



Brief paper

On pinning synchronization of complex dynamical networks[☆]Wenwu Yu^{a,*}, Guanrong Chen^a, Jinhua Lü^b^a Department of Electronic Engineering, City University of Hong Kong, Hong Kong, China^b Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

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ABSTRACT

There exist some fundamental and yet challenging problems in pinning control of complex networks: (1) What types of pinning schemes may be chosen for a given complex network to realize synchronization? (2) What kinds of controllers may be designed to ensure the network synchronization? (3) How large should the coupling strength be used in a given complex network to achieve synchronization? This paper addresses these technique questions. Surprisingly, it is found that a network under a typical framework can realize synchronization subject to any linear feedback pinning scheme by using adaptive tuning of the coupling strength. In addition, it is found that the nodes with low degrees should be pinned first when the coupling strength is small, which is contrary to the common view that the most-highly-connected nodes should be pinned first. Furthermore, it is interesting to find that the derived pinning condition with controllers given in a high-dimensional setting can be reduced to a low-dimensional condition without the pinning controllers involved. Finally, simulation examples of scale-free networks are given to verify the theoretical results.

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1. Introduction

Many large-scale systems in nature and human societies, such as biological neural networks, ecosystems, metabolic pathways, the Internet, the WWW, electrical power grids, etc., can be described by networks with the nodes representing individuals in the system and the edges representing the connections among them. Recently, the study of various complex networks has attracted increasing attention from researchers in various fields of physics, mathematics, engineering, biology, and sociology.

In the early 1960s, Erdős and Rényi (1959, 1960) proposed a random-graph model, which had laid a solid foundation of modern network theory. In a random network, each pair of nodes is connected with a certain probability. To describe a transition from a regular network to a random network, Watts and Strogatz

(1998) proposed an interesting small-world network model. Then, Newman and Watts (1999) modified it to generate another variant of the small-world model. Later, Barabási and Albert (1999) proposed a scale-free network model, in which the degree distribution of the nodes follows a power-law form. Thereafter, small-world and scale-free networks have been extensively investigated.

Synchronization, on the other hand, is a typical collective behavior in nature. Since the pioneering work of Pecora and Carroll (1990), chaos control and synchronization have received a great deal of attention (Yu & Cao, 2007; Yu, Cao, Wong, & Lü, 2007; Yu, Chen, Cao, Lü, & Parlitz, 2007) due to their potential applications in secure communications, chemical reactions, biological systems, and so on. Typically, there are large numbers of nodes in real-world complex networks. Therefore, a large amount of work has been devoted to the study of synchronization in various large-scale complex networks (Cao, Li, & Wang, 2006; W. Lu & T. Chen, 2004; Wang & Cao, 2006; Wang & Chen, 2002a,b; Yu, Cao, & Lü, 2008). In Wang and Chen (2002a,b), local synchronization was investigated by the transverse stability to the synchronization manifold, where synchronization was discussed on small-world and scale-free networks. In Wu (2005) and Wu and Chua (1995), a distance from the collective states to the synchronization manifold was defined, based on which some results were obtained for global synchronization of coupled systems (Cao et al., 2006; W. Lu & T. Chen, 2004; Wang & Cao, 2006). A general criterion was given in Yu et al. (2008), where the network sizes can be extended to be much larger compared to those very small ones studied in Cao

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et al. (2006), W. Lu and T. Chen (2004) and Wang and Cao (2006). However, it is still very difficult to ensure global synchronization in very large-scale networks due to the computational complexity.

In the case where the whole network cannot synchronize by itself, some controllers may be designed and applied to force the network to synchronize. However, it is literally impossible to add controllers to all nodes. To reduce the number of controlled nodes, some local feedback injections may be applied to a fraction of network nodes, which is known as pinning control. In Grigoriev, Cross, and Schuster (1997), pinning control of spatiotemporal chaos, and later in Parekh, Parthasarathy, and Sinha (1998) global and local control of spatiotemporal chaos in coupled map lattices, were discussed. Very recently, in Wang and Chen (2002c), both specific and random pinning schemes were studied, where specific pinning of the nodes with large degrees is shown to require a smaller number of controlled nodes than the random pinning scheme. However, there exist some fundamental and challenging problems which have not been solved to date. A key problem is how the local controllers on the pinned nodes affect the global network synchronization. This paper aims to address the following questions: (1) What kinds of pinning schemes may be chosen for a given complex network to realize synchronization? (2) What types of controllers may be designed to ensure the synchronization? (3) How large should the coupling strength be used for a network with a fixed topological structure to effectively achieve global network synchronization?

