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Synchronization control of switched linearly coupled neural networks with delay

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ABSTRACT

In this paper, synchronization control of switched linearly coupled delayed neural networks is investigated by using the Lyapunov functional method, synchronization manifold and linear matrix inequality (LMI) approach. A sufficient condition is derived to ensure the global synchronization of switched linearly coupled complex neural networks, which are controlled by some designed controllers. A globally convergent algorithm involving convex optimization is also presented to construct such controllers effectively. In many cases, it is desirable to control the whole network by changing the connections of some nodes in the complex network, and this paper provides an applicable approach. It is even applicable to the case when the derivative of the time-varying delay takes arbitrary. Finally, some simulations are constructed to justify the theoretical analysis.

1. Introduction

Synchronization and stability control of dynamical systems is an important topic in nonlinear system control [1–10,44–46] in the past decades. Recently, arrays of coupled systems have attracted much attention of researchers in different research fields. The study of synchronization of coupled neural networks is an important step for both understanding brain science and designing coupled neural networks for practical use.

Networks of coupled connection have been widely investigated [11–26] since Wang and Chen introduced an array of N linearly coupled connected complex network model [27,28]. Consider a complex dynamical network consisting of N identical linearly and diffusively coupled nodes, with each node being an n-dimensional dynamical system in [27,28] as follows:

$$\dot{x}_i(t)=f(x_i(t))+c\sum_{j=1,j\neq i}^{N}G_{ij}\Gamma(x_j(t)-x_i(t)),\quad i=1,2,\ldots,N, \tag{1}$$

where $x_i(t)=(x_{i1}(t),x_{i2}(t),\ldots,x_{in}(t))^T\in R^n$ $(i=1,2,\ldots,N)$ is the state vector representing the state variables of node i, $f: R^n\longrightarrow R^n$ is continuously differentiable, the constant c is the coupling strength, $\Gamma=\mathrm{diag}(\gamma_1,\gamma_2,\ldots,\gamma_n)\in R^{n\times n}$ is a constant 0–1 matrix linking the coupled variables with $\gamma_i=1$ for a specific i and $\gamma_j=0$ $(j\neq i)$, that is, there is only one 1 in the diagonal of matrix Γ and all the other components of Γ are zeros, $G=(G_{ij})_{N\times N}$ is the coupling configuration matrix representing the topological structure of the network, in which G_{ij} is defined as follows: if there is a connection between node i and node j $(j\neq i)$, then the coupling strength $G_{ij}=G_{ji}>0$; otherwise, $G_{ij}=G_{ji}=0$ $(j\neq i)$, and the diagonal elements of matrix G are defined by

$$G_{ii}=-\sum_{j=1,j\neq i}^{N}G_{ij}. \tag{2}$$

Then, in this case, the complex network (1) reduces to the model

$$\dot{x}_i(t)=f(x_i(t))+c\sum_{j=1}^{N}G_{ij}\Gamma x_j(t),\quad i=1,2,\ldots,N. \tag{3}$$

Hereafter, suppose that the network (3) is connected in the sense that there are no isolate clusters. Thus, the coupling configuration G is an irreducible matrix.

In the following, a brief introduction of recent works about synchronization of linearly coupled complex networks are given based on the model (1)–(3).

For the case that the coupling matrix G is irreducible, symmetric, and all the off-diagonal elements of G are nonegative and satisfies (2), local synchronization analysis via linearlization technique was studied in [16–20], where the eigenvalues and

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Jacobian matrix are given in the criteria of ensuring the local synchronization of the complex network. Wu and Chua [29,30] investigated the synchronization in an array of linearly coupled dynamical systems, and then a lot of works [11–13,15,21] were devoted to investigating the global asymptotical synchronization of the complex network by using the synchronization manifold and Lyapunov method. In [29], the authors defined a distance between the collective states and the synchronization manifold, based on which the a methodology was proposed to discuss global synchronization of the coupled systems.

In [11,12,15], the following linearly coupled neural networks satisfying (2) was studied:

$$\dot{x}_i(t) = -Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau)) + I(t) + \sum_{j=1}^{N} G_{ij}\Gamma x_j(t), \quad i = 1,2,\ldots,N, \tag{4}$$

where $C = \mathrm{diag}(c_1, c_2, \ldots, c_n) \in R^{n\times n}$ is a diagonal matrix with positive diagonal entries $c_i > 0$, $i = 1,2,\ldots,n$, $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are weight and delayed weight matrices, respectively. $I(t) = (I_1(t), I_2(t), \ldots, I_n(t))^T \in R^n$ is an external input vector, $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n) \in R^{n\times n}$. $f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)), \ldots, f_n(x_{in}(t)))^T \in R^n$ corresponds to the activation functions of neurons. Equivalently, system (4) can be written as

$$\dot{x}_{ik}(t) = -c_k x_{ik}(t) + \sum_{l=1}^{n} a_{kl} f_l(x_{il}(t)) + \sum_{l=1}^{n} b_{kl} f_l(x_{il}(t-\tau)) + I_k(t) + \sum_{j=1}^{N} G_{ij}\gamma_k x_{jk}(t), \quad i = 1,2,\ldots,N, \ k = 1,2,\ldots,n. \tag{5}$$

With the rapid development of intelligent control, hybrid systems have been widely investigated. It is found that many physical and biological models are governed by more than one dynamical system and these systems are changed depending on time. Switched systems [31–34], a special case in hybrid systems, are regarded as nonlinear systems, which are composed of a family subsystems and a rule that orchestrates the switching between the subsystems. Recently, switched systems have numerous applications in communication systems [6,35,36], control of mechanical systems, automotive industry, aircraft and air traffic control, electric power systems [37] and many other fields. In [31–34], the stability of switching system was investigated, which is a combination of discrete and continuous dynamical systems.

The main contribution of this paper is of threefolds. Firstly, we studied global synchronization of coupled systems with time-varying delay by using LMI and distance function from collective states to the synchronization manifold [26]. A delay-dependent condition is given to ensure the synchronization of coupled systems in this paper based on free-weighting matrix approach and cone complementarity linearization algorithm [47–49]. It is noted that the derivative of time delay can take any value. Secondly, the feedback matrix of the network is designed to adjust the configuration matrix, i.e., the connections among the nodes. Thirdly, it is very difficult to design the feedback matrix due to the complexity of the systems. So, a globally convergent algorithm involving convex optimization is presented.

The rest of the paper is organized as follows: In Section 2, preliminaries are given. In Section 3, the main results are derived. A sufficient condition is given to ensure the synchronization of switched coupled networks and a globally convergent algorithm involving convex optimization is also presented to design such controllers effectively. In Section 4, numerical simulations are constructed to justify the theoretical analysis in this paper. Finally, the conclusion is drawn.

2. Preliminaries

A set of coupled complex neural networks is considered as the individual subsystems of the switched system and the switched coupled neural network is described as follows:

$$\dot{x}_i(t) = -C_\alpha x_i(t) + A_\alpha f(x_i(t)) + B_\alpha f(x_i(t-\tau)) + I_\alpha(t) + \sum_{j=1}^{N} G_{\alpha ij} D x_j(t), \quad i = 1,2,\ldots,N, \tag{6}$$

where D is inner coupling matrix and α is a switching signal which takes its value in the finite set $\mathcal{I} = \{1,2,\ldots,\overline{N}\}$. This means that the matrices $(C_\alpha, A_\alpha, B_\alpha, I_\alpha, G_\alpha)$ are allowed to take values, at particular time, in a finite set $\{(C_1, A_1, B_1, I_1, G_1), (C_2, A_2, B_2, I_2, G_2), \ldots, (C_{\overline{N}}, A_{\overline{N}}, B_{\overline{N}}, I_{\overline{N}}, G_{\overline{N}})\}$. Throughout this paper, we assume that the switching rule α is not known priori and its instantaneous value is available in real time.

Since in most cases the time delay is not a constant, in this paper, the coupled neural network with time-varying delay is studied. Consider the state-feedback control law

$$u_{\alpha i}(t) = \sum_{j=1}^{N} K_{\alpha ij} D x_j(t), \quad i = 1,2,\ldots,N, \tag{7}$$

where

$$K_{\alpha ii} = -\sum_{j=1, j\neq i}^{N} K_{\alpha ij}. \tag{8}$$

It is useful to design a memoryless state-feedback controller $u_{\alpha i}(t)$ so that the coupled system (6) is globally synchronized. In this paper, a linearly feedback controller (7) is added to the coupled dynamical system (6)

$$\dot{x}_i(t) = -C_\alpha x_i(t) + A_\alpha f(x_i(t)) + B_\alpha f(x_i(t-\tau(t))) + I_\alpha(t) + \sum_{j=1}^{N} (G_{\alpha ij} + K_{\alpha ij}) D x_j(t), \quad i = 1,2,\ldots,N. \tag{9}$$

It is easy to see that one can control the synchronization of coupled neural network by adjusting the configuration coupling matrix, that is, the network topology can be changed to achieve synchronization. The architecture for such switched coupled neural networks is shown in Fig. 1. Next, we focus on global asymptotical synchronization of coupled feedback system (9).

