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Robust H_∞ control and uniformly bounded control for genetic regulatory network with stochastic disturbance

W. Yu[1,2] *J. Lü*[3,4] *Z.D. Wang*[5] *J. Cao*[1] *Q. Zhou*[6]

[1]Department of Mathematics, Southeast University, Nanjing 210096, People's Republic of China
[2]Department of Electronic Engineering, City University of Hong Kong, Hong Kong
[3]Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, , Beijing 100190, People's Republic of China
[4]School of Electrical and Computer Engineering, RMIT University, Melbourne VIC 3001, Australia
[5]Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex UB8 3PH, UK
[6]Department of Biological Sciences, Columbia University, New York, NY 10027, USA
E-mail: wenwuyu@gmail.com

Abstract: Robust H_∞ control and uniformly bounded control for a genetic regulatory network (GRN) with stochastic disturbance are considered, where the GRN is delayed with SUM regulatory functions. A sufficient condition is derived to ensure the mean-square stability of stochastic GRN. Robust H_∞ controller is then designed to stabilise the stochastic GRN in the mean square sense. Furthermore, some robust uniformly bounded controllers are also proposed to overcome the flaws where H_∞ control cannot be used. Finally, numerical results are given to verify the theoretical analysis in this paper.

1 Introduction

Genome sequencing and gene recognition have been intensively investigated recently. However, a huge gap still exists between sequencing of the total genome of an organism and understanding the gene functions. For example, we still do not fully understand how the genes are expressed in the right time and right place, and at the right a mount throughout the development of the organism. Genetic regulatory networks (GRNs) are the mechanisms that have evolved to regulate the expression of genes, where the expression of a gene is regulated by its production. Actually, each gene contains some regulatory sequences, which are called *cis* elements. Transcription factors and their cofactors as well as other proteins can bind to those elements, and increase or reduce the gene expression level, which results in the change of the corresponding protein level, and in turn, affects other genes' expression levels.

Regulatory networks have become an important new research area in the biological and biomedical sciences and received great attention over the past few years [1–10]. In particular, since GRNs are high-dimensional and non-linear biochemically dynamical systems, it is natural to consider the GRNs from the view point of non-linear system theory, which provides a useful tool for studying gene regulation processes in living organisms.

Recently, there has been great interest to model the GRNs. Mathematical models are useful for investigating the mechanisms of organisms and studying behaviours of gene networks from the observed data. Basically, there are two types of genetic network models, that is, the Boolean model [11] and the differential equation model [1, 9, 10, 12–14]. In Boolean model, the activity of each gene is expressed in one of two states, ON or OFF, which is determined by a Boolean function of its own and other related states. On the other hand, the differential equation model describes the concentrations of gene products, such as mRNA and proteins, as the variables in GRNs.

It is well known that time delay is ubiquitous in most biological, physical, chemical, neural and other dynamical systems because of finite propagation speeds of signals, finite processing times, finite reaction times and finite switching speed of amplifiers. Therefore in biological systems especially GRNs, time delays are inevitable because of the slow process of transcription, translation and translocation process. It has been observed both experimentally and numerically that time delay could derail the stability of the system and even cause sustained oscillations, such as bifurcation or chaos [15, 16]. Therefore it is very important to consider delayed GRNs. Mathematical models without addressing time delays may even provide the wrong predictions of the mRNA and protein concentrations.

When modelling the GRNs, it should be noted that molecular noise has been shown to play important roles in biological functions of GRNs since molecular activities in cells are subject to significant thermal fluctuations, and noise process with transcriptional control and translation. Generally, the stochastic noise arising in gene expression can be described in one of the two ways, that is, intracelluar noise and extracelluar noise. The intracelluar noise is inevitable in reactions of transcription, translation and translocation process. On the other hand, the extracelluar noise originates from external fluctuations. Recently, some researchers studied the stochastic GRN models [4, 12, 14, 17, 18]. In addition, in practice, some system parameters are not accurately known, but it is possible to explore the range of system parameters by experience even from incomplete information, which paves the way for introducing the theory of interval matrices and interval dynamics to study the GRN. Therefore robust stability of uncertain regulatory network was investigated in [13].

In real biological organisms, mRNA and protein concentrations are regulated by the elegant GRN system. The steady stage of those molecular concentrations is often essential for normal life functions. Although the elegantly designed GRN can absorb some fluctuation of molecular concentrations caused by some intracelluar or extracelluar factors through negative feedback loops existing in GRN, sometimes the fluctuation will become so dramatic that it exceeds the ability of regulation by GRN, and in turn leads to very serious consequences. For example, cancer is a well-known disease that is caused by this inability of regulation by GRN. One of the most important factors that causes cancer is the up-regulation of antiapoptosis proteins [19, 20]. Apoptosis is also called programmed cell death, which exists in the whole life of living organisms. There is a fine-tuned balance in the process of apoptosis. Genetic mutations caused by intracelluar and extracelluar factors take place all the time. It is the apoptosis that forces the cells containing those mutations to die and therefore, protect the whole living organisms. However, in the body of some cancer patients, the anti-apoptosis protein level is up-regulated to a very high level caused by various factors such as chromosome translocations, which exceeds the ability of regulation by GRN. Then the cells with genetic mutations cannot be eliminated by apoptosis because of the high amount of anti-apoptosis proteins. This high amount of anti-apoptosis proteins contributes a lot to the eventual cancer formation. From this, it is clear that the limited ability of GRN cannot shutdown the over-active anti-apoptosis proteins and it is natural to think that artificially controlling the right amount of anti-apoptosis proteins is one way to fight against cancer, which is exactly what the current bio-medical researchers are doing. Thanks to the Nobel Prize winner 2006 Andrew Z. Fire and Craig C. Mello, bio-medical researchers now have a tool (RNA interference) to directly reduce the target mRNA level [21], and various researches are being conducted to test this potential cancer therapy [22–24]. Therefore, by providing an artificial control to some abnormal mRNA levels, it is possible to bring those molecules to a range where normal GRN can control.

RNA interference (also called ‘RNA-mediated interference’, abbreviated RNAi) is a mechanism for RNA-guided regulation of gene expression in which double-stranded ribonucleic acid inhibits the expression of genes with complementary nucleotide sequences. Usually, there are two ways to introduce RNA interference into cells. One way is to synthesise the small interfereing RNA, double-stranded, and introduce it to the cell [25–27]. The other way is to make a vector encoding a transcript which is the precursor of SiRNA, introduce this to the cell and it will be processed into functional SiRNA. The resulting SiRNA will form a complex with other proteins as well as the target mRNA, and lead to the target mRNA degradation [28]. Thus, from this, we can clearly regard the function of RNA interference as an artificial control, which is designed to reduce the specific mRNA level. Generally, if a protein is mis-expressed to a very high level, which is beyond the control of intrinsic GRNs, a proper RNA interference targeting this specific mRNA will reduce the level of that protein, which will lead to a steady state of the GRNs. It is natural to consider the artificial RNA interference as a control of the intrinsic GRNs. Therefore the design of feedback controller, which can be considered as artificial regulation, is a useful tool to stabilise the GRN. Motivated by this, we can study control of GRN from the viewpoint of control theory. We have already proposed some controllers to stabilise the non-linear systems, such as memoryless linear feedback controller [29], delayed feedback controller [30], adaptive controller [31] and other controllers [32, 33].

Here, we focus on a genetic network model where each transcription factor acts additively to regulate a gene, that is, the regulation function sums over all the inputs, which is called SUM logic [34, 35]. Such a regulation function is indeed found in many GRNs. In [2], the authors studied the local stability and bifurcation for delayed GRNs. In [13], robust stability of GRNs with time-varying delay was investigated. Considering the stochastic disturbance, the authors considered the mean-square stability of genetic networks [12, 14].

In this paper, we will study the mean-square stability, robust H_∞ control and robust uniformly bounded control of delayed GRN with both the parameter uncertainties and stochastic disturbance. Some feedback controllers are designed to ensure the mean-square stability of GRN. From the definition of mean-square stability [36], the perturbations should vanish when the genetic network is in steady state. However, from experimental results, we can see that GRN usually do not achieve mean-square stability, but perturbs around the steady states. Then robust H_∞ control is an effective approach to control the genetic network [37, 38]. Recently, the improved H_2/H_∞ control were also studied in [39, 40]. The so-called H_2/H_∞ control requires one to look for a controller, which not only satisfies the H_∞ performance, but also minimise the H_2 cost function when in the worst case disturbance is implemented, simultaneously. In addition, in some real GRNs, the energy of stochastic disturbance is infinite, which means that the perturbations exist and do not converge to 0 almost every time as the time goes to infinity. Therefore the assumption used in H_∞ is not satisfied. However, we can study the robust uniformly bounded control based on boundedness of stochastic noise [41, 42]. To the best of our knowledge, the control of GRN and the robust uniformly bounded control of delayed stochastic system are rarely investigated anywhere.