The rest of the paper is organized as follows. In Section 2, some preliminaries are briefly outlined. The main theorems and corollaries for pinning synchronization on complex networks are given in Section 3. In Section 4, the pinning schemes on some scale-free networks are simulated to verify the theoretical analysis. Conclusions are finally drawn in Section 5.

2. Preliminaries

Consider a complex dynamical network consisting of N identical nodes with linearly diffusive coupling (Cao et al., 2006; W. Lu & T. Chen, 2004; J. Lü & G. Chen, 2005; Wang & Chen, 2002a,b; Yu et al., 2008), described by

$$\dot{x}_i(t) = f(x_i(t), t) + c \sum_{j=1, j \neq i}^N G_{ij} \Gamma (x_j(t) - x_i(t)), \quad i = 1, 2, \dots, N, \quad (1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^T \in \mathbb{R}^n$ is the state vector of the i th node, $f: \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is a continuously differentiable vector function, c is the coupling strength, $\Gamma \in \mathbb{R}^{n \times n}$ is the inner coupling matrix, $G = (G_{ij})_{N \times N}$ is the coupling configuration matrix representing the topological structure of the network, where G_{ij} are defined as follows: if there exists a connection between node i and node j , then $G_{ij} = G_{ji} > 0$; otherwise, $G_{ij} = G_{ji} = 0$ ($j \neq i$); and the diagonal elements of matrix G are defined by

$$G_{ii} = - \sum_{j=1, j \neq i}^N G_{ij}, \quad (2)$$

which ensures the diffusion that $\sum_{j=1}^N G_{ij} = 0$. Equivalently, network (1) can be rewritten in a simpler form as follows:

$$\dot{x}_i(t) = f(x_i(t), t) + c \sum_{j=1}^N G_{ij} \Gamma x_j(t), \quad i = 1, 2, \dots, N. \quad (3)$$

Note that a solution $s(t)$ of an isolated node satisfies

$$\dot{s}(t) = f(s(t), t). \quad (4)$$

Here, $s(t)$ may be an equilibrium point, a periodic orbit, or even a chaotic orbit. In this paper, the undirected networks are considered.

In the following, the synchronization of complex network model (3) is investigated. To realize the synchronization of network (3), some pinning controllers will be added to part of its nodes if the network is not self-synchronized. Here, the pinning strategy is applied on a small fraction δ ($0 < \delta < 1$) of the nodes in network (3). Suppose that the nodes i_1, i_2, \dots, i_l are selected, where $l = \lfloor \delta N \rfloor$ represents the integer part of the real number δN . Without loss of generality, rearrange the order of the nodes in the network, and let the first l nodes be controlled. Thus, the pinning controlled network can be described by

$$\begin{aligned} \dot{x}_i(t) &= f(x_i(t), t) + c \sum_{j=1}^N G_{ij} \Gamma x_j(t) + u_i, \quad i = 1, 2, \dots, l, \\ \dot{x}_i(t) &= f(x_i(t), t) + c \sum_{j=1}^N G_{ij} \Gamma x_j(t), \\ & \quad i = l + 1, 2, \dots, N, \end{aligned} \quad (5)$$

where

$$u_i = -cd_i \Gamma (x_i - s(t)) \in \mathbb{R}^n, \quad i = 1, 2, \dots, l, \quad (6)$$

are n -dimensional linear feedback controllers with all the control gains $d_i > 0$.

The objective of control here is to find some appropriate controllers (6) such that the solutions of the controlled network (5) synchronize with the solution of (4), in the sense that

$$\lim_{t \rightarrow \infty} \|x_i(t) - s(t)\| = 0, \quad i = 1, 2, \dots, N. \quad (7)$$

When the controlled complex network (5) achieves synchronization, the coupling terms and control inputs will automatically vanish due to the diffusive condition $\sum_{j=1}^N G_{ij} = 0$. This indicates that any solution $x_i(t)$ of any single node is also a solution of the synchronized coupled network.