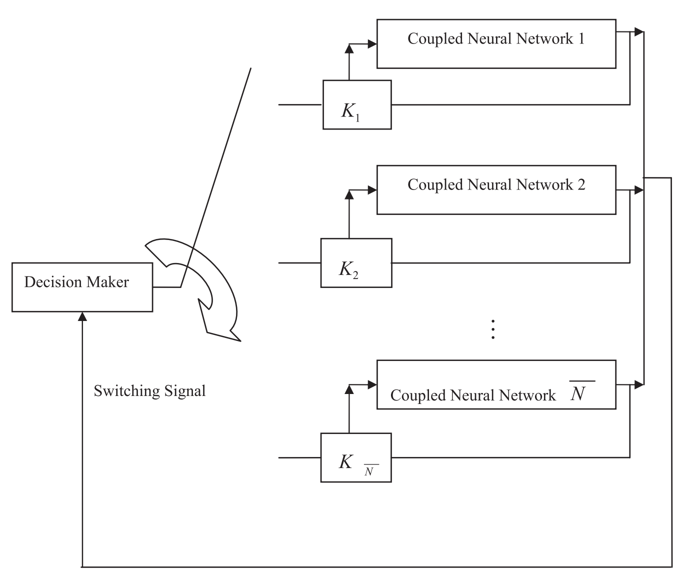

Fig. 1. Architecture of the switched coupled neural networks.

Define an indicator function $\xi(t)=(\xi_1(t),\xi_2(t),\ldots,\xi_{\overline{N}}(t))^T$, where

$$\xi_k(t)=\begin{cases}1 & \text{when the switched system is described}\\ & \text{by the } k\text{th mode } (C_k,A_k,B_k,I_k,G_k),\\ 0 & \text{otherwise,}\end{cases} \tag{10}$$

with $k=1,2,\ldots,\overline{N}$. The model of switched coupled neural network model (9) can be written as

$$\dot{x}_i(t)=\sum_{k=1}^{\overline{N}}\xi_k(t)[-C_kx_i(t)+A_kf(x_i(t))+B_kf(x_i(t-\tau(t)))+I_k(t) + \sum_{j=1}^{N}(G_{kij}+K_{kij})Dx_j(t)],\quad i=1,2,\ldots,N. \tag{11}$$

It follows that $\sum_{k=1}^{\overline{N}}\xi_k(t)=1$ under any switching rules.

We assume that the system (11) satisfies the following initial conditions: $x_i(t)=\phi_i(t)\in\mathcal{C}([-r,0],R)$ $(i=1,2,\ldots,N)$ with $r=\max_{t\in R}\{\tau(t)\}$, where $\mathcal{C}([-r,0],R)$ denotes the set of all continuous functions from $[-r,0]$ to R.

In order to derive the main results, it is necessary to make the following assumptions:

A_1: The activation functions $f_i(x_i)$ $(i=1,2,\ldots,n)$ are Lipschitz continuous, that is, there exist constants $F_i>0$ such that

$$|f_i(\alpha_1)-f_i(\alpha_2)|\le F_i|\alpha_1-\alpha_2|,\quad \forall\alpha_1,\alpha_2\in R. \tag{12}$$

A_2: $\tau(t)$ is a bounded differential function of time t, i.e., $r=\max_{t\in R}\{\tau(t)\}$, and the following condition is satisfied:

$$0\le\dot{\tau}(t)\le h, \tag{13}$$

where h is a positive real constant.

A_3: The coupling matrix G_α and the feedback gain matrix K_α are defined by

$$G_{\alpha ii}=-\sum_{j=1,j\ne i}^{N}G_{\alpha ij},\quad i=1,2,\ldots,N,\ \alpha=1,2,\ldots,\overline{N}, \tag{14}$$

and

$$K_{\alpha ii}=-\sum_{j=1,j\ne i}^{N}K_{\alpha ij},\quad i=1,2,\ldots,N,\ \alpha=1,2,\ldots,\overline{N}. \tag{15}$$

Let I_n be the n-dimensional identity matrix.

Definition 1 (*Lu and Chen [11]*). Let $r=\max_{t\in R}\{\tau(t)\}$, the set $\mathbf{S}=\{x=(x_1(s),x_2(s),\ldots,x_N(s)):x_i(s)\in\mathcal{C}([-r,0],R),\ x_i(s)=x_j(s),\ i,j=1,2,\ldots,N\}$ is called the synchronization manifold.

Definition 2. Synchronization manifold $\mathbf{S}$ is said to be globally asymptotically stable, equivalently, the coupled system (11) is globally asymptotically synchronized, if for any $\varepsilon>0$, for each initial data $\phi_i(s)$, $s\in[-r,0]$, $i=1,2,\ldots,N$, there exists $T>0$, such that

$$\|x_i(t)-x_j(t)\|\le\varepsilon, \tag{16}$$

holds for all $t>T, i,j=1,2,\ldots,N$.

Definition 3 (*Wu and Chua [29]*). Let $\hat{R}$ denote a ring, and define $T(\hat{R},K)=\{$the set of matrices with entries $\hat{R}$ such that the sum of the entries in each row is equal to K for some $K\in\hat{R}.\}$

Definition 4 (*Wu and Chua [29]*). Set of $M_1^N(1)$: $M_1^N(1)$ is composed of matrices with N columns. Each row (for instance, the i th row) of $\tilde{M}\in M_1^N(1)$ has exactly one entry α_i and one entry $-\alpha_i$, where $\alpha_i\ne0$. All the other entries are zeros.

Definition 5 (*Wu and Chua [29]*). Set of $M_1^N(n)$: $M_1^N(n)$ are matrices $\mathbf{M}$ obtained by replacing entry m_{ij} in $\tilde{M}\in M_1^N(1)$ with $m_{ij}I_n$, i.e., $M_1^N(n)=\{\mathbf{M}=\tilde{M}\otimes I_n:\tilde{M}\in M_1^N(1)\}$,where $\otimes$ is Kronecker product.

Definition 6 (*Wu and Chua [29]*). $M_2^N(n)\subset M_1^N(n)$: If $M\in M_2^N(n)$, then, for any pair of indices i and j, there exist indices $j_1,j_2,\ldots,j_l$, where $j_1=i$ and $j_l=j$, and $p_1,p_2,\ldots,p_{l-1}$ such that $M_{p_qj_q}\ne0$ and $M_{p_qj_{q+1}}\ne0$ for all $1\le q<l$.

Definition 7 (*Kronecker product*). For matrices A and B, the notation $A\otimes B$ stands for the matrix composed of submatrices $A_{ij}B$, i.e.,

$$A\otimes B=\begin{pmatrix}A_{11}B & A_{12}B & \cdots & A_{1n}B\\ A_{21}B & A_{22}B & \cdots & A_{2n}B\\ \cdots & \cdots & \ddots & \cdots\\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B\end{pmatrix},$$

where A_{ij}, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$ stands for the ij th entry of the $m\times n$ matrix A.

Lemma 1 (*Wu and Chua [29]*). *Let G be a $N\times N$ matrix in $T(\hat{R},K)$. Then the $(N-1)\times(N-1)$ matrix H is defined by $H=MGJ$ satisfying $MG=HM$, where M is the $(N-1)\times N$ matrix*

$$M=\begin{pmatrix}\mathbf{1} & -\mathbf{1} & & & \\ & \mathbf{1} & -\mathbf{1} & & \\ & & & \ddots & \\ & & & \mathbf{1} & -\mathbf{1}\end{pmatrix}, \tag{17}$$

and J is the $N\times(N-1)$ matrix

$$J=\begin{pmatrix}\mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1}\\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1}\\ & & \ddots & & \mathbf{1}\\ & & \cdots & \mathbf{1} & \mathbf{1}\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\end{pmatrix}, \tag{18}$$

in which $\mathbf{1}$ is the multiplicative identity of $\hat{R}$.