The remaining parts in this paper is organised as follows: In Section 2, the main background of GRN is briefly outlined and a generally delayed GRN with parameter uncertainties and stochastic disturbance is proposed. Some controllers are designed to stabilise the delayed uncertain GRN in the mean-square sense in Section 3. In Sections 4 and 5, some robust H_∞ and robust uniformly bounded controllers are further designed. In Section 6, numerical examples are constructed to show the effectiveness of proposed controllers. The conclusions are finally drawn in Section 7.

2 Preliminaries and problem formulation

The activities of a gene are regulated by other genes through the interactions between them, that is, the transcription and translation factors. Here, regulation can be regarded as the feedback, that is, the level of gene expressions as a function of the concentration of transcription factors. In [1, 12, 13], the following GRN model was considered

$$\begin{aligned}\frac{\mathrm{d}m_i(t)}{\mathrm{d}t} &= -a_i m_i(t) + G_i(p_1(t), p_2(t), \ldots, p_n(t))\\ \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} &= -c_i p_i(t) + d_i m_i(t)\end{aligned} \tag{1}$$

where $m_i(t)$, $p_i(t) \in R$ are concentrations of mRNA and protein of the i the node at time t, respectively, a_i and c_i are the degradation rates of the mRNA and protein, d_i is the translation rate, and the functions G_i represent the feedback regulation of the protein on the transcription, which is generally a non-linear monotonically increasing function [1–5, 12–14]. Taking the time delay into account, the following delayed GRN model was proposed [2, 12, 13]

$$\begin{aligned}\frac{\mathrm{d}m_i(t)}{\mathrm{d}t} &= -a_i m_i(t) + G_i(p_1(t-\tau), p_2(t-\tau), \ldots, p_n(t-\tau))\\ \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} &= -c_i p_i(t) + d_i m_i(t-\tau)\end{aligned} \tag{2}$$

where τ is a time delay. Here, the time delay is supposed to be the same for all mRNA and protein concentrations for simplicity. However, the proposed results can easily be extended to study the GRNs with multiple time delays.

The gene activity is controlled in a cell, and gene regulation function G_i plays a key role in the dynamical behaviour of the GRN. Generally, in order to accurately represent the dynamics in GRN, the term of G_i can be very complex, depending on all the biological reactions involved in the genetic network. In [12, 13], each transcription factor acts additively to regulate the ith gene, and the regulatory function is of the form $G_i = \sum_{j=1}^{n} G_{ij}(p_j(t))$, which is also called SUM logic [34, 35]. G_{ij} is a monotonic function of the Hill form [3]. If transcription factor j is an activator of gene i, then

$$G_{ij}(p_j(t)) = \alpha_{ij}\frac{(p_j(t)/\beta_j)^{H_j}}{1 + (p_j(t)/\beta_j)^{H_j}} \tag{3}$$

if transcription factor j is a repressor of gene i, then

$$G_{ij}(p_j(t)) = \alpha_{ij}\frac{1}{1 + (p_j(t)/\beta_j)^{H_j}} \tag{4}$$

where H_j is the Hill coefficient, β_j are positive constants, and α_{ij} are the dimensionless transcriptional rate of transcription factor j to i. Hence, system (2) can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}m_i(t)}{\mathrm{d}t} &= -a_i m_i(t) + \sum_{j=1}^{n} w_{ij} f_j(p_j(t-\tau)) + L_i\\ \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} &= -c_i p_i(t) + d_i m_i(t-\tau)\end{aligned} \tag{5}$$

where $f_j(x) = (x/\beta_j)^{H_j}/(1 + (x/\beta_j)^{H_j})$, $L_i = \sum_{j\in I_i} \alpha_{ij}$ and I_i is the set of all the repressors of gene i, $W = (w_{ij}) \in R^{n\times n}$ is defined as follows

$$w_{ij} = \begin{cases}\alpha_{ij} & \text{if transcription factor } j \text{ is an activator of gene } i,\\ 0 & \text{if there is no link from node } j \text{ to } i,\\ -\alpha_{ij} & \text{if transcription factor } j \text{ is a repressor of gene } i.\end{cases}$$

Then system (5) can be written into compact matrix form

$$\begin{aligned}\frac{\mathrm{d}m(t)}{\mathrm{d}t} &= -Am(t) + Wf(p(t-\tau)) + L \\ \frac{\mathrm{d}p(t)}{\mathrm{d}t} &= -Cp(t) + Dm(t-\tau)\end{aligned} \tag{6}$$

where $m(t) = [m_1(t), m_2(t), \ldots, m_n(t)]^{\mathrm{T}}$, $p(t) = [p_1(t), p_2(t), \ldots, p_n(t)]^{\mathrm{T}}$, $f(p(t-\tau)) = [f_1(p_1(t-\tau)), f_2(p_2(t-\tau)), \ldots, f_n(p_n(t-\tau))]^{\mathrm{T}}$, $m(t-\tau) = [m_1(t-\tau), m_2(t-\tau), \ldots, m_n(t-\tau)]^{\mathrm{T}}$, $L = (L_1, L_2, \ldots, L_n)^{\mathrm{T}}$, $A = \mathrm{diag}\{a_1, a_2, \ldots, a_n\}$, $C = \mathrm{diag}\{c_1, c_2, \ldots, c_n\}$ and $D = \mathrm{diag}\{d_1, d_2, \ldots, d_n\}$.

Let $[(p^*)^{\mathrm{T}}, (m^*)^{\mathrm{T}}]^{\mathrm{T}}$ be an equilibrium of (6), and the following equations are satisfied

$$\begin{aligned}-Am^* + Wf(p^*) + L &= 0 \\ -Cp^* + Dm^* &= 0\end{aligned} \tag{7}$$

Next, we will shift the equilibrium $[(p^*)^{\mathrm{T}}, (m^*)^{\mathrm{T}}]^{\mathrm{T}}$ of system (6) to the origin. Using the transformation $\widehat{m}(t) = m(t) - m^*, \widehat{p}(t) = p(t) - p^*$, system (6) can be transformed into the following form

$$\begin{aligned}\frac{\mathrm{d}\widehat{m}(t)}{\mathrm{d}t} &= -A\widehat{m}(t) + Wg(\widehat{p}(t-\tau)) \\ \frac{\mathrm{d}\widehat{p}(t)}{\mathrm{d}t} &= -C\widehat{p}(t) + D\widehat{m}(t-\tau)\end{aligned} \tag{8}$$

where $\widehat{m}(t) = [\widehat{m}_1(t), \widehat{m}_2(t), \ldots, \widehat{m}_n(t)]^{\mathrm{T}}$, $\widehat{p}(t) = [\widehat{p}_1(t), \widehat{p}_2(t), \ldots, \widehat{p}_n(t)]^{\mathrm{T}}$, $g(\widehat{p}(t-\tau)) = [g_1(\widehat{p}_1(t-\tau)), g_2(\widehat{p}_2(t-\tau)), \ldots, g_n(\widehat{p}_n(t-\tau))]^{\mathrm{T}}$ with $g_i(\widehat{p}_i(t)) = f_i(\widehat{p}_i(t) + p_i^*) - f_i(p_i^*)$. In the real system, the parameter of the system are not accurately known. Then delayed uncertain GRN is obtained [4]

$$\begin{aligned}\frac{\mathrm{d}\widehat{m}(t)}{\mathrm{d}t} &= -(A + \Delta A)\widehat{m}(t) + (W + \Delta W)g(\widehat{p}(t-\tau)) \\ \frac{\mathrm{d}\widehat{p}(t)}{\mathrm{d}t} &= -(C + \Delta C)\widehat{p}(t) + (D + \Delta D)\widehat{m}(t-\tau)\end{aligned} \tag{9}$$

where ΔA, ΔW, ΔC and ΔD are unknown matrices. Let $x(t) = [\widehat{m}^{\mathrm{T}}(t), \widehat{p}^{\mathrm{T}}(t)]^{\mathrm{T}}$. Next we consider the following delayed GRN with stochastic disturbance

$$\begin{aligned}\mathrm{d}x(t) &= [-(\widetilde{A} + \Delta\widetilde{A})x(t) + (\widetilde{D} + \Delta\widetilde{D})x(t-\tau) \\ &\quad + (\widetilde{W} + \Delta\widetilde{W})g(x(t-\tau))]\mathrm{d}t + \sigma(x(t), x(t-\tau))\,\mathrm{d}\omega \\ &\quad + v(t)\,\mathrm{d}\nu + Bu(t)\,\mathrm{d}t \\ y(t) &= C_0 x(t) + D_0 u(t)\end{aligned} \tag{10}$$