Next, the pinning synchronization of network (5) is investigated. Subtracting (4) from (5) yields the following error dynamical network:

$$\begin{aligned} \dot{e}_i(t) &= f(x_i(t), t) - f(s(t), t) + c \sum_{j=1}^N G_{ij} \Gamma e_j(t) \\ & \quad - cd_i \Gamma e_i(t), \quad i = 1, 2, \dots, l, \\ \dot{e}_i(t) &= f(x_i(t), t) - f(s(t), t) + c \sum_{j=1}^N G_{ij} \Gamma e_j(t), \\ & \quad i = l + 1, 2, \dots, N, \end{aligned} \quad (8)$$

where $e_i(t) = x_i(t) - s(t)$, $i = 1, 2, \dots, N$. Throughout the rest of the paper, the following assumption is needed:

Assumption. There exists a constant matrix K such that

$$\begin{aligned} (x - y)^T (f(x, t) - f(y, t)) &\leq (x - y)^T K \Gamma (x - y), \\ \forall x, y \in \mathbb{R}^n. \end{aligned} \quad (9)$$

Note that Assumption (9) is very mild. For example, all linear and piecewise linear functions satisfy this condition. In addition, if $\partial f_i / \partial x_j$ ($i, j = 1, 2, \dots, n$) are bounded and Γ is positive definite, the above condition is satisfied. So, it includes many well-known systems, such as the Lorenz system (Lorenz, 1963), Chen system (Chen & Ueta, 1999), Lü system (Lü & Chen, 2002), recurrent neural networks (Yu, Cao, & Chen, 2008), Chua's circuit, and so on, some of which will be used for simulation in Section 4 below.

Hereafter, suppose that network (3) is connected, therefore there are no isolated clusters. From Theorem 6.2.24 (Horn & Johnson, 1985), a connected network is equivalent to an irreducible

network. Thus, only connected networks are considered throughout the paper. Let $\lambda_{\max}(F)$ be the largest eigenvalue of matrix F , and F^T be the transpose of matrix F . For matrices A and B with the same order, $A > B$ denotes that $A - B$ is a positive definite matrix.

Lemma 1 (Schur complement (Boyd, Ghaoui, Feron, & Balakrishnan, 1994)). The following linear matrix inequality (LMI)

$$\begin{pmatrix} \mathcal{Q}(x) & \mathcal{S}(x) \\ \mathcal{S}(x)^T & \mathcal{R}(x) \end{pmatrix} > 0,$$

where $\mathcal{Q}(x) = \mathcal{Q}(x)^T$, $\mathcal{R}(x) = \mathcal{R}(x)^T$, is equivalent to one of the following conditions:

- (i) $\mathcal{Q}(x) > 0$, $\mathcal{R}(x) - \mathcal{S}(x)^T \mathcal{Q}(x)^{-1} \mathcal{S}(x) > 0$;
- (ii) $\mathcal{R}(x) > 0$, $\mathcal{Q}(x) - \mathcal{S}(x) \mathcal{R}(x)^{-1} \mathcal{S}(x)^T > 0$.

Lemma 2 (Chen, Liu, & Lu, 2007). If G is irreducible, $G_{ij} = G_{ji} \geq 0$ for $i \neq j$, and $\sum_{j=1}^N G_{ij} = 0$, for all $i = 1, 2, \dots, N$, then all eigenvalues of the matrix

$$\begin{pmatrix} G_{11} - \varepsilon & G_{12} & \dots & G_{1N} \\ G_{21} & G_{22} & \dots & G_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1} & G_{N2} & \dots & G_{NN} \end{pmatrix}$$

are negative for any positive constant ε .

3. Pinning synchronization criteria for complex networks

In this section, some pinning criteria are established to ensure the global synchronization of complex dynamical networks.

3.1. General pinning synchronization criteria

In this subsection, some general pinning synchronization criteria are derived.

Theorem 1. Suppose that Assumption (9) holds. The controlled undirected network (5) is globally synchronized if the following condition is satisfied:

$$I_N \otimes (K\Gamma) + c(G - D) \otimes \Gamma < 0, \tag{10}$$

where \otimes is the Kronecker product, $D = \text{diag}(\underbrace{d_1, \dots, d_l}_{l}, \underbrace{0, \dots, 0}_{N-l})$, and I_N is the N -dimensional identity matrix.