Lemma 2 (*Wu and Chua [29]*). *Let $x=(x_1,x_2,\ldots,x_N)^T$, where $x_i\in R^n$, $i=1,2,\ldots,N$. Then $x\in\mathbf{S}$ if and only if*

$$\|\mathbf{M}x\|=0 \tag{19}$$

holds for some $\mathbf{M}\in M_2^N(n)$. We use $d(x)$ to denote a nonnegative real-valued function that measures the distance between the various nodes. In particular, $d(x)$ is of the following form:

$$d(x)=\|\mathbf{M}x\|^2=x^T\mathbf{M}^T\mathbf{M}x,\quad \mathbf{M}\in M_2^N(n). \tag{20}$$

Because of the assumptions on $\mathbf{M}$, the crucial property of $d(x)$ is that $d(x)\longrightarrow0$ if and only if $\|x_i(t)-x_j(t)\|\longrightarrow0$ for all i and j.

Lemma 3 (*Chen and Chen [38]*). *By the definition of Kronecker product, the following properties can be satisfied for appropriate dimensions*:

(1) $(\alpha A)\otimes B=A\otimes(\alpha B)$;
(2) $(A+B)\otimes C=A\otimes C+B\otimes C$;
(3) $(A\otimes B)(C\otimes D)=(AC)\otimes(BD)$.

Lemma 4 (*Schur complement [39]*). *The following linear matrix inequality (LMI)*:

$$\begin{pmatrix}Q(x) & S(x)\\ S(x)^T & R(x)\end{pmatrix}>0,$$

where $Q(x)=Q(x)^T$, $R(x)=R(x)^T$, is equivalent to one of the following conditions:

(i) $Q(x)>0,\ R(x)-S(x)^TQ(x)^{-1}S(x)>0$,
(ii) $R(x)>0,\ Q(x)-S(x)R(x)^{-1}S(x)^T>0$.

Lemma 5 (*Gu et al. [40]*). *For any constant matrix* $W \in R^{m\times m}$, $W = W^T$, *scalar* $r > 0$, *vector function* $\omega : [0,r] \in R^m$ *such that the integrations concerned are well defined, then*

$$r\int_0^r \omega^T(s)W\omega(s)\,ds \geq \left(\int_0^r \omega(s)\,ds\right)^T W\left(\int_0^r \omega(s)\,ds\right).$$

3. Main results

In this section, new criteria are presented for the global synchronization of system (11) based on Lyapunov functional method and linear matrix inequality (LMI) approach.

First, some notations are given to simplify the proof. M is defined in (17). Let $\mathbf{M} = M \otimes I_n \in M_2^N(n)$ be the matrix defined in Definition 6.

Let $A \otimes B$ denote the Kroneker product of matrices A and B and also let

$$\mathbf{C}_k = I_N \otimes C_k, \quad \mathbf{C}_k^1 = I_{N-1} \otimes C_k, \quad \mathbf{A}_k = I_N \otimes A_k, \quad \mathbf{A}_k^1 = I_{N-1} \otimes A_k,$$

$$\mathbf{B}_k = I_N \otimes B_k, \quad \mathbf{B}_k^1 = I_{N-1} \otimes B_k, \quad \mathbf{G}_k = G_k \otimes D, \quad \mathbf{K}_k = K_k \otimes D,$$

$$x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T, \quad \forall i = 1,2,\ldots,N,$$

$$x(t) = (x_1^T(t), x_2^T(t), \ldots, x_N^T(t))^T,$$

$$\mathbf{f}(x(t)) = (f^T(x_1(t)), f^T(x_2(t)), \ldots, f^T(x_N(t)))^T,$$

$$\mathbf{I}_k(t) = (I_k^T(t), I_k^T(t), \ldots, I_k^T(t))^T,$$

where $k \in \mathcal{I} = \{1,2,\ldots,\overline{N}\}$.

The linearly coupled dynamical system (11) can be rewritten as

$$\dot{x}(t) = \sum_{k=1}^{\overline{N}} \xi_k(t)[-\mathbf{C}_k x(t) + \mathbf{A}_k \mathbf{f}(x(t)) + \mathbf{B}_k \mathbf{f}(x(t-\tau(t))) + \mathbf{I}_k(t) + (\mathbf{G}_k + \mathbf{K}_k)x(t)]. \tag{21}$$

Next, a theorem is established to ensure the global asymptotical synchronization of system (21).

Theorem 1. *Under assumptions* (A_1)–(A_3), *the dynamical system* (21) *is globally asymptotically synchronized if there are positive definite matrices* $\mathbf{P} \in R^{(N-1)n\times(N-1)n}$, $\mathbf{Q} \in R^{(N-1)n\times(N-1)n}$, $\mathbf{R} \in R^{(N-1)n\times(N-1)n}$, $\mathbf{T} \in R^{(N-1)n\times(N-1)n}$, *positive definite diagonal matrices* $\mathbf{\Sigma} = \mathrm{diag}\,(\Sigma_1, \Sigma_2, \ldots, \Sigma_{(N-1)n}) \in R^{(N-1)n\times(N-1)n}$, $\mathbf{\Lambda} = \mathrm{diag}(\Lambda_1, \Lambda_2, \ldots, \Lambda_{(N-1)n}) \in R^{(N-1)n\times(N-1)n}$, *a matrix* $\mathbf{W} = (\mathbf{W}_1^T\ \mathbf{W}_2^T\ \mathbf{W}_3^T\ \mathbf{W}_4^T\ \mathbf{W}_5^T)^T \in R^{5(N-1)n\times(N-1)n}$ *and the feedback gain matrix* $K_k \in R^{N\times N}$, *for each* $k \in \mathcal{I} = \{1,2,\ldots,\overline{N}\}$ *such that*

$$\mathbf{\Phi}_k = \begin{pmatrix} \mathbf{\Phi}_{k11} & \mathbf{\Phi}_{k12} & \mathbf{\Phi}_{k13} & \mathbf{\Phi}_{k14} & \mathbf{\Phi}_{k15} & \mathbf{\Phi}_{k16} \\ \mathbf{\Phi}_{k12}^T & \mathbf{\Phi}_{k22} & -\mathbf{W}_3^T & -\mathbf{W}_4^T & -\mathbf{W}_2-\mathbf{W}_5^T & 0 \\ \mathbf{\Phi}_{k13}^T & -\mathbf{W}_3 & -\mathbf{\Sigma}+\mathbf{Q} & 0 & -\mathbf{W}_3 & \mathbf{A}_k^{1T}\mathbf{T} \\ \mathbf{\Phi}_{k14}^T & -\mathbf{W}_4 & 0 & -(1-h)\mathbf{Q}-\mathbf{\Lambda} & -\mathbf{W}_4 & \mathbf{B}_k^{1T}\mathbf{T} \\ \mathbf{\Phi}_{k15}^T & -\mathbf{W}_2^T-\mathbf{W}_5 & -\mathbf{W}_3^T & -\mathbf{W}_4^T & -\frac{1}{r}\mathbf{T}-\mathbf{W}_5-\mathbf{W}_5^T & 0 \\ \mathbf{\Phi}_{k16}^T & 0 & \mathbf{T}\mathbf{A}_k^1 & \mathbf{T}\mathbf{B}_k^1 & 0 & -\frac{1}{r}\mathbf{T} \end{pmatrix} < 0, \tag{22}$$

where

$$\mathbf{\Phi}_{k11} = \mathbf{P}(-\mathbf{C}_k^1 + \mathbf{H}_k + \mathbf{U}_k) + (-\mathbf{C}_k^1 + \mathbf{H}_k + \mathbf{U}_k)^T\mathbf{P} + \mathbf{R} + \mathbf{F}\mathbf{\Sigma}\mathbf{F} + \mathbf{W}_1 + \mathbf{W}_1^T,$$

$$\mathbf{\Phi}_{k12} = \mathbf{W}_2^T - \mathbf{W}_1,$$

$$\mathbf{\Phi}_{k13} = \mathbf{P}\mathbf{A}_k^1 + \mathbf{W}_3^T,$$

$$\mathbf{\Phi}_{k14} = \mathbf{P}\mathbf{B}_k^1 + \mathbf{W}_4^T,$$

$$\mathbf{\Phi}_{k15} = \mathbf{W}_5^T - \mathbf{W}_1,$$

$$\mathbf{\Phi}_{k16} = (-\mathbf{C}_k^1 + \mathbf{H}_k + \mathbf{U}_k)^T\mathbf{T},$$

$$\mathbf{\Phi}_{k22} = -(1-h)\mathbf{R} + \mathbf{F}\mathbf{\Lambda}\mathbf{F} - \mathbf{W}_2 - \mathbf{W}_2^T,$$

$F = \mathrm{diag}(F_1, F_2, \ldots, F_n) \in R^{n\times n}$, $\mathbf{F} = I_{N-1} \otimes F$, $H_k = MG_kJ$, $\mathbf{H}_k = H_k \otimes D$, $U_k = MK_kJ$, $\mathbf{U}_k = U_k \otimes D$, *M and J are defined in* (17) *and* (18).