where
$\widetilde{A} = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$, $\Delta\widetilde{A} = \begin{pmatrix} \Delta A & 0 \\ 0 & \Delta C \end{pmatrix}$, $\widetilde{D} = \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}$, $\Delta\widetilde{D} = \begin{pmatrix} 0 & 0 \\ \Delta D & 0 \end{pmatrix}$, $\widetilde{W} = \begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix}$, $\Delta\widetilde{W} = \begin{pmatrix} 0 & \Delta W \\ 0 & 0 \end{pmatrix}$
$g(x(t-\tau)) = [g^{\mathrm{T}}(\widehat{m}(t)), g^{\mathrm{T}}(\widehat{p}(t))]^{\mathrm{T}}$, $\sigma(x(t), x(t-\tau)) \in R^{2n}$ and $v(t) \in R^{2n}$ are states and external noise intensity functions, $\omega(t)$ and $\nu(t)$ are two independent one-dimensional Brownian motion satisfying $\mathbf{E}\{\mathrm{d}\omega(t)\} = 0$, $\mathbf{E}\{\mathrm{d}\nu(t)\} = 0$, $\mathbf{E}\{\mathrm{d}\omega(t)^2\} = 1$ and $\mathbf{E}\{\mathrm{d}\nu(t)^2\} = 1$ ($\mathbf{E}\{\cdot\}$ is the mathematical expectation), $y(t) \in R^m$ is the control output, $u(t) \in R^q$ is the control input, B, C_0 and D_0 are matrices with appropriate dimensions, and the initial conditions of (10) are given by $x_i(t) = \phi_i(t) \in \mathcal{C}([-\tau, 0], R)$, where $\mathcal{C}([-\tau, 0], R)$ denotes the set of all continuous functions from $[-\tau, 0]$ to R. This type of stochastic perturbation can be regarded as a result from the occurrence of random uncertainties from the GRN. If the network (10) is unstable, in this paper, we aim to design some controllers to stabilise the system (10). Here, we mainly consider the simple linear memoryless feedback controller $u(t) = Kx(t)$, where K is the gain matrix with appropriate dimensions.

To establish our main results, it is necessary to make the following assumptions:

A$_1$: Each function f_i: $R \rightarrow R$ is monotonically increasing with saturation $h_i > 0$, that is

$$|f_i(u) - f_i(v)| \le h_i|u - v| \forall u, \quad v \in R, \quad i = 1, 2, \ldots, n$$

Clearly, the function g_i satisfies

$$0 \le \frac{g_i(u)}{u} \le h_i, \quad \forall u \in R, \quad i = 1, 2, \ldots, n \tag{11}$$

A$_2$: The uncertain matrices ΔA, ΔW, ΔC and ΔD are assumed to be of the form

$$\begin{aligned}\Delta A &= MF(t)N_A, & \Delta W &= MF(t)N_W, \\ \Delta C &= MF(t)N_C, & \Delta D &= MF(t)N_D\end{aligned} \tag{12}$$

where M, N_A, N_W, N_C and N_D are known real matrices and F: $R \longrightarrow R^{k \times l}$ is an unknown time-varying matrix function satisfying

$$F^{\mathrm{T}}(t)F(t) \le I, \quad \forall t \tag{13}$$

A$_3$: σ: $R^{2n} \times R^{2n} \longrightarrow R^{2n}$ satisfies

$$\begin{aligned}&\sigma^{\mathrm{T}}(x(t), x(t-\tau))\sigma(x(t), x(t-\tau)) \\ &\quad \le x^{\mathrm{T}}(t)P_1^{\mathrm{T}}P_1x(t) + x^{\mathrm{T}}(t-\tau)P_2^{\mathrm{T}}P_2x(t-\tau)\end{aligned} \tag{14}$$

where P_1 and P_2 are matrices with appropriate dimensions.

A$_4$: $v(t) \in R^{2n}$ belongs to $L_2[0, \infty)$.

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A$_5$: $v(t) \in R^{2n}$ belongs to $L_\infty[0, \infty)$, that is, $v(t)$ is a bounded vector function satisfying

$$v^{\mathrm{T}}(t)v(t) \leq \beta^2, \quad \forall t \tag{15}$$

where β is a positive constant.

The Assumptions A$_1$–A$_5$ are quite common in both the control and biological fields. Assumption 1 is very mild and many regulation functions satisfy such an assumption, for example, the Hill function in (3). It is also reasonable in Assumption 2 that the unknown matrices in the GRN are norm bounded. The assumptions on the states and external noise intensity functions in A$_3$–A$_5$ are satisfied in the real biological GRNs since the intensity of the molecular noise from the reactions of transcription, translation, translocation process, and also from external fluctuations is typically faded or bounded.

Definition 1: The system (10) with $u(t) = 0$ and $v(t) = 0$ is said to be mean-square stable if $\forall \varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that

$$\mathbf{E}\|x(t)\|^2 < \varepsilon^2, \quad t > 0$$

when $\sup_{-\tau \leq s \leq 0} \mathbf{E}\|\phi(s)\|^2 < \delta(\varepsilon)$. If, in addition

$$\lim_{t \to \infty} \mathbf{E}\|x(t)\|^2 \longrightarrow 0$$

for any initial conditions, then the system (10) with $u(t) = 0$ and $v(t) = 0$ is said to be mean-square asymptotically stable. Furthermore, the system (10) with $u(t) = 0$ and $v(t) = 0$ is said to be mean-square robustly asymptotically stable if it is mean-square asymptotically stable for all system matrices under Assumption A$_2$.

Definition 2: The system (10) with $v(t) = 0$ is said to be mean-square robustly asymptotically stabilisable if there exists a memoryless state feedback controller in the form of $u(t) = Kx(t)$ such that system (10) with $v(t) = 0$ is mean-square robustly asymptotically stable.

Definition 3: Consider the system (10), $u(t) = Kx(t)$ is said to be a robust H_∞ controller if

(i) system (10) with $v(t) = 0$ is mean-square robustly asymptotically stabilisable;

(ii) Under zero initial conditions, there exists a scalar γ

$$\mathbf{E}\int_0^\infty \|y(s)\|^2 \, \mathrm{d}s \leq \gamma^2 \int_0^\infty \|v(s)\|^2 \, \mathrm{d}s$$

for all non-zero $v \in L_2[0, \infty)$, and Assumptions A$_2$ and A$_4$ are satisfied. The parameter γ is said to be the H_∞-norm bound for the H_∞ state feedback control.

Definition 4: Consider the system (10), $u(t) = Kx(t)$ is said to be a robust uniformly bounded controller if

(i) system (10) with $v(t) = 0$ is mean-square robustly asymptotically stabilisable;

(ii) there exists a scalar $\alpha > 0$

$$\lim_{t \to \infty} \mathbf{E}\|y(t)\|^2 \leq \alpha^2 \beta^2$$

for all non-zero $v \in L_\infty[0, \infty)$, and Assumptions A$_2$ and A$_5$ are satisfied.

In order to derive the main results, the following four lemmas are needed:

Lemma 1 [43]: For any vectors $x, y \in R^n$ and positive-definite matrix $G \in R^{n \times n}$, the following matrix inequality holds

$$2x^{\mathrm{T}}y \leq x^{\mathrm{T}}Gx + y^{\mathrm{T}}G^{-1}y$$

Lemma 2 [37]: For any scalar $\varepsilon > 0$, vectors $x, y \in R^n$ and matrices M, N, and $F^{\mathrm{T}}F \leq I$ with appropriate dimensions, the following matrix inequality holds

$$2x^{\mathrm{T}}MFNy \leq \varepsilon x^{\mathrm{T}}MM^{\mathrm{T}}x + \varepsilon^{-1}y^{\mathrm{T}}N^{\mathrm{T}}Ny$$

Lemma 3 (Schur complement [44]): The following linear matrix inequality (LMI)

$$\begin{pmatrix} Q(x) & S(x) \\ S(x)^{\mathrm{T}} & R(x) \end{pmatrix} > 0$$

where $Q(x) = Q(x)^{\mathrm{T}}$, $R(x) = R(x)^{\mathrm{T}}$, is equivalent to one of the following conditions

(i) $Q(x) > 0$, $R(x) - S(x)^{\mathrm{T}}Q(x)^{-1}S(x) > 0$,

(ii) $R(x) > 0$, $Q(x) - S(x)R(x)^{-1}S(x)^{\mathrm{T}} > 0$.