Proof. Consider the Lyapunov functional candidate:

$$V(t) = \frac{1}{2} \sum_{i=1}^N e_i^T(t) e_i(t). \tag{11}$$

The derivative of $V(t)$ along the trajectories of (8) gives

$$\begin{aligned} \dot{V} &= \sum_{i=1}^N e_i^T(t) \dot{e}_i(t) \\ &= \sum_{i=1}^N e_i^T(t) \left[f(x_i(t), t) - f(s(t), t) + c \sum_{j=1}^N G_{ij} \Gamma e_j(t) \right] \\ &\quad - c \sum_{i=1}^l d_i e_i^T(t) \Gamma e_i(t) \\ &\leq \sum_{i=1}^N e_i^T(t) \left[K \Gamma e_i(t) + c \sum_{j=1}^N G_{ij} \Gamma e_j(t) \right] - c \sum_{i=1}^l d_i e_i^T(t) \Gamma e_i(t) \\ &= e^T(t) [(I_N \otimes K\Gamma) + c(G \otimes \Gamma) - c(D \otimes \Gamma)] e(t), \end{aligned} \tag{12}$$

where $e(t) = (e_1^T(t), e_2^T(t), \dots, e_N^T(t))^T$.

From (10), it is easy to see that (7) is satisfied, so the undirected network (5) is globally synchronized under the given linear feedback pinning controllers. The proof is completed.

Note that the dimension of the matrix condition in (10) is $N \times n$, so it is not easy to verify this condition. Let Γ be a positive definite matrix. If K and Γ are commutable and let $\theta = \lambda_{\max}(\frac{K+K^T}{2})$, then it can be considered as the passivity degree (Xiang & Chen, 2007), where $\theta > 0$ means that each node needs energy from outside to stabilize the network, while $\theta < 0$ means that each node itself is already stable. Here, only $\theta = \|K\| > 0$ is considered throughout the paper for simplicity; otherwise, the network is synchronized even without pinning control.

Corollary 1. Suppose that Assumption (9) holds and Γ is a positive definite matrix. The controlled undirected network (5) is globally synchronized if the following condition is satisfied:

$$C < 0, \tag{13}$$

where $C = \theta I_N + c(G - D)$ with $\theta = \|K\|$ and K defined in (9).

Proof. First, one has

$$I_N \otimes (K\Gamma) + c(G - D) \otimes \Gamma \leq (\theta I_N + cG - cD) \otimes \Gamma. \tag{14}$$

If $\theta I_N + cG - D < 0$ in (13), and $\Gamma > 0$, then one can easily show (Lütkepohl, 1996) that

$$(\theta I_N + cG - cD) \otimes \Gamma < 0.$$

So, condition (10) is satisfied, completing the proof. \square

Condition (13) is a general criterion to ensure the network pinning synchronization (Chen et al., 2007; Li, Wang, & Chen, 2004; Wang & Chen, 2002c).

Remark 1. Though the condition $C < 0$ in (13) is very simple, it is still difficult to choose an appropriate gain matrix D . One may attempt to use the MATLAB LMI Toolbox to solve (13). In the previous works (Cao et al., 2006; W. Lu & T. Chen, 2004; Wang & Cao, 2006), LMI approach was used to study the global synchronization of linearly coupled networks with delays, and it is solvable only for networks of extremely small sizes. However, in Yu et al. (2008), global synchronization of linearly hybrid coupled networks with time-varying delays was investigated, where the network size can be extended to be much larger. If the network size N is very large, the computation of LMI conditions is difficult in finite time. Therefore, it is still a challenging problem to find some appropriate pinning controllers to achieve global network synchronization with low computational complexity.

3.2. Coupling strength in pinning control

In this subsection, some criteria for choosing an appropriate coupling strength are established.

