.e., $k = 0$, so by Corollary 1, a stable periodic solution exists.

Finally, we show a bifurcation diagram in a local region to verify the theoretical analysis, which is shown in Fig. 6. Here, τ_2 and c_1 are considered as parameters. In some region of Fig. 6, system (35) is locally asymptotically stable, while in some other region, it is unstable. Hopf bifurcation occurs when τ_2 and c_1 pass through some critical values where the stability condition of the equilibrium is not satisfied.

In [28], some delay-dependent conditions are given to ensure the stability of the equilibrium point. However, these delay-dependent conditions are too conservative and depend on the maximum bound of all the time delays. The method used in many papers [10]–[17] cannot be used to solve the aforementioned problem, where more than one time delays (which are system bifurcation parameters) are greater than τ_0. The method developed in this paper, however, can be applied. In addition, a general high-dimensional Hopf bifurcation is analyzed.

V. Conclusion

In this paper, we have discussed the local asymptotical stability and Hopf bifurcation of a general model of delayed recurrent neural networks, which gives a better understanding of the situation when the stability of delayed recurrent neural networks is not guaranteed and a Hopf bifurcation occurs. To the best of our knowledge, there are very few results available about the bifurcation analysis of higher dimensional system with multiple delays, which may lead the system to instability. The analytical results obtained in this paper may, therefore, give new insights on the dynamics of multidelayed recurrent neural networks.

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