Lemma 4 [45]: Suppose that function $V(t)$ is non-negative when $t \in (-\tau, \infty)$ and satisfies the following

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leq -k_1V(t) + k_2V(t-\tau) + \alpha, \quad t \geq 0 \tag{16}$$

where α, k_1 and k_2 are positive constants, and $k_1 > k_2$. Then

$$V(t) < \|V(0)\|_\tau \mathrm{e}^{-r\tau} + \frac{\alpha}{r}, \quad t \geq 0 \tag{17}$$

where $\|V(0)\|_\tau = \sup_{-\tau \leq s \leq 0} |V(s)|$ and r is the unique positive solution of

$$-r = -k_1 + k_2 e^{r\tau} \tag{18}$$

3 Mean-square robust asymptotical stabilisation

In this section, we will derive some controllers to ensure mean-square robust asymptotical stability of GRN (10) with $v(t) = 0$.

Theorem 1: Under Assumptions $A_1 - A_3$, system (10) with $v(t) = 0$ is mean-square robustly asymptotically stabilisable if there are positive definite diagonal matrix $\overline{\Lambda} = \text{diag}(\overline{\Lambda}_1, \overline{\Lambda}_2, \ldots, \overline{\Lambda}_{2n}) > 0$, positive-definite matrices $\overline{P} = (\overline{p}_{ij})_{2n\times 2n}$, $\overline{S} = (\overline{s}_{ij})_{2n\times 2n}$, $\overline{Q} = (q_{ij})_{2n\times 2n}$, positive constants $\overline{\varepsilon}_1$, $\overline{\varepsilon}_2$, $\overline{\varepsilon}_3$, $\overline{\rho}$ and matrix $\overline{K}$, such that (see (19))

$$\overline{P} \geq \overline{\rho} I \tag{20}$$

where

$$\Theta_{11} = -2\widetilde{A}\overline{P} + 2B\overline{K} + \overline{S} + (\overline{\varepsilon}_1 + \overline{\varepsilon}_2 + \overline{\varepsilon}_3)\begin{pmatrix} MM^{\mathrm{T}} & 0 \\ 0 & MM^{\mathrm{T}} \end{pmatrix},$$

$$\widetilde{N}_A = \begin{pmatrix} N_A & 0 \\ 0 & N_C \end{pmatrix}^{\mathrm{T}}, \quad \widetilde{N}_D = \begin{pmatrix} 0 & 0 \\ N_D & 0 \end{pmatrix}^{\mathrm{T}},$$

$$\widetilde{N}_W = \begin{pmatrix} 0 & N_W \\ 0 & 0 \end{pmatrix}^{\mathrm{T}}$$

and i is the identity matrix. In this case, the appropriate state feedback controller can be chosen by

$$u(t) = Kx(t), \quad K = \overline{KP}^{-1} \tag{21}$$

Proof: Let $\overline{P} = P^{-1}$, $\overline{\rho} = \rho^{-1}$, $\overline{K} = KP^{-1}$, $\overline{\varepsilon}_1 = \varepsilon_1^{-1}$, $\overline{\varepsilon}_2 = \varepsilon_2^{-1}$, $\overline{\varepsilon}_3 = \varepsilon_3^{-1}$, $\overline{\Lambda} = \Lambda^{-1}$, $\overline{S} = P^{-1}SP^{-1}$, $\overline{Q} = \Lambda^{-1}Q\Lambda^{-1}$. Consider the Lyapunov candidate

$$V(t) = x^{\mathrm{T}}(t)Px(t) + \int_{t-\tau}^{t} x^{\mathrm{T}}(l)Sx(l)\,\mathrm{d}l + \int_{t-\tau}^{t} g^{\mathrm{T}}(x(l))Qg(l)\,\mathrm{d}l \tag{22}$$

where $P = \overline{P}^{-1}$, $S = \overline{P}^{-1}\overline{SP}^{-1}$ and $Q = \overline{\Lambda}^{-1}\overline{Q\Lambda}^{-1}$ are positive-definite matrices.

From Itô formula [36], we obtain the following stochastic differential

$$\mathrm{d}V(t) = \mathcal{L}V(t)\,\mathrm{d}t + 2x^{\mathrm{T}}(t)P[\sigma(x(t), x(t-\tau))\,\mathrm{d}\omega] \tag{23}$$

The weak infinitesimal operator $\mathcal{L}$ of the stochastic process $\{x_t = x(t+s), t \geq 0, -r \leq s \leq 0\}$ is given by

$$\begin{aligned} \mathcal{L}V(t) = {} & 2x^{\mathrm{T}}(t)P[-(\widetilde{A} + \Delta\widetilde{A})x(t) \\ & + (\widetilde{D} + \Delta\widetilde{D})x(t-\tau) + (\widetilde{W} + \Delta\widetilde{W})g(x(t-\tau))] \\ & + \sigma^{\mathrm{T}}(x(t), x(t-\tau(t))P\sigma(x(t), x(t-\tau(t)) \\ & + x^{\mathrm{T}}(t)Sx(t) - x^{\mathrm{T}}(t-\tau)Sx(t-\tau) \\ & + g^{\mathrm{T}}(x(t))Qg(x(t)) - g^{\mathrm{T}}(x(t-\tau))Qg(x(t-\tau)) \\ & + 2x^{\mathrm{T}}PBKx(t) \end{aligned} \tag{24}$$

By Assumption A_3 and (20), we have

$$\begin{aligned} & \sigma^{\mathrm{T}}(x(t), x(t-\tau(t))P\sigma(x(t), x(t-\tau(t)) \\ & \quad \leq \rho\sigma^{\mathrm{T}}(x(t), x(t-\tau(t))\sigma(x(t), x(t-\tau(t)) \\ & \quad \leq \rho[x^{\mathrm{T}}(t)P_1^{\mathrm{T}}P_1x(t) + x^{\mathrm{T}}(t-\tau)P_2^{\mathrm{T}}P_2x^{\mathrm{T}}(t-\tau)] \end{aligned} \tag{25}$$

From Assumption A_1, it is obvious that

$$\begin{aligned} g^{\mathrm{T}}(x(t))\Lambda g(x(t)) &= \sum_{i=1}^{n} g_i(x_i(t))\Lambda_i g_i(x_i(t)) \leq \sum_{i=1}^{n} \Lambda_i h_i^2(x_i(t)) \\ &= x^{\mathrm{T}}(t)H\Lambda Hx(t) \end{aligned} \tag{26}$$

where $H = \text{diag}(h_1, h_2, \ldots, h_n)$ and $\Lambda = \text{diag}(\Lambda_1, \Lambda_2, \ldots, \Lambda_n)$ are positive-definite diagonal matrices.

$$\Theta = \begin{pmatrix} \Theta_{11} & \widetilde{D}\overline{P} & 0 & \widetilde{W}\overline{\Lambda} & \overline{P}P_1^{\mathrm{T}} & 0 & \overline{P}H & \overline{P}\widetilde{N}_A & 0 & 0 \\ * & -\overline{S} & 0 & 0 & 0 & \overline{P}P_2^{\mathrm{T}} & 0 & 0 & \overline{P}\widetilde{N}_D & 0 \\ * & * & \overline{Q} - \overline{\Lambda} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\overline{Q} & 0 & 0 & 0 & 0 & 0 & \overline{\Lambda}\widetilde{N}_W \\ * & * & * & * & -\overline{\rho}I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\overline{\rho}I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\overline{\Lambda} & 0 & 0 & \\ * & * & * & * & * & * & * & -\overline{\varepsilon}_1 I & 0 & 0 \\ * & * & * & * & * & * & * & * & -\overline{\varepsilon}_2 I & \\ * & * & * & * & * & * & * & * & * & -\overline{\varepsilon}_3 I \end{pmatrix} < 0 \tag{19}$$

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From Lemma 2 and Assumption A_2, we obtain

$$\begin{aligned}
&-2x^{\mathrm{T}}(t)P\Delta\widetilde{A}x(t)\\
&= 2x^{\mathrm{T}}(t)P\begin{pmatrix} M & 0\\ 0 & M\end{pmatrix}\begin{pmatrix} F(t) & 0\\ 0 & F(t)\end{pmatrix}\begin{pmatrix} -N_A & 0\\ 0 & -N_C\end{pmatrix}x(t)\\
&\le \varepsilon_1^{-1}x^{\mathrm{T}}(t)P\begin{pmatrix} MM^{\mathrm{T}} & 0\\ 0 & MM^{\mathrm{T}}\end{pmatrix}Px(t)\\
&\quad + \varepsilon_1 x^{\mathrm{T}}(t)\begin{pmatrix} N_A^{\mathrm{T}}N_A & 0\\ 0 & N_C^{\mathrm{T}}N_C\end{pmatrix}x(t)
\end{aligned}\tag{27}$$