Corollary 2. Suppose that Assumption (9) holds and Γ is a positive definite matrix. The controlled undirected network (5) is globally synchronized if the following condition is satisfied:

$$c > \frac{\theta}{|\lambda_{\max}(G - D)|}. \tag{15}$$

Proof. By Lemma 2, it is easy to see that $G - D^*$ is negative definite, where D^* corresponds to only one controller, i.e., $l = 1$. It is easy to see that $\lambda_{\max}(G - D) \leq \lambda_{\max}(G - D^*) < 0$. From (15), one has $c(G - D) + \theta I_N \leq c\lambda_{\max}(G - D) + \theta < 0$, so (13) is satisfied. The proof is completed. \square

Remark 2. In Chen et al. (2007), pinning complex networks by a single controller, i.e., $l = 1$, was studied. However, it requires a very large coupling strength in general, which may not be very practical.

It is easy to see that the theoretical coupling strength given in (15) is too conservative, usually much larger than the needed value. Clearly, it is desirable to make the coupling strength as small as possible. Here, the adaptive technique (Chen et al., 2007; Yu, Cao, & Chen, 2007; Yu, Cao, Wong et al., 2007; Yu, Chen et al., 2007; Zhou, Lu, & Lü, 2006) is adopted to achieve this goal. The selected pinning controllers in (5), associated with the adaptive coupling law, lead to

$$\begin{aligned} \dot{x}_i(t) &= f(x_i(t), t) + c(t) \sum_{j=1}^N G_{ij} \Gamma x_j(t) \\ &\quad - c(t) d_i \Gamma (x_i(t) - s(t)), \quad i = 1, 2, \dots, l, \\ \dot{x}_i(t) &= f(x_i(t), t) + c(t) \sum_{j=1}^N G_{ij} \Gamma x_j(t), \\ &\quad i = l + 1, 2, \dots, N, \\ \dot{c}(t) &= \alpha \sum_{j=1}^N (x_j(t) - s(t))^T \Gamma (x_j(t) - s(t)), \end{aligned} \tag{16}$$

where α is a small positive constant.

Theorem 2. Suppose that Assumption (9) holds and Γ is a positive definite matrix. Then, the adaptively controlled undirected network (16) is globally synchronized for a small constant $\alpha > 0$.

Proof. Consider the Lyapunov functional candidate:

$$V(t) = \frac{1}{2} \sum_{i=1}^N e_i^T(t) e_i(t) + \frac{\beta}{2\alpha} (c(t) - \tilde{c})^2, \tag{17}$$

where β and \tilde{c} are positive constants to be determined below.

The derivative of $V(t)$ along the trajectories of (16) gives

$$\begin{aligned} \dot{V} &= \sum_{i=1}^N e_i^T(t) \dot{e}_i(t) + \beta(c(t) - \tilde{c}) \sum_{j=1}^N e_j(t)^T \Gamma e_j(t) \\ &= \sum_{i=1}^N e_i^T(t) \left[f(x_i(t), t) - f(s(t), t) \right. \\ &\quad \left. + c(t) \sum_{j=1}^N G_{ij} \Gamma e_j(t) \right] - c(t) \sum_{i=1}^l d_i e_i^T(t) \Gamma e_i(t) \\ &\quad + \beta(c(t) - \tilde{c}) \sum_{j=1}^N e_j(t)^T \Gamma e_j(t) \\ &\leq e^T(t) [(\theta I_N + c(t)(G - D + \beta I_N) - \beta \tilde{c}) \otimes \Gamma] e(t), \end{aligned} \tag{18}$$

where $e(t) = (e_1^T(t), e_2^T(t), \dots, e_N^T(t))^T$.

From Lemma 2, one can see that $G - D + \beta I_N$ is negative definite when β is sufficiently small. If one chooses a large initial condition $c(0)$ and let α be small enough, then $c(t)$ increases very slowly. In this case, the synchronization can be achieved by Corollary 2 when c is very large. After reaching synchronization, $c(t)$ converges to a constant and is bounded. Since $c(t)$ is bounded, one can always choose a sufficiently large \tilde{c} such that $\theta I_N + c(t)(G - D + \beta I_N) - \beta \tilde{c} < 0$. Then, in the whole time evolution process, \dot{V} is always negative definite. Consequently, the network is synchronized, so the third equation in (16) will vanish, leading to $c(t) \rightarrow \text{constant}$. This completes the proof. \square

Remark 3. It is noted that the condition in Theorem 2 is very mild, which is independent of the network structure (G) and the fraction of the nodes (l). Under any fixed network structure and pinning controllers in the form of (6), the coupling strengths can be self-adaptively determined according to (16) by Theorem 2 to realize network synchronization.