Proof. Consider the following Lyapunov functional:

$$V(t) = \sum_{i=1}^{i=4} V_i(t), \tag{23}$$

where

$$V_1(t) = x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{M}x(t), \tag{24}$$

$$V_2(t) = \int_{t-\tau(t)}^t \mathbf{f}^T(x(s))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(s))\,ds, \tag{25}$$

$$V_3(t) = \int_{t-\tau(t)}^t x^T(s)\mathbf{M}^T\mathbf{R}\mathbf{M}x(s)\,ds, \tag{26}$$

$$V_4(t) = \int_{-r}^0 d\theta \int_{t+\theta}^t \dot{x}^T(s)\mathbf{M}^T\mathbf{T}\mathbf{M}\dot{x}(s)\,ds. \tag{27}$$

Taking the derivative of $V(t)$ along the trajectories of (21) and by Lemma 5, one has

$$\begin{aligned}\dot{V}(t)|_{(21)} &= 2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{M}\dot{x}(t) + \mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(t)) \\ &\quad -(1-\dot{\tau}(t))\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(t-\tau(t))) + x^T(t)\mathbf{M}^T\mathbf{R}\mathbf{M}x(t) \\ &\quad -(1-\dot{\tau}(t))x^T(t-\tau(t))\mathbf{M}^T\mathbf{R}\mathbf{M}x(t-\tau(t)) + r\dot{x}^T(t)\mathbf{M}^T\mathbf{T}\mathbf{M}\dot{x}(t) \\ &\quad -\int_{t-r}^t \dot{x}^T(\theta)\mathbf{M}^T\mathbf{T}\mathbf{M}\dot{x}(\theta)\,d\theta \\ &\leq \sum_{k=1}^{\overline{N}} \xi_k(t)\Big\{2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{M}[(-\mathbf{C}_k + \mathbf{G}_k + \mathbf{K}_k)x(t) + \mathbf{A}_k\mathbf{f}(x(t)) \\ &\quad + \mathbf{B}_k\mathbf{f}(x(t-\tau(t))) + \mathbf{I}_k(t)] + \mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(t)) \\ &\quad -(1-h)\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(t-\tau(t))) + x^T(t)\mathbf{M}^T\mathbf{R}\mathbf{M}x(t) \\ &\quad -(1-h)x^T(t-\tau(t))\mathbf{M}^T\mathbf{R}\mathbf{M}x(t-\tau(t)) + r\dot{x}^T(t)\mathbf{M}^T\mathbf{T}\mathbf{M}\dot{x}(t) \\ &\quad -\frac{1}{r}\left(\int_{t-\tau(t)}^t \mathbf{M}\dot{x}(\theta)\,d\theta\right)^T\mathbf{T}\left(\int_{t-\tau(t)}^t \mathbf{M}\dot{x}(\theta)\,d\theta\right)\Big\}. \end{aligned} \tag{28}$$

By the structure of $\mathbf{M}$, following equalities are easy to verify:

$$\mathbf{M}\mathbf{C}_k = \mathbf{C}_k^1\mathbf{M}, \quad \mathbf{M}\mathbf{A}_k = \mathbf{A}_k^1\mathbf{M}, \quad \mathbf{M}\mathbf{B}_k = \mathbf{B}_k^1\mathbf{M}, \quad \mathbf{M}\mathbf{I}_k(t) = 0.$$

Therefore, from (28) one obtains

$$\begin{aligned}\dot{V}(t)|_{(21)} &\leq \sum_{k=1}^{\overline{N}} \xi_k(t)\Big\{2x^T(t)\mathbf{M}^T\mathbf{P}[(-\mathbf{C}_k^1\mathbf{M} + \mathbf{M}\mathbf{G}_k + \mathbf{M}\mathbf{K}_k)x(t) \\ &\quad + \mathbf{A}_k^1\mathbf{M}\mathbf{f}(x(t)) + \mathbf{B}_k^1\mathbf{M}\mathbf{f}(x(t-\tau(t)))] + \mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(t)) \\ &\quad -(1-h)\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(t-\tau(t))) + x^T(t)\mathbf{M}^T\mathbf{R}\mathbf{M}x(t) \\ &\quad -(1-h)x^T(t-\tau(t))\mathbf{M}^T\mathbf{R}\mathbf{M}x(t-\tau(t)) + r\dot{x}^T(t)\mathbf{M}^T\mathbf{T}\mathbf{M}\dot{x}(t) \\ &\quad -\frac{1}{r}\left(\int_{t-\tau(t)}^t \mathbf{M}\dot{x}(\theta)\,d\theta\right)^T\mathbf{T}\left(\int_{t-\tau(t)}^t \mathbf{M}\dot{x}(\theta)\,d\theta\right)\Big\}. \end{aligned} \tag{29}$$

By assumption A_1, it is obvious that

$$\begin{aligned}\mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{\Sigma}\mathbf{M}\mathbf{f}(x(t)) &= \sum_{j=1}^{N-1}[f(x_j(t)) - f(x_{j+1}(t))]^T\mathbf{\Sigma}_j[f(x_j(t)) - f(x_{j+1}(t))] \\ &\leq \sum_{j=1}^{N-1}[x_j(t) - x_{j+1}(t)]^T F\mathbf{\Sigma}_j F[x_j(t) - x_{j+1}(t)] \\ &= x^T(t)\mathbf{M}^T\mathbf{F}\mathbf{\Sigma}\mathbf{F}\mathbf{M}x(t), \end{aligned} \tag{30}$$

where $\mathbf{\Sigma}_j = \mathrm{diag}(\Sigma_{(j-1)n+1}, \ldots, \Sigma_{jn})$, and

$$\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\mathbf{\Lambda}\mathbf{M}\mathbf{f}(x(t-\tau(t))) \le x^T(t-\tau(t))\mathbf{M}^T\mathbf{F}\mathbf{\Lambda}\mathbf{F}\mathbf{M}x(t-\tau(t)). \tag{31}$$

From Lemmas 1 and 3, we obtain

$$\begin{aligned} 2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{M}\mathbf{G}_k x(t) &= 2x^T(t)\mathbf{M}^T\mathbf{P}[(M\otimes I_n)(G_k\otimes D)]x(t) \\ &= 2x^T(t)\mathbf{M}^T\mathbf{P}[MG_k\otimes D]x(t) \\ &= 2x^T(t)\mathbf{M}^T\mathbf{P}[H_kM\otimes D]x(t) \\ &= 2x^T(t)\mathbf{M}^T\mathbf{P}[(H_k\otimes D)(M\otimes I_n)]x(t) \\ &= 2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{H}_k\mathbf{M}x(t), \end{aligned} \tag{32}$$

and

$$2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{M}\mathbf{K}_k x(t) = 2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{U}_k\mathbf{M}x(t), \tag{33}$$

where $H_k = MG_kJ$, $\mathbf{H}_k = (MG_kJ)\otimes D$, $U_k = MK_kJ$, $\mathbf{U}_k = (MK_kJ)\otimes D$, M and J are defined in (17) and (18).