$$\begin{aligned}
&-2x^{\mathrm{T}}(t)P\Delta\widetilde{D}x(t-\tau)\\
&= 2x^{\mathrm{T}}(t)P\begin{pmatrix} M & 0\\ 0 & M\end{pmatrix}\begin{pmatrix} F(t) & 0\\ 0 & F(t)\end{pmatrix}\begin{pmatrix} 0 & 0\\ N_D & 0\end{pmatrix}x(t-\tau)\\
&\le \varepsilon_2^{-1}x^{\mathrm{T}}(t)P\begin{pmatrix} MM^{\mathrm{T}} & 0\\ 0 & MM^{\mathrm{T}}\end{pmatrix}Px(t)\\
&\quad + \varepsilon_2 x^{\mathrm{T}}(t-\tau)\begin{pmatrix} N_D^{\mathrm{T}}N_D & 0\\ 0 & 0\end{pmatrix}x(t-\tau)
\end{aligned}\tag{28}$$

$$\begin{aligned}
&-2x^{\mathrm{T}}(t)P\Delta\widetilde{W}g(x(t-\tau))\\
&= 2x^{\mathrm{T}}(t)P\begin{pmatrix} M & 0\\ 0 & M\end{pmatrix}\begin{pmatrix} F(t) & 0\\ 0 & F(t)\end{pmatrix}\\
&\quad\times\begin{pmatrix} 0 & N_W\\ 0 & 0\end{pmatrix}g(x(t-\tau))\\
&\le \varepsilon_3^{-1}x^{\mathrm{T}}(t)P\begin{pmatrix} MM^{\mathrm{T}} & 0\\ 0 & MM^{\mathrm{T}}\end{pmatrix}Px(t)\\
&\quad + \varepsilon_3 g^{\mathrm{T}}(x(t-\tau))\begin{pmatrix} 0 & 0\\ 0 & N_W^{\mathrm{T}}N_W\end{pmatrix}g(x(t-\tau))
\end{aligned}\tag{29}$$

Therefore combining (24)–(29) we have

$$\begin{aligned}
\mathcal{L}V(t) &\le -2x^{\mathrm{T}}(t)P\widetilde{A}x(t) + 2x^{\mathrm{T}}(t)PBKx(t)\\
&\quad + 2x^{\mathrm{T}}(t)P\widetilde{D}x(t-\tau) + 2x^{\mathrm{T}}(t)P\widetilde{W}g(x(t-\tau))\\
&\quad + \rho[x^{\mathrm{T}}(t)P_1^{\mathrm{T}}P_1x(t) + x^{\mathrm{T}}(t-\tau)P_2^{\mathrm{T}}P_2x(t-\tau)]\\
&\quad + x^{\mathrm{T}}(t)Sx(t) - x^{\mathrm{T}}(t-\tau)Sx(t-\tau)\\
&\quad + g^{\mathrm{T}}(x(t))Qg(x(t)) - g^{\mathrm{T}}(x(t-\tau))Qg(x(t-\tau))\\
&\quad + x^{\mathrm{T}}(t)H\Lambda Hx(t) - g^{\mathrm{T}}(x(t))\Lambda g(x(t))\\
&\quad + (\varepsilon_1^{-1}+\varepsilon_2^{-1}+\varepsilon_3^{-1})x^{\mathrm{T}}(t)P\begin{pmatrix} MM^{\mathrm{T}} & 0\\ 0 & MM^{\mathrm{T}}\end{pmatrix}Px(t)\\
&\quad + \varepsilon_1 x^{\mathrm{T}}(t)\begin{pmatrix} N_A^{\mathrm{T}}N_A & 0\\ 0 & N_C^{\mathrm{T}}N_C\end{pmatrix}x(t)\\
&\quad + \varepsilon_2 x^{\mathrm{T}}(t-\tau)\begin{pmatrix} N_D^{\mathrm{T}}N_D & 0\\ 0 & 0\end{pmatrix}x(t-\tau)\\
&\quad + \varepsilon_3 g^{\mathrm{T}}(x(t-\tau))\begin{pmatrix} 0 & 0\\ 0 & N_W^{\mathrm{T}}N_W\end{pmatrix}g(x(t-\tau))\\
&= \xi^{\mathrm{T}}(t)\overline{\Theta}\xi(t)
\end{aligned}\tag{30}$$

where (see (31))

$$\begin{aligned}
\overline{\Theta}_{11} &= -2P\widetilde{A} + 2PBK + \rho P_1^{\mathrm{T}}P_1 + S + \varepsilon_1\begin{pmatrix} N_A^{\mathrm{T}}N_A & 0\\ 0 & N_C^{\mathrm{T}}N_C\end{pmatrix}\\
&\quad + H\Lambda H + (\varepsilon_1^{-1}+\varepsilon_2^{-1}+\varepsilon_3^{-1})P\begin{pmatrix} MM^{\mathrm{T}} & 0\\ 0 & MM^{\mathrm{T}}\end{pmatrix}P,
\end{aligned}$$

and $\xi(t) = \left(x^{\mathrm{T}}(t)\ \ x^{\mathrm{T}}(t-\tau)\ \ g^{\mathrm{T}}(x(t))\ \ g^{\mathrm{T}}(x(t-\tau))\right)^{\mathrm{T}}$

Pre- and post-multiplying $\overline{\Theta}$ in (31) by $\mathrm{diag}(P^{-1}, P^{-1}, \Lambda^{-1}, \Lambda^{-1})$ and $\mathrm{diag}(P^{-1}, P^{-1}, \Lambda^{-1}, \Lambda^{-1})$, respectively, and we obtain (see (32))

$$\overline{\Theta} = \begin{pmatrix} \overline{\Theta}_{11} & P\widetilde{D} & 0 & P\widetilde{W}\\ * & \rho P_2^{\mathrm{T}}P_2 + \varepsilon_2\begin{pmatrix} N_D^{\mathrm{T}}N_D & 0\\ 0 & 0\end{pmatrix} - S & 0 & 0\\ * & * & Q-\Lambda & 0\\ * & * & * & \varepsilon_3\begin{pmatrix} 0 & 0\\ 0 & N_W^{\mathrm{T}}N_W\end{pmatrix} - Q\end{pmatrix}\tag{31}$$

$$\sum = \begin{pmatrix} \sum_{11} & \widetilde{D}P^{-1} & 0 & \widetilde{W}\Lambda^{-1}\\ * & \sum_{22} & 0 & 0\\ * & * & \Lambda^{-1}Q\Lambda^{-1} - \Lambda^{-1} & 0\\ * & * & * & \varepsilon_3\Lambda^{-1}\begin{pmatrix} 0 & 0\\ 0 & N_W^{\mathrm{T}}N_W\end{pmatrix}\Lambda^{-1} - \Lambda^{-1}Q\Lambda^{-1}\end{pmatrix}\tag{32}$$

where

$$\sum_{11} = -2(\tilde{A} - BK)P^{-1} + \rho P^{-1}P_1^{\mathrm{T}}P_1P^{-1} + P^{-1}SP^{-1}$$

$$+ \varepsilon_1 P^{-1}\begin{pmatrix} N_A^{\mathrm{T}}N_A & 0 \\ 0 & N_C^{\mathrm{T}}N_C \end{pmatrix}P^{-1} + P^{-1}H\Lambda HP^{-1}$$

$$+ (\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_3^{-1})\begin{pmatrix} MM^{\mathrm{T}} & 0 \\ 0 & MM^{\mathrm{T}} \end{pmatrix},$$

$$\sum_{22} = \rho P^{-1}P_2^{\mathrm{T}}P_2P^{-1} + \varepsilon_2 P^{-1}\begin{pmatrix} N_D^{\mathrm{T}}N_D & 0 \\ 0 & 0 \end{pmatrix}P^{-1} - P^{-1}SP^{-1}$$

Then $\sum < 0$ in (32) is equivalent to

$$\overline{\sum} = \begin{pmatrix} \overline{\sum}_{11} & \tilde{D}\overline{P} & 0 & \tilde{W}\overline{\Lambda} \\ * & \overline{\sum}_{22} & 0 & 0 \\ * & * & \overline{Q} - \overline{\Lambda} & 0 \\ * & * & * & \overline{\varepsilon}_3^{-1}\overline{\Lambda}\begin{pmatrix} 0 & 0 \\ 0 & N_W^{\mathrm{T}}N_W \end{pmatrix}\overline{\Lambda} - \overline{Q} \end{pmatrix} < 0 \tag{33}$$

where

$$\overline{\sum}_{11} = -2\tilde{A}\overline{P} + 2B\overline{K} + \overline{\rho}^{-1}\overline{P}P_1^{\mathrm{T}}P_1\overline{P} + \overline{S}$$

$$+ \overline{\varepsilon}_1^{-1}\overline{P}\begin{pmatrix} N_A^{\mathrm{T}}N_A & 0 \\ 0 & N_C^{\mathrm{T}}N_C \end{pmatrix}\overline{P} + \overline{P}H\overline{\Lambda}^{-1}H\overline{P}$$

$$+ (\overline{\varepsilon}_1 + \overline{\varepsilon}_2 + \overline{\varepsilon}_3)\begin{pmatrix} MM^{\mathrm{T}} & 0 \\ 0 & MM^{\mathrm{T}} \end{pmatrix},$$

$$\overline{\sum}_{22} = \overline{\rho}^{-1}\overline{P}P_2^{\mathrm{T}}P_2\overline{P} + \overline{\varepsilon}_2^{-1}\overline{P}\begin{pmatrix} N_D^{T}N_D & 0 \\ 0 & 0 \end{pmatrix}\overline{P} - \overline{S}$$