3.3. Selective pinning scheme

In this subsection, the pinning scheme is designed when the network structure and coupling strength are fixed.

Let $C = (C_{ij})_{N \times N}$ in (13) and $k_i = \sum_{j=1, j \neq i}^N G_{ij}$ be the total weights between node i and all the other nodes, where k_i is the degree of node i .

It is easy to see that $C_{ij} = cG_{ij}$ ($i \neq j$), $C_{ii} = \theta - ck_i - cd_i$ ($1 \leq i \leq l$), and $C_{ii} = \theta - ck_i$ ($l + 1 \leq i \leq N$), where $\theta = \|K\|$ with K defined in (9).

Corollary 3. Suppose that Assumption (9) holds and Γ is a positive definite matrix. To satisfy condition (13), it is necessary that

$$\begin{aligned} C_{ii} &= \theta - ck_i - cd_i < 0, \quad 1 \leq i \leq l, \\ C_{ii} &= \theta - ck_i < 0, \quad l + 1 \leq i \leq N, \end{aligned} \tag{19}$$

where $C = \theta I_N + cG - cD$, and $\theta = \|K\|$ with K defined in (9).

Proof. To ensure $C = \theta I_N + cG - D < 0$, it is necessary that $C_{ii} < 0$. This completes the proof. \square

Remark 4. From (19), one can see that for the nodes without pinning controllers, it is necessary that their degrees $k_i > \theta/c$ under condition (13). When the coupling strength c is very large, the degree of each node satisfies (19). In Chen et al. (2007), Li et al. (2004), Wang and Cao (2006), Xiang and Chen (2007, in press), Xiang, Liu, Chen, Chen, and Yuan (2007) and Zhou, Lu, and Lü (2008), condition (19) is assumed to be satisfied in the proposed pinning schemes, namely, the coupling strength c is very large therein. As seen here, interestingly, the nodes with low degrees should be controlled first, if c is not very large. It is common that the nodes with low degrees receive very little information from other nodes which can not be used to synchronize with the network due to the nonlinearity of the single node.

Up to this point, it is still not very clear as how to determine the control gains. Next, a theorem is given to address this concern.

Let $C = \theta I_N + cG - cD = \begin{pmatrix} A - \tilde{D} & B \\ B^T & \tilde{C} \end{pmatrix}$, where A and B are matrices with appropriate dimensions, $\tilde{D} = \text{diag}(d_1, \dots, d_l)$, and \tilde{C} is obtained by removing the $1, 2, \dots, l$ row-column pairs of matrix C .

Theorem 3. Suppose that Assumption (9) holds and Γ is a positive definite matrix. If the control gains d_i can be sufficiently large, condition (13), namely, $C < 0$, is equivalent to

$$\tilde{C} < 0. \tag{20}$$

For simplicity, one can select $\tilde{D} > \lambda_{\max}(A - B\tilde{C}^{-1}B^T)I_l$.

Proof. It is easy to see that if $C < 0$ then $\tilde{C} < 0$. So, one only needs to prove that if $\tilde{C} < 0$ then $C < 0$. Choose $\tilde{D} > A - B\tilde{C}^{-1}B^T$. Then, by Lemma 1, one indeed has $C < 0$. The proof is completed. \square

From the definition of \tilde{C} , one has $\tilde{C} = \theta I_{N-l} + c\tilde{G}$, where $\tilde{G}_{ij} = G_{l+i, l+j}$ for $i, j = l + 1, \dots, N$.

Corollary 4. Suppose that Assumption (9) holds and Γ is a positive definite matrix. The controlled undirected network (5) is globally synchronized if $\tilde{D} > \lambda_{\max}(A - B\tilde{C}^{-1}B^T)I_l$ and one of the following conditions holds:

- (i) $c\lambda_{\max}(\tilde{G}) + \theta < 0$,
- (ii) $\lambda_{\max}(\tilde{G}) < -\theta/c$,
- (iii) $c > \frac{\theta}{|\lambda_{\max}(\tilde{G})|}$.

Remark 5. Condition (20) is very simple since the pinning controllers are not involved, which provides some guidance to choose the control gains in \tilde{D} and therefore is very useful.