From the Leibniz–Newton formula, the following equation is true for any matrix $\mathbf{W}$ with appropriate dimensions:

$$2\eta^T(t)\mathbf{W}\mathbf{M}\left(x(t)-x(t-\tau(t))-\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds\right)=0, \tag{34}$$

where $\eta(t) = (x^T(t)\mathbf{M}^T \;\; x^T(t-\tau(t))\mathbf{M}^T \;\; \mathbf{f}^T(x(t))\mathbf{M}^T \;\; \mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T \;\; \int_{t-\tau(t)}^{t}\dot{x}^T(s)\,ds\mathbf{M}^T)^T$. Let $\Pi_k = (-\mathbf{C}_k^1+\mathbf{H}_k+\mathbf{U}_k \;\; 0 \;\; \mathbf{A}_k^1 \;\; \mathbf{B}_k^1 \;\; 0)$, then we have

$$\dot{x}^T(t)\mathbf{M}^T\mathbf{T}\mathbf{M}\dot{x}(t) = \eta^T(t)\Pi_k^T\mathbf{T}\Pi_k\eta(t). \tag{35}$$

Combining (29)–(35), we obtain

$$\begin{aligned} \dot{V}(t)|_{(21)} \le \sum_{k=1}^{\overline{N}}\xi_k(t)\Big\{&2x^T(t)\mathbf{M}^T\mathbf{P}(-\mathbf{C}_k^1+\mathbf{H}_k+\mathbf{U}_k)\mathbf{M}x(t) \\ &+2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{A}_k^1\mathbf{M}\mathbf{f}(x(t))+2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{B}_k^1\mathbf{M}\mathbf{f}(x(t-\tau(t))) \\ &+x^T(t)\mathbf{M}^T\mathbf{F}\mathbf{\Sigma}\mathbf{F}\mathbf{M}x(t)-\mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{\Sigma}\mathbf{M}\mathbf{f}(x(t)) \\ &+x^T(t-\tau(t))\mathbf{M}^T\mathbf{F}\mathbf{\Lambda}\mathbf{F}\mathbf{M}x(t-\tau(t))-\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\mathbf{\Lambda}\mathbf{M}\mathbf{f}(x(t-\tau(t))) \\ &+\mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(t))-(1-h)\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(t-\tau(t))) \\ &+x^T(t)\mathbf{M}^T\mathbf{R}\mathbf{M}x(t)-(1-h)x^T(t-\tau(t))\mathbf{M}^T\mathbf{R}\mathbf{M}x(t-\tau(t)) \\ &+r\eta^T(t)\Pi_k^T\mathbf{T}\Pi_k\eta(t)-\frac{1}{r}\left(\int_{t-\tau(t)}^{t}\mathbf{M}\dot{x}(\theta)\,d\theta\right)^T\mathbf{T}\left(\int_{t-\tau(t)}^{t}\mathbf{M}\dot{x}(\theta)\,d\theta\right) \\ &+2\eta^T(t)\mathbf{W}\mathbf{M}\left(x(t)-x(t-\tau(t))-\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds\right)\Big\} \\ = \sum_{k=1}^{\overline{N}}\xi_k(t)\Big\{&\eta^T(t)(\mathbf{\Psi}_k+r\Pi_k^T\mathbf{T}\Pi_k)\eta(t)\Big\}, \end{aligned} \tag{36}$$

where

$$\mathbf{\Psi}_k = \begin{pmatrix} \mathbf{\Phi}_{k11} & \mathbf{\Phi}_{k12} & \mathbf{\Phi}_{k13} & \mathbf{\Phi}_{k14} & \mathbf{\Phi}_{k15} \\ \mathbf{\Phi}_{k12}^T & \mathbf{\Phi}_{k22} & -\mathbf{W}_3^T & -\mathbf{W}_4^T & -\mathbf{W}_2-\mathbf{W}_5^T \\ \mathbf{\Phi}_{k13}^T & -\mathbf{W}_3 & -\mathbf{\Sigma}+\mathbf{Q} & 0 & -\mathbf{W}_3 \\ \mathbf{\Phi}_{k14}^T & -\mathbf{W}_4 & 0 & -(1-h)\mathbf{Q}-\mathbf{\Lambda} & -\mathbf{W}_4 \\ \mathbf{\Phi}_{k15}^T & -\mathbf{W}_2^T-\mathbf{W}_5 & -\mathbf{W}_3^T & -\mathbf{W}_4^T & -\frac{1}{r}\mathbf{T}-\mathbf{W}_5-\mathbf{W}_5^T \end{pmatrix}. \tag{37}$$

It is easy to see that $\mathbf{\Psi}_k+r\Pi_k^T\mathbf{T}\Pi_k<0$ is equivalent to the condition (22) $\mathbf{\Phi}_k<0$ by Lemma 4 (Schur complement). So by Lemma 2 and from (36), we know that under the given condition (22), $\dot{V}(t)\le 0$ and we obtain $V(t)\le V(0)$, namely, $V(t)$ is a bounded function. Thus, $\|\mathbf{M}x(t)\|\longrightarrow 0$. This completes the proof. □

The term K_k is both involved in $\mathbf{\Psi}_{11}$ and $\mathbf{\Psi}_{16}$, so it is difficult to solve this by Matlab LMI Toolbox. In order to solve the feedback gain matrix K_k, a simple transformation is made to derive the following theorem.

Theorem 2. *Under assumptions* $(A_1)-(A_3)$, *the dynamical system* (21) *is globally asymptotically synchronized if there are positive definite matrices* $\mathbf{P}\in R^{(N-1)n\times(N-1)n}$, $\mathbf{Q}\in R^{(N-1)n\times(N-1)n}$, $\mathbf{R}\in R^{(N-1)n\times(N-1)n}$, $\mathbf{T}\in R^{(N-1)n\times(N-1)n}$, *positive definite diagonal matrices* $\mathbf{\Sigma}=\mathrm{diag}(\Sigma_1,\Sigma_2,\ldots,\Sigma_{(N-1)n})\in R^{(N-1)n\times(N-1)n}$, $\mathbf{\Lambda}=\mathrm{diag}(\Lambda_1,\Lambda_2,\ldots,\Lambda_{(N-1)n})\in R^{(N-1)n\times(N-1)n}$, *a matrix* $\mathbf{W}=(\mathbf{W}_1^T\;\mathbf{W}_2^T\;\mathbf{W}_3^T\;\mathbf{W}_4^T\;\mathbf{W}_5^T)^T\in R^{5(N-1)n\times(N-1)n}$ *and a matrix* $\mathbf{O}_k\in R^{(N-1)n\times(N-1)n}$, *for each* $k\in\mathcal{I}=\{1,2,\ldots,\overline{N}\}$ *such that*

$$\hat{\mathbf{\Phi}}_k = \begin{pmatrix} \hat{\mathbf{\Phi}}_{k11} & \mathbf{\Phi}_{k12} & \mathbf{\Phi}_{k13} & \mathbf{\Phi}_{k14} & \mathbf{\Phi}_{k15} & \hat{\mathbf{\Phi}}_{k16} \\ \mathbf{\Phi}_{k12}^T & \mathbf{\Phi}_{k22} & -\mathbf{W}_3^T & -\mathbf{W}_4^T & -\mathbf{W}_2-\mathbf{W}_5^T & 0 \\ \mathbf{\Phi}_{k13}^T & -\mathbf{W}_3 & -\mathbf{\Sigma}+\mathbf{Q} & 0 & -\mathbf{W}_3 & \mathbf{A}_k^{1T}\mathbf{P} \\ \mathbf{\Phi}_{k14}^T & -\mathbf{W}_4 & 0 & -(1-h)\mathbf{Q}-\mathbf{\Lambda} & -\mathbf{W}_4 & \mathbf{B}_k^{1T}\mathbf{P} \\ \mathbf{\Phi}_{k15}^T & -\mathbf{W}_2^T-\mathbf{W}_5 & -\mathbf{W}_3^T & -\mathbf{W}_4^T & -\frac{1}{r}\mathbf{T}-\mathbf{W}_5-\mathbf{W}_5^T & 0 \\ \hat{\mathbf{\Phi}}_{k16}^T & 0 & \mathbf{P}\mathbf{A}_k^1 & \mathbf{P}\mathbf{B}_k^1 & 0 & -\frac{1}{r}\mathbf{P}\mathbf{T}^{-1}\mathbf{P} \end{pmatrix}<0, \tag{38}$$

where

$$\hat{\mathbf{\Phi}}_{k11} = \mathbf{P}(-\mathbf{C}_k^1+\mathbf{H}_k)+\mathbf{O}_k+(-\mathbf{C}_k^1+\mathbf{H}_k)^T\mathbf{P}+\mathbf{O}_k^T+\mathbf{R}+\mathbf{F}\mathbf{\Sigma}\mathbf{F}+\mathbf{W}_1+\mathbf{W}_1^T,$$

$$\hat{\mathbf{\Phi}}_{k16} = (-\mathbf{C}_k^1+\mathbf{H}_k)^T\mathbf{P}+\mathbf{O}_k^T,$$

$F=\mathrm{diag}(F_1,F_2,\ldots,F_n)\in R^{n\times n}$, $\mathbf{F}=I_{N-1}\otimes F$, $H_k=MG_kJ$, $\mathbf{H}_k=H_k\otimes D$, $U_k=MK_kJ$, $\mathbf{U}_k=U_k\otimes D$, M *and* J *are defined in* (17) *and* (18). *Moreover, the estimation gain matrix* $\mathbf{U}_k=\mathbf{P}^{-1}\mathbf{O}_k$.