From Lemma 3, it is easy to see that $\overline{\sum} < 0$ in (33) is equivalent to $\theta < 0$ in (19). From (23) and Itô formula, it is obvious to see that

$$\mathbf{E}V(t) - \mathbf{E}V(t_0) = \mathbf{E}\int_{t_0}^{t} \mathcal{L}V(s)\,\mathrm{d}s$$

From the definition of $V(t)$ in (22), there exists a positive constant λ_1 such that

$$\lambda_1\mathbf{E}\|x(t)\|^2 \le \mathbf{E}V(t) \le \mathbf{E}V(t_0) + \mathbf{E}\int_{t_0}^{t} \mathcal{L}V(s)\,\mathrm{d}s$$

$$\le \mathbf{E}V(t_0) + \lambda_{\max}(\Theta)\mathbf{E}\int_{t_0}^{t} \|x(s)\|^2\,\mathrm{d}s \tag{34}$$

where $\lambda_{\max}(\Theta)$ is the maximal eigenvalue of θ and it is negative.

Therefore from (34) and the discussion in [36], we know that the equilibrium of (10) is mean-square robustly asymptotically stabilisable. This completes the proof. □

Corollary 1: Under the Assumptions A$_1$–A$_3$, system (10) with $u(t) = 0$ and $v(t) = 0$ is mean-square robustly asymptotically stable if there are positive-definite diagonal matrix $\overline{\Lambda} = \mathrm{diag}(\overline{\Lambda}_1, \overline{\Lambda}_2, \ldots, \overline{\Lambda}_{2n}) > 0$, positive definite matrices $\overline{P} = (\overline{p}_{ij})_{2n\times 2n}$, $\overline{S} = (\overline{s}_{ij})_{2n\times 2n}$, $\overline{Q} = (q_{ij})_{2n\times 2n}$, and positive constants $\overline{\varepsilon}_1$, $\overline{\varepsilon}_2$, $\overline{\varepsilon}_3$, $\overline{\rho}$, such that (see equation at the bottom of the page)

where

$$\Upsilon_{11} = -2\tilde{A}\overline{P} + \overline{S} + (\overline{\varepsilon}_1 + \overline{\varepsilon}_2 + \overline{\varepsilon}_3)\begin{pmatrix} MM^{\mathrm{T}} & 0 \\ 0 & MM^{\mathrm{T}} \end{pmatrix},$$

$$\tilde{N}_A = \begin{pmatrix} N_A & 0 \\ 0 & N_C \end{pmatrix}^{\mathrm{T}}, \quad \tilde{N}_D = \begin{pmatrix} 0 & 0 \\ N_D & 0 \end{pmatrix}^{\mathrm{T}},$$

$$\tilde{N}_W = \begin{pmatrix} 0 & N_W \\ 0 & 0 \end{pmatrix}^{\mathrm{T}}$$

and I is the identity matrix.

Proof: Let $u(t) = Kx(t) = 0$ in Theorem 1, we can get the Corollary 1 directly. □

Remark 1: In [12, 14], the stability of stochastic GRN was investigated by the authors. However, the uncertainties of the system parameters are not involved. And in [13], the authors studied robust stability of deterministic GRN without concerning the stochastic disturbance. Therefore in this paper we consider the mean-square robust stability of GRN with stochastic disturbance and parameter uncertainties, and also design controllers to stabilise its mean-square stability. The

$$\Upsilon = \begin{pmatrix} \Upsilon_{11} & \tilde{D}\overline{P} & 0 & \tilde{W}\overline{\Lambda} & \overline{P}P_1^{\mathrm{T}} & 0 & \overline{P}H & \overline{P}\tilde{N}_A & 0 & 0 \\ * & -\overline{S} & 0 & 0 & 0 & \overline{P}P_2^{\mathrm{T}} & 0 & 0 & \overline{P}\tilde{N}_D & 0 \\ * & * & \overline{Q} - \overline{\Lambda} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\overline{Q} & 0 & 0 & 0 & 0 & 0 & \overline{\Lambda}\tilde{N}_W \\ * & * & * & * & -\overline{\rho}I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\overline{\rho}I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\overline{\Lambda} & 0 & 0 & \\ * & * & * & * & * & * & * & -\overline{\varepsilon}_1 I & 0 & 0 \\ * & * & * & * & * & * & * & * & -\overline{\varepsilon}_2 I & \\ * & * & * & * & * & * & * & * & * & -\overline{\varepsilon}_3 I \end{pmatrix} < 0 \quad \overline{P} \ge \overline{\rho}I$$

proposed results in this paper can be extended to study the GRNs with multiple time delays. As far as we know, there are not many works that study the control of this delayed GRN. From the LMI conditions in Theorem 1 and Corollary 1, the size of matrices in LMIs becomes large if many variables of the network are involved, which is quite common since the network system becomes more complex with more variables. The Matlab LMI Toolbox implements state-of-the-art interior-point LMI solvers. Although these solvers are significantly faster than classical convex optimisation algorithms, it should be kept in mind that the complexity of LMI computations remains higher than that of solving, say, a Riccati equation. For instance, problems with a thousand designed variables typically take over an hour on today's workstations. However, research on LMI optimisation is a very active area in the applied math, optimisation and the operations research communities, and substantial speed-ups for high-dimensional LMIs can be expected in the future.

4 Robust H_∞ control

In this section, a sufficient condition for the solvability of robust H_∞ control problem is proposed to ensure mean-square robust asymptotical stability of GRN (10).

Theorem 2: Under the Assumptions A_1–A_4, $u = Kx(t)$ is a robust H_∞ controller in system (10) if there are positive-definite diagonal matrix $\overline{\Lambda} = \mathrm{diag}(\overline{\Lambda}_1, \overline{\Lambda}_2, \ldots, \overline{\Lambda}_{2n}) > 0$, positive-definite matrices $\overline{P} = (\overline{p}_{ij})_{2n\times 2n}$, $\overline{S} = (\overline{s}_{ij})_{2n\times 2n}$, $\overline{Q} = (q_{ij})_{2n\times 2n}$, positive constants $\overline{\varepsilon}_1$, $\overline{\varepsilon}_2$, $\overline{\varepsilon}_3$, $\overline{\rho}$, $\overline{\gamma}$, and matrix $\overline{K}$, such that (see (35))

$$\overline{P} \geq \overline{\rho} I \geq \overline{\gamma} I \tag{36}$$

where

$$\Omega_{11} = -2\widetilde{A}\overline{P} + 2B\overline{K} + \overline{S} + (\overline{\varepsilon}_1 + \overline{\varepsilon}_2 + \overline{\varepsilon}_3)\begin{pmatrix} MM^{\mathrm{T}} & 0 \\ 0 & MM^{\mathrm{T}} \end{pmatrix} + (C_0\overline{P} + D_0\overline{K})^{\mathrm{T}}(C_0\overline{P} + D_0\overline{K}),$$

$$\widetilde{N}_A = \begin{pmatrix} N_A & 0 \\ 0 & N_C \end{pmatrix}^{\mathrm{T}}, \quad \widetilde{N}_D = \begin{pmatrix} 0 & 0 \\ N_D & 0 \end{pmatrix}^{\mathrm{T}},$$

$$\widetilde{N}_W = \begin{pmatrix} 0 & N_W \\ 0 & 0 \end{pmatrix}^{\mathrm{T}} \overline{\gamma} = \gamma^{-2}$$

and I is the identity matrix. In this case, the appropriate state feedback controller can be chosen by

$$u(t) = Kx(t), \quad K = \overline{KP}^{-1} \tag{37}$$

Proof: By the feedback controller $u(t) = Kx(t)$, system (10) can be written as

$$\begin{aligned} \mathrm{d}x(t) &= [-(\widetilde{A}_k + \Delta\widetilde{A})x(t) + (\widetilde{D} + \Delta\widetilde{D})x(t-\tau) \\ &\quad + (\widetilde{W} + \Delta\widetilde{W})g(x(t-\tau))]\,\mathrm{d}t \\ &\quad + \sigma(x(t), x(t-\tau))\,\mathrm{d}\omega + v(t)\,\mathrm{d}\nu \\ y(t) &= C_k x(t) \end{aligned} \tag{38}$$

where $\widetilde{A}_k = \widetilde{A} - BK$, and $C_k = C_0 + D_0K$.

Let $\overline{P} = P^{-1}$, $\overline{\rho} = \rho^{-1}$, $\overline{K} = KP^{-1}$, $\overline{\varepsilon}_1 = \varepsilon_1^{-1}$, $\overline{\varepsilon}_2 = \varepsilon_2^{-1}$, $\overline{\varepsilon}_3 = \varepsilon_3^{-1}$, $\overline{\Lambda} = \Lambda^{-1}$, $\overline{S} = P^{-1}SP^{-1}$, $\overline{Q} = \Lambda^{-1}Q\Lambda^{-1}$.

Next, we shall design a robust H_∞ controller to show that system (38) satisfies

$$\mathbf{E}\int_0^\infty \|y(t)\|^2 \leq \gamma^2 \int_0^\infty \|v(t)\|^2 \tag{39}$$

for all non-zero $v \in L_2[0, \infty)$ and under zero initial conditions, that is, we assume the initial condition of system (38) is $\phi(t) = 0$ for $t \in [-\tau, 0]$. By Itô formula, it is obvious to see that

$$\mathbf{E}V(t) = \mathbf{E}\int_0^t \mathcal{L}V(s)\,\mathrm{d}s \tag{40}$$

where the Lyapunov functional candidate $V(t)$ is given in (22), and similarly as in (24), we obtain

$$\begin{aligned} \mathcal{L}V(t) = 2x^{\mathrm{T}}(t)P[&-(\widetilde{A}_k + \Delta\widetilde{A})x(t) \\ &+ (\widetilde{D} + \Delta\widetilde{D})x(t-\tau) + (\widetilde{W} + \Delta\widetilde{W})g(x(t-\tau))] \\ &+ \sigma^{\mathrm{T}}(x(t), x(t-\tau(t)))P\sigma(x(t), x(t-\tau(t))) \\ &+ x^{\mathrm{T}}(t)Sx(t) - x^{\mathrm{T}}(t-\tau)Sx(t-\tau) \\ &+ g^{\mathrm{T}}(x(t))Qg(x(t)) - g^{\mathrm{T}}(x(t-\tau))Qg(x(t-\tau)) \\ &+ v^{\mathrm{T}}(t)Pv(t) \end{aligned} \tag{41}$$

$$\Omega = \begin{pmatrix} \Omega_{11} & \widetilde{D}\overline{P} & 0 & \widetilde{W}\overline{\Lambda} & \overline{P}P_1^{\mathrm{T}} & 0 & \overline{P}H & \overline{P}\widetilde{N}_A & 0 & 0 & (C_0\overline{P} + D_0\overline{K})^{\mathrm{T}} \\ * & -\overline{S} & 0 & 0 & 0 & \overline{P}P_2^{\mathrm{T}} & 0 & 0 & \overline{P}\widetilde{N}_D & 0 & 0 \\ * & * & \overline{Q} - \overline{\Lambda} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\overline{Q} & 0 & 0 & 0 & 0 & 0 & \overline{\Lambda}\widetilde{N}_W & 0 \\ * & * & * & * & -\overline{\rho}I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\overline{\rho}I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\overline{\Lambda} & 0 & 0 & 0 & \\ * & * & * & * & * & * & * & -\overline{\varepsilon}_1 I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -\overline{\varepsilon}_2 I & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -\overline{\varepsilon}_3 I & 0 \\ * & * & * & * & * & * & * & * & * & * & -I \end{pmatrix} < 0 \tag{35}$$

For $\gamma > 0$, set

$$J(t) = \mathbf{E}\int_0^t [y^{\mathrm{T}}(s)y(s) - \gamma^2 v^{\mathrm{T}}(s)v(s)]\,\mathrm{d}s \tag{42}$$

Then from (30), (40) and (41), we have

$$\begin{aligned} J(t) &\le \mathbf{E}\int_0^t [y^{\mathrm{T}}(s)y(s) - \gamma^2 v^{\mathrm{T}}(s)v(s) + \mathcal{L}V(s)]\,\mathrm{d}s \\ &\le \mathbf{E}\int_0^t [y^{\mathrm{T}}(s)y(s) - \gamma^2 v^{\mathrm{T}}(s)v(s) + \xi^{\mathrm{T}}(s)\overline{\Theta}\xi(s) \\ &\quad + \rho v^{\mathrm{T}}(s)v(s)]\,\mathrm{d}s \end{aligned}$$

By (36), we have

$$\begin{aligned} J(t) &\le \mathbf{E}\int_0^t [y^{\mathrm{T}}(s)y(s) + \xi^{\mathrm{T}}(s)\overline{\Theta}\xi(s)]\,\mathrm{d}s \\ &\le \mathbf{E}\int_0^t [\xi^{\mathrm{T}}(s)\widehat{\Theta}\xi(s)]\,\mathrm{d}s \end{aligned} \tag{43}$$

where (see (44))

$$\begin{aligned} \widehat{\Theta}_{11} &= -2P(\widetilde{A} - BK) + \rho P_1^{\mathrm{T}}P_1 + S + \varepsilon_1\begin{pmatrix} N_A^{\mathrm{T}}N_A & 0 \\ 0 & N_C^{\mathrm{T}}N_C \end{pmatrix} \\ &\quad + H\Lambda H + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_3^{-1})P\begin{pmatrix} MM^{\mathrm{T}} & 0 \\ 0 & MM^{\mathrm{T}} \end{pmatrix}P \\ &\quad + (C_0 + D_0K)^T(C_0 + D_0K), \text{ and} \end{aligned}$$

$$\xi(t) = \begin{pmatrix} x^{\mathrm{T}}(t) & x^{\mathrm{T}}(t-\tau) & g^{\mathrm{T}}(x(t)) & g^{\mathrm{T}}(x(t-\tau)) \end{pmatrix}^{\mathrm{T}}$$

By (35), (36) and (44), it is easy to see that (19) and (20) hold if $v(t) = 0$. Therefore the system (38) is mean-square robustly asymptotically stabilisable. Pre- and post-multiplying $\widehat{\Theta}$ in (44) by $\mathrm{diag}(P^{-1}, P^{-1}, \Lambda^{-1}, \Lambda^{-1})$ and $\mathrm{diag}(P^{-1}, P^{-1}, \Lambda^{-1}, \Lambda^{-1})$, respectively, and we obtain (see (45))

where

$$\begin{aligned} \Omega_{11}O &= -2(\widetilde{A} - BK)P^{-1} + \rho P^{-1}P_1^{\mathrm{T}}P_1P^{-1} + P^{-1}SP^{-1} \\ &\quad + \varepsilon_1 P^{-1}\begin{pmatrix} N_A^{\mathrm{T}}N_A & 0 \\ 0 & N_C^{\mathrm{T}}N_C \end{pmatrix}P^{-1} + P^{-1}H\Lambda HP^{-1} \\ &\quad + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_3^{-1})\begin{pmatrix} MM^{\mathrm{T}} & 0 \\ 0 & MM^{\mathrm{T}} \end{pmatrix} \\ &\quad + P^{-1}(C_0 + D_0K)^{\mathrm{T}}(C_0 + D_0K)P^{-1}, \end{aligned}$$

$$\Omega_{22} = \rho P^{-1}P_2^{\mathrm{T}}P_2P^{-1} + \varepsilon_2 P^{-1}\begin{pmatrix} N_D^{\mathrm{T}}N_D & 0 \\ 0 & 0 \end{pmatrix}P^{-1} - P^{-1}SP^{-1}$$

Then $\overline{\Omega} < 0$ in (45) is equivalent to

$$\overline{\Omega} = \begin{pmatrix} \overline{\Omega}_{11} & \widetilde{D}\overline{P} & 0 & \widetilde{W}\overline{\Lambda} \\ * & \Omega_{22} & 0 & 0 \\ * & * & \overline{Q} - \overline{\Lambda} & 0 \\ * & * & * & \overline{\varepsilon}_3^{-1}\overline{\Lambda}\begin{pmatrix} 0 & 0 \\ 0 & N_W^{\mathrm{T}}N_W \end{pmatrix}\overline{\Lambda} - \overline{Q} \end{pmatrix} < 0 \tag{46}$$

where

$$\begin{aligned} \overline{\Omega}_{11} &= -2\widetilde{A}\overline{P} + 2B\overline{K} + \overline{\rho}^{-1}\overline{P}P_1^TP_1\overline{P} + \overline{S} \\ &\quad + \overline{\varepsilon}_1^{-1}\overline{P}\begin{pmatrix} N_A^{\mathrm{T}}N_A & 0 \\ 0 & N_C^{\mathrm{T}}N_C \end{pmatrix}\overline{P} + \overline{P}H\overline{\Lambda}^{-1}H\overline{P} \\ &\quad + (\overline{\varepsilon}_1 + \overline{\varepsilon}_2 + \overline{\varepsilon}_3)\begin{pmatrix} MM^{\mathrm{T}} & 0 \\ 0 & MM^{\mathrm{T}} \end{pmatrix} \\ &\quad + (C_0\overline{P} + D_0\overline{K})^{\mathrm{T}}(C_0\overline{P} + D_0\overline{K}), \end{aligned}$$

$$\Omega_{22} = \overline{\rho}^{-1}\overline{P}P_2^{\mathrm{T}}P_2\overline{P} + \overline{\varepsilon}_2^{-1}\overline{P}\begin{pmatrix} N_D^{\mathrm{T}}N_D & 0 \\ 0 & 0 \end{pmatrix}\overline{P} - \overline{S}$$