Proof. Pre- and post-multiplying $\mathbf{\Phi}$ in (22) by $\mathrm{diag}(\mathbf{I},\mathbf{I},\mathbf{I},\mathbf{I},\mathbf{I},\mathbf{P}\mathbf{T}^{-1})$ and $\mathrm{diag}(\mathbf{I},\mathbf{I},\mathbf{I},\mathbf{I},\mathbf{I},\mathbf{T}^{-1}\mathbf{P})$, respectively, and introducing a new variable $\mathbf{O}_k=\mathbf{P}\mathbf{U}_k$ yield (38), where $\mathbf{I}$ is identical matrix with appropriate dimensions. □

It is noted that the resulting condition in Theorem 2 are no longer LMI conditions due to term $\mathbf{P}\mathbf{T}^{-1}\mathbf{P}$ in (38). As a result, we cannot solve (38) by using Matlab LMI Toolbox. However, this non-convex problem can be solved by using an iterative algorithm based on the algorithms in [41–43].

First, we define a new positive definite matrix $\mathbf{L}$ such that $\mathbf{P}\mathbf{T}^{-1}\mathbf{P}\ge\mathbf{L}$ and replace (38) with

$$\hat{\mathbf{\Phi}}_k = \begin{pmatrix} \hat{\mathbf{\Phi}}_{k11} & \mathbf{\Phi}_{k12} & \mathbf{\Phi}_{k13} & \mathbf{\Phi}_{k14} & \mathbf{\Phi}_{k15} & \hat{\mathbf{\Phi}}_{k16} \\ \mathbf{\Phi}_{k12}^T & \mathbf{\Phi}_{k22} & -\mathbf{W}_3^T & -\mathbf{W}_4^T & -\mathbf{W}_2-\mathbf{W}_5^T & 0 \\ \mathbf{\Phi}_{k13}^T & -\mathbf{W}_3 & -\mathbf{\Sigma}+\mathbf{Q} & 0 & -\mathbf{W}_3 & \mathbf{A}_k^{1T}\mathbf{P} \\ \mathbf{\Phi}_{k14}^T & -\mathbf{W}_4 & 0 & -(1-h)\mathbf{Q}-\mathbf{\Lambda} & -\mathbf{W}_4 & \mathbf{B}_k^{1T}\mathbf{P} \\ \mathbf{\Phi}_{k15}^T & -\mathbf{W}_2^T-\mathbf{W}_5 & -\mathbf{W}_3^T & -\mathbf{W}_4^T & -\frac{1}{r}\mathbf{T}-\mathbf{W}_5-\mathbf{W}_5^T & 0 \\ \hat{\mathbf{\Phi}}_{k16}^T & 0 & \mathbf{P}\mathbf{A}_k^1 & \mathbf{P}\mathbf{B}_k^1 & 0 & -\frac{1}{r}\mathbf{L} \end{pmatrix}<0, \tag{39}$$

and

$$\mathbf{P}\mathbf{T}^{-1}\mathbf{P}\ge\mathbf{L}. \tag{40}$$

Since (40) is equivalent to $\mathbf{P}^{-1}\mathbf{T}\mathbf{P}^{-1}\le\mathbf{L}^{-1}$, it is expressed as

$$\begin{pmatrix} \mathbf{L}^{-1} & \mathbf{P}^{-1} \\ \mathbf{P}^{-1} & \mathbf{T}^{-1} \end{pmatrix}\ge 0 \tag{41}$$

by Lemma 2 (Schur complement), then, by introducing new variables $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$, the original condition (38) can be represented as (39) and

$$\begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Y} & \mathbf{Z} \end{pmatrix} \geq 0, \quad \mathbf{X} = \mathbf{L}^{-1}, \quad \mathbf{Y} = \mathbf{P}^{-1}, \quad \mathbf{Z} = \mathbf{T}^{-1}. \tag{42}$$

Using a cone complementary problem, this problem is converted to the following LMI-based nonlinear optimization problem:

Minimize $\mathrm{tr}(\mathbf{LX}+\mathbf{PY}+\mathbf{TZ})$

subject to (39) and

$$\begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Y} & \mathbf{Z} \end{pmatrix} \geq 0, \quad \begin{pmatrix} \mathbf{L} & \mathbf{I} \\ \mathbf{I} & \mathbf{X} \end{pmatrix} \geq 0, \quad \begin{pmatrix} \mathbf{P} & \mathbf{I} \\ \mathbf{I} & \mathbf{Y} \end{pmatrix} \geq 0, \quad \begin{pmatrix} \mathbf{T} & \mathbf{I} \\ \mathbf{I} & \mathbf{Z} \end{pmatrix} \geq 0. \tag{43}$$

The algorithm proposed in this paper is demonstrated as follows:

Algorithm 1. For a given precision $\delta > 0$, let $\hat{N}$ be the maximum number of iterations.

Step 1: Find a feasible set $(\mathbf{P}^0, \mathbf{Q}^0, \mathbf{R}^0, \mathbf{T}^0, \mathbf{\Sigma}^0, \mathbf{\Lambda}^0, \mathbf{W}^0, \mathbf{O}_k^0, \mathbf{L}^0, \mathbf{X}^0, \mathbf{Y}^0, \mathbf{Z}^0)$ satisfying (39) and (43). Set $j=0$.

Step 2: Solve the following LMI problem for variables $(\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{T}, \mathbf{\Sigma}, \mathbf{\Lambda}, \mathbf{W}, \mathbf{L}, \mathbf{O}_k, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$

Minimize $\mathrm{tr}(\mathbf{L}^j\mathbf{X}+\mathbf{L}\mathbf{X}^j+\mathbf{P}^j\mathbf{Y}+\mathbf{P}\mathbf{Y}^j+\mathbf{T}^j\mathbf{Z}+\mathbf{T}\mathbf{Z}^j)$

subject to (38) and (43).

Set $\hat{\mathbf{L}}^j = \mathbf{L}$, $\hat{\mathbf{P}}^j = \mathbf{P}$, $\hat{\mathbf{T}}^j = \mathbf{T}$, $\hat{\mathbf{X}}^j = \mathbf{X}$, $\hat{\mathbf{Y}}^j = \mathbf{Y}$, $\hat{\mathbf{Z}}^j = \mathbf{Z}$.

Step 3: If the condition (38) is satisfied, then exit. If the condition (38) is not satisfied, within a specified number of iterations, i.e., $j = \hat{N}$, then exit. Otherwise, set $j = j+1$ and go to Step 4.

Step 4: If $|\mathrm{tr}(\mathbf{L}^j\hat{\mathbf{X}}^j+\hat{\mathbf{L}}^j\mathbf{X}^j+\mathbf{P}^j\hat{\mathbf{Y}}^j+\hat{\mathbf{P}}^1\mathbf{Y}^j+\mathbf{T}^j\hat{\mathbf{Z}}^j+\hat{\mathbf{T}}^j\mathbf{Z}^j)-2\,\mathrm{tr}(\mathbf{L}^j\mathbf{X}^j+\mathbf{P}^j\mathbf{Y}^j+\mathbf{T}^j\mathbf{Z}^j)| < \delta$ then go to Step 2, else go to Step 5.

Step 5. Compute $\theta^* \in [0,1]$ by solving:

Minimize$_{\theta \in [0,1]}$ $\mathrm{tr}[\mathbf{L}^j+\theta(\hat{\mathbf{L}}^j-\mathbf{L}^j)][\mathbf{X}^j+\theta(\hat{\mathbf{X}}^j-\mathbf{X}^j)]+[\mathbf{P}^j+\theta(\hat{\mathbf{P}}^j-\mathbf{P}^j)][\mathbf{Y}^j-+\theta(\hat{\mathbf{Y}}^j-\mathbf{Y}^j)]+[\mathbf{T}^j+\theta(\hat{\mathbf{T}}^j-\mathbf{T}^j)][\mathbf{Z}^j+\theta(\hat{\mathbf{Z}}^j-\mathbf{Z}^j)]$.

Set $\mathbf{L}^{j+1} = \mathbf{L}^j+\theta^*(\hat{\mathbf{L}}^j-\mathbf{L}^j)$, $\mathbf{X}^{j+1} = \mathbf{X}^j+\theta^*(\hat{\mathbf{X}}^j-\mathbf{X}^j)$, $\mathbf{P}^{j+1} = \mathbf{P}^j+\theta^*(\hat{\mathbf{P}}^j-\mathbf{P}^j)$, $\mathbf{Y}^{j+1} = \mathbf{Y}^j+\theta^*(\hat{\mathbf{Y}}^j-\mathbf{Y}^j)$, $\mathbf{T}^{j+1} = \mathbf{T}^j+\theta^*(\hat{\mathbf{T}}^j-\mathbf{T}^j)$, $\mathbf{Z}^{j+1} = \mathbf{Z}^j+\theta^*(\hat{\mathbf{Z}}^j-\mathbf{Z}^j)$, then go to Step 2.

Note that we obtain $\mathbf{P}$ and $\mathbf{O}_k$ from Algorithm 1, our main purpose is to choose feedback gain matrix K_k. As $\mathbf{U}_k = (MK_kJ) \otimes D$, so we cannot solve K_k directly from Theorem 2. Since it is easy to see from Lemma 4 (Schur complement) that if (38) and

$$\mathbf{P}[(MK_kJ) \otimes D] \leq \mathbf{O}_k \tag{44}$$

are satisfied, then (22) is satisfied. If (38) is solved by Algorithm 1, we can use the following algorithm to solve (44):

Algorithm 2. Minimize $\mathrm{tr}(\mathbf{O}_k-\mathbf{P}[(MK_kJ) \otimes D])$

subject to (44).

Note that under condition (44), the lower bound of the object is zero, which means that gain matrix should not be very large.

Moreover, the adopted approaches here can also be used to control the linearly coupled neural network (21) if there are no switching signals ($\overline{N}=1$), which is reduced to one coupled complex network. It is noted that an applicable method is proposed to control the synchronization of coupled networks by changing the connection structure.

Corollary 1. *Under assumptions* $(A_1)-(A_3)$, *the dynamical system* (21) ($\overline{N}=1$) *is globally asymptotically synchronized if there are positive definite matrices* $\mathbf{P} \in R^{(N-1)n \times (N-1)n}$, $\mathbf{Q} \in R^{(N-1)n \times (N-1)n}$, $\mathbf{R} \in R^{(N-1)n \times (N-1)n}$, $\mathbf{T} \in R^{(N-1)n \times (N-1)n}$, *positive definite diagonal matrices* $\mathbf{\Sigma} = \mathrm{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_{(N-1)n}) \in R^{(N-1)n \times (N-1)n}$, $\mathbf{\Lambda} = \mathrm{diag}(\Lambda_1, \Lambda_2, \ldots, \Lambda_{(N-1)n}) \in R^{(N-1)n \times (N-1)n}$, a matrix $\mathbf{W} = (\mathbf{W}_1^T\ \mathbf{W}_2^T\ \mathbf{W}_3^T\ \mathbf{W}_4^T\ \mathbf{W}_5^T)^T \in R^{5(N-1)n \times (N-1)n}$ *and the feedback gain matrix* $K_k \in R^{N \times N}$, *for* $k=1$, *such that*

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{\Phi}_{k11} & \mathbf{\Phi}_{k12} & \mathbf{\Phi}_{k13} & \mathbf{\Phi}_{k14} & \mathbf{\Phi}_{k15} & \mathbf{\Phi}_{k16} \\ \mathbf{\Phi}_{k12}^T & \mathbf{\Phi}_{22} & -\mathbf{W}_3^T & -\mathbf{W}_4^T & -\mathbf{W}_2-\mathbf{W}_5^T & 0 \\ \mathbf{\Phi}_{k13}^T & -\mathbf{W}_3 & -\mathbf{\Sigma}+\mathbf{Q} & 0 & -\mathbf{W}_3 & \mathbf{A}_k^{1T}\mathbf{T} \\ \mathbf{\Phi}_{k14}^T & -\mathbf{W}_4 & 0 & -(1-h)\mathbf{Q}-\mathbf{\Lambda} & -\mathbf{W}_4 & \mathbf{B}_k^{1T}\mathbf{T} \\ \mathbf{\Phi}_{k15}^T & -\mathbf{W}_2^T-\mathbf{W}_5 & -\mathbf{W}_3^T & -\mathbf{W}_4^T & -\frac{1}{r}\mathbf{T}-\mathbf{W}_5-\mathbf{W}_5^T & 0 \\ \mathbf{\Phi}_{k16}^T & 0 & \mathbf{TA}_k^1 & \mathbf{TB}_k^1 & 0 & -\frac{1}{r}\mathbf{T} \end{pmatrix} < 0, \tag{45}$$

where

$$\mathbf{\Phi}_{k11} = \mathbf{P}(-\mathbf{C}_k^1+\mathbf{H}_k+\mathbf{U}_k)+(-\mathbf{C}_k^1+\mathbf{H}_k+\mathbf{U}_k)^T\mathbf{P}+\mathbf{R}+\mathbf{F\Sigma F}+\mathbf{W}_1+\mathbf{W}_1^T,$$

$$\mathbf{\Phi}_{k12} = \mathbf{W}_2^T-\mathbf{W}_1,$$

$$\mathbf{\Phi}_{k13} = \mathbf{PA}_k^1+\mathbf{W}_3^T,$$

$$\mathbf{\Phi}_{k14} = \mathbf{PB}_k^1+\mathbf{W}_4^T,$$

$$\mathbf{\Phi}_{k15} = \mathbf{W}_5^T-\mathbf{W}_1,$$

$$\mathbf{\Phi}_{k16} = (-\mathbf{C}_k^1+\mathbf{H}_k+\mathbf{U}_k)^T\mathbf{T},$$

$$\mathbf{\Phi}_{k22} = -(1-h)\mathbf{R}+\mathbf{F\Lambda F}-\mathbf{W}_2-\mathbf{W}_2^T,$$

$F = \mathrm{diag}(F_1, F_2, \ldots, F_n) \in R^{n \times n}$, $\mathbf{F} = I_{N-1} \otimes F$, $H_k = MG_kJ$, $\mathbf{H}_k = H_k \otimes D$, $U_k = MK_kJ$, $\mathbf{U}_k = U_k \otimes D$, *M and J are defined in* (17) *and* (18).

Remark 1. To the best of our knowledge, there are few works about the synchronization control of switched linearly coupled dynamical systems. In this paper, we consider global synchronization of switched linearly coupled delayed neural network. In addition, some controllers are designed to ensure the global synchronization of coupled dynamical system based on the convex optimization algorithm.

Remark 2. It is noted that when there is only one node in the switched coupled neural network, i.e., $\overline{N}=1$, the obtained results can also be satisfied for the linearly coupled neural networks. It is even applicable to the case that the derivative of the time-varying delay takes any value compared to the assumption $\dot{\tau}(t) < 1$ of earlier works.

4. Numerical examples

In this section, simulation examples are presented to illustrate the utility of theoretical analysis in this paper.

Example 1. Consider the following linearly coupled neural network model:

$$\dot{x}_i(t) = -C_\alpha x_i(t)+A_\alpha f(x_i(t))+B_\alpha f(x_i(t-\tau_\alpha(t)))+I_\alpha(t) + \sum_{j=1}^{N}(G_{\alpha ij}+K_{\alpha ij})Dx_j(t), \quad i=1,2,3, \ \ \alpha=1,2, \tag{46}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t))^T$, $f(x_i(t)) = (\tanh(x_{i1}(t)), \tanh(x_{i2}(t)))^T$, $I_1(t) = I_2(t) = (0,0)^T$,

$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 2.0 & -0.1 \\ -5.0 & 3.0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} -0.2 & 0.1 & 0.1 \\ 0.1 & -0.1 & 0 \\ 0.1 & 0 & -0.1 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1.8 & -0.1 \\ -4.5 & 4.0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -1.6 & -0.1 \\ -0.2 & -2.8 \end{pmatrix},$$

$$G_2=\begin{pmatrix}-0.3 & 0.2 & 0.1\\ 0.2 & -0.2 & 0\\ 0.1 & 0 & -0.1\end{pmatrix},$$

$D=\begin{pmatrix}1 & 0.5\\ 0.8 & 1\end{pmatrix}$, $\tau(t)=1+0.1\sin(12t)$. It is obvious that $0<\tau(t)<1.1=r$, $\dot{\tau}(t)\leq 1.2=h$. Clearly, assumptions A_1-A_3 are satisfied ($F=I_2$). Note that many earlier works assume $\dot{\tau}(t)<1$ and assumption A_3 is just needed in this paper. It is even applicable to the case that the derivative of the time-varying delay takes any value.

Choose the following initial conditions:

$$x_1(s)=\begin{pmatrix}0.1\\ -0.3\end{pmatrix},\quad x_2(s)=\begin{pmatrix}0.5\\ -1\end{pmatrix},\quad x_3(s)=\begin{pmatrix}1\\ -0.5\end{pmatrix}.$$

By Algorithms 1 and 2, the condition (22) in Theorem 1 is satisfied. The following coupling gain matrices are obtained:

$$K_1=\begin{pmatrix}-15.3789 & 7.4186 & 7.9603\\ 7.4186 & -15.3088 & 7.8902\\ 7.9603 & 7.8902 & -15.8505\end{pmatrix},$$

$$K_2=\begin{pmatrix}-14.0595 & 6.8074 & 7.2521\\ 6.8074 & -13.8852 & 7.0778\\ 7.2521 & 7.0778 & -14.3298\end{pmatrix}.$$

If there is no switched rule, the trajectories of one node in mode 1 ($\alpha=1$) and mode 2 ($\alpha=2$) are shown in Figs. 2 and 3 by choosing the initial conditions:

$x_1(s)=0.4,\quad x_2(s)=0.6,\quad \forall s\in[-1,0].$

Next a random switching rule is used for the two coupled neural networks. The error distance among the nodes of trajectories in the coupled networks are

$$err(t)=\sum_{i=1}^{2}\sqrt{[x_{1i}(t)-x_{2i}(t)]^2+[x_{1i}(t)-x_{3i}(t)]^2}.$$

It is shown from Fig. 4 that the trajectory of error distance of coupled system (46) without control does not converge to zero, which means that the switched coupled systems without control are not synchronized. The trajectories of the switched coupled neural networks with control are illustrated in Fig. 5, and the corresponding trajectory of error distance is illustrated in Fig. 6. It

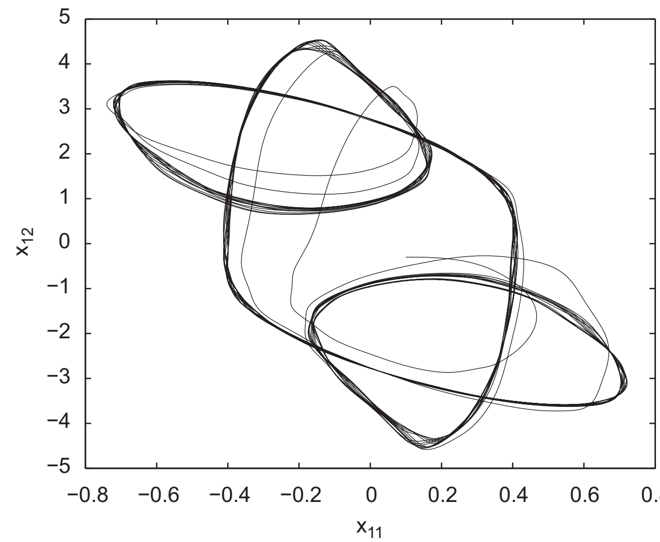

Fig. 3. Trajectories of one node in the coupled networks of mode 2.

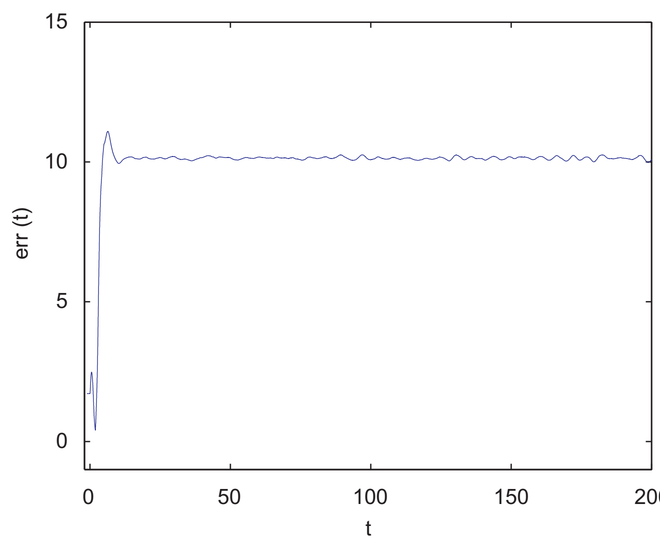

Fig. 4. Error distance of the switched coupled networks without control.

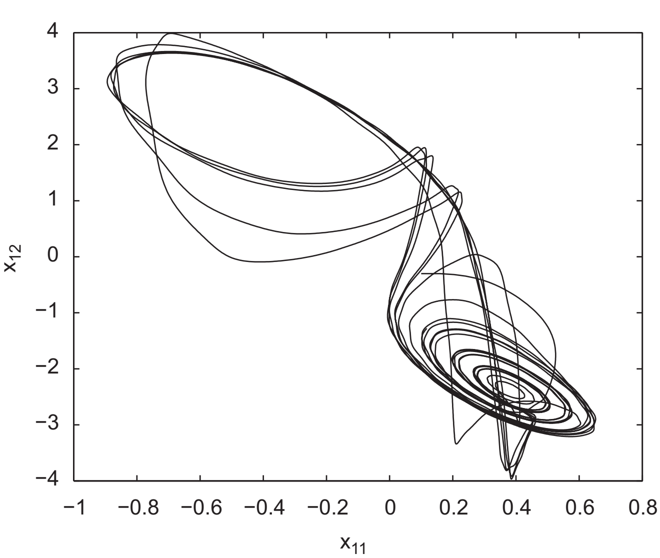

Fig. 2. Trajectories of one node in the coupled networks of mode 1.

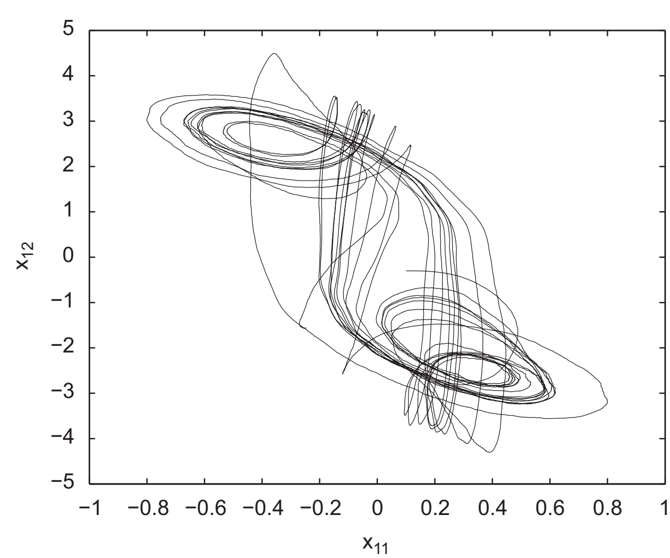

Fig. 5. Trajectories of one node in the switched coupled networks.

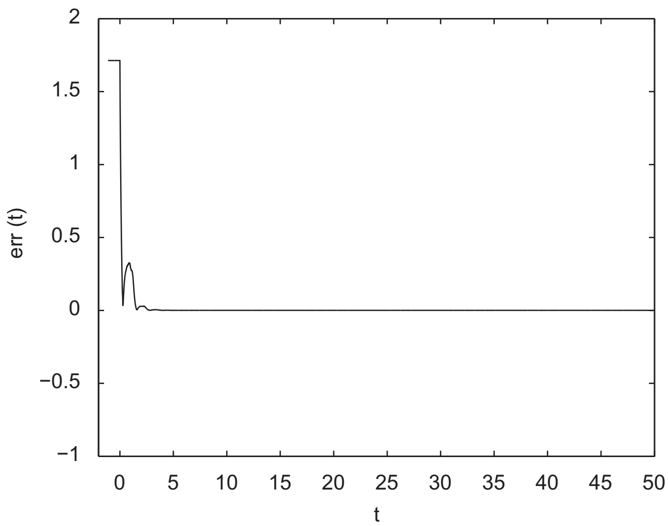

Fig. 6. Error distance of the switched coupled networks with control.

is easy to see that the switched coupled system with control (46) is globally synchronized in Fig. 6.

5. Conclusions

In this paper, synchronization control of switched linearly coupled delayed neural networks is considered based on Lyapunov functional method and linear matrix inequality (LMI) approach. The obtained results are easy to apply.

To the best of our knowledge, there are few works about synchronization control of switched coupled delayed systems. A globally convergent algorithm involving convex optimization is also presented to construct such controllers effectively. In many cases, we want to control the whole network by changing the weights of some nodes in the complex network, and this paper provides an applicable approach.

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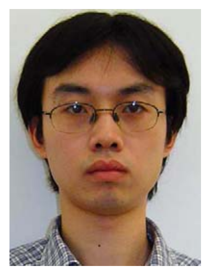

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