From Lemma 3, it is easy to see that $\overline{\Omega} < 0$ is equivalent to $\Omega < 0$ in (35). From (46) and Itô formula, it is obvious to see that

$$J(t) < 0$$

It follows that

$$J(t) = \mathbf{E}\int_0^t [y^{\mathrm{T}}(s)y(s)]\,\mathrm{d}s \le \gamma^2\int_0^t [v^{\mathrm{T}}(s)v(s)]\,\mathrm{d}s$$

Therefore the condition (ii) in Definition 3 is satisfied. $u(t)$ in (37) is a robust H_∞ controller. This completes the proof. □

$$\widehat{\Theta} = \begin{pmatrix} \widehat{\Theta}_{11} & P\widetilde{D} & 0 & P\widetilde{W} \\ * & \rho P_2^{\mathrm{T}}P_2 + \varepsilon_2\begin{pmatrix} N_D^{\mathrm{T}}N_D & 0 \\ 0 & 0 \end{pmatrix} - S & 0 & 0 \\ * & * & Q - \Lambda & 0 \\ * & * & * & \varepsilon_3\begin{pmatrix} 0 & 0 \\ 0 & N_W^{\mathrm{T}}N_W \end{pmatrix} - Q \end{pmatrix} \tag{44}$$

$$\overline{\Omega} = \begin{pmatrix} \Omega_{11} & \widetilde{D}P^{-1} & 0 & \widetilde{W}\Lambda^{-1} \\ * & \Omega_{22} & 0 & 0 \\ * & * & \Lambda^{-1}Q\Lambda^{-1} - \Lambda^{-1} & 0 \\ * & * & * & \varepsilon_3\Lambda^{-1}\begin{pmatrix} 0 & 0 \\ 0 & N_W^{\mathrm{T}}N_W \end{pmatrix}\Lambda^{-1} - \Lambda^{-1}Q\Lambda^{-1} \end{pmatrix} \tag{45}$$

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Sometimes, we are interested in finding out the minimal value γ where the conditions in (35) and (36) are satisfied. In the following, a corollary is proposed to find the minimal γ:

Corollary 2: Under the Assumptions A_1–A_4, $u = Kx(t)$ is a robust H_∞ controller in system (10) if there are positive-definite diagonal matrix $\overline{\Lambda} = \mathrm{diag}(\overline{\Lambda}_1, \overline{\Lambda}_2, \ldots, \overline{\Lambda}_{2n}) > 0$, positive-definite matrices $\overline{P} = (\overline{p}_{ij})_{2n\times 2n}$, $\overline{S} = (\overline{s}_{ij})_{2n\times 2n}$, $\overline{Q} = (q_{ij})_{2n\times 2n}$, positive constants $\overline{\varepsilon}_1$, $\overline{\varepsilon}_2$, $\overline{\varepsilon}_3$, $\overline{\rho}$, $\overline{\gamma}$ and matrix $\overline{K}$, the following optimisation problem (see (47))

$$\overline{P} \geq \overline{\rho} I \geq \overline{\gamma} I \tag{48}$$

has a solution, where

$$\Omega_{11} = -2\widetilde{A}\overline{P} + 2B\overline{K} + \overline{S} + (\overline{\varepsilon}_1 + \overline{\varepsilon}_2 + \overline{\varepsilon}_3)\begin{pmatrix} MM^{\mathrm{T}} & 0 \\ 0 & MM^{\mathrm{T}} \end{pmatrix}$$
$$+ (C_0\overline{P} + D_0\overline{K})^{\mathrm{T}}(C_0\overline{P} + D_0\overline{K}), \quad \widetilde{N}_A = \begin{pmatrix} N_A & 0 \\ 0 & N_C \end{pmatrix}^{\mathrm{T}},$$

$$\widetilde{N}_D = \begin{pmatrix} 0 & 0 \\ N_D & 0 \end{pmatrix}^{\mathrm{T}}, \quad \widetilde{N}_W = \begin{pmatrix} 0 & N_W \\ 0 & 0 \end{pmatrix}^{\mathrm{T}} \overline{\gamma} = \gamma^{-2}$$

and I is the identity matrix. In this case, the appropriate state feedback controller can be chosen by

$$u(t) = Kx(t), K = \overline{KP}^{-1} \tag{49}$$

However, the minimal γ may lead to high feedback gain matrix K. Therefore we can fix the feedback gain matrix K. For example, we can use the designed controller used in Theorem 1, and then find out the minimal γ.

Corollary 3: Under the Assumptions A_1–A_4, $u = Kx(t)$ is a robust H_∞ controller in system (10) if there are positive-definite diagonal matrix $\overline{\Lambda} = \mathrm{diag}(\overline{\Lambda}_1, \overline{\Lambda}_2, \ldots, \overline{\Lambda}_{2n}) > 0$, positive-definite matrices $\overline{P} = (\overline{p}_{ij})_{2n\times 2n}$, $\overline{S} = (\overline{s}_{ij})_{2n\times 2n}$, $\overline{Q} = (q_{ij})_{2n\times 2n}$ and positive constants $\overline{\varepsilon}_1$, $\overline{\varepsilon}_2$, $\overline{\varepsilon}_3$, $\overline{\rho}$, $\overline{\gamma}$, such that the following optimisation problem (see (50))

$$\overline{P} \geq \overline{\rho} I \geq \overline{\gamma} I \tag{51}$$

---

maximise $\overline{\gamma}$

subject to the following LMI:

$$\Omega = \begin{pmatrix}
\Omega_{11} & \widetilde{D}\overline{P} & 0 & \widetilde{W}\overline{\Lambda} & \overline{P}P_1^{\mathrm{T}} & 0 & \overline{P}H & \overline{P}\widetilde{N}_A & 0 & 0 & (C_0\overline{P} + D_0\overline{K})^{\mathrm{T}} \\
* & -\overline{S} & 0 & 0 & 0 & \overline{P}P_2^{\mathrm{T}} & 0 & 0 & \overline{P}\widetilde{N}_D & 0 & 0 \\
* & * & \overline{Q} - \overline{\Lambda} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\overline{Q} & 0 & 0 & 0 & 0 & 0 & \overline{\Lambda}\widetilde{N}_W & 0 \\
* & * & * & * & -\overline{\rho}I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\overline{\rho}I & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\overline{\Lambda} & 0 & 0 & 0 & \\
* & * & * & * & * & * & * & -\overline{\varepsilon}_1 I & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\overline{\varepsilon}_2 I & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -\overline{\varepsilon}_3 I & 0 \\
* & * & * & * & * & * & * & * & * & * & -I
\end{pmatrix} < 0 \tag{47}$$

---

maximise $\overline{\gamma}$

subject to the following LMI:

$$\Omega = \begin{pmatrix}
\Omega_{11} & \widetilde{D}\overline{P} & 0 & \widetilde{W}\overline{\Lambda} & \overline{P}P_1^{\mathrm{T}} & 0 & \overline{P}H & \overline{P}\widetilde{N}_A & 0 & 0 & (C_0\overline{P} + D_0K\overline{P})^{\mathrm{T}} \\
* & -\overline{S} & 0 & 0 & 0 & \overline{P}P_2^{\mathrm{T}} & 0 & 0 & \overline{P}\widetilde{N}_D & 0 & 0 \\
* & * & \overline{Q} - \overline{\Lambda} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\overline{Q} & 0 & 0 & 0 & 0 & 0 & \overline{\Lambda}\widetilde{N}_W & 0 \\
* & * & * & * & -\overline{\rho}I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\overline{\rho}I & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\overline{\Lambda} & 0 & 0 & 0 & \\
* & * & * & * & * & * & * & -\overline{\varepsilon}_1 I & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\overline{\varepsilon}_2 I & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -\overline{\varepsilon}_3 I & 0 \\
* & * & * & * & * & * & * & * & * & * & -I
\end{pmatrix} < 0 \tag{50}$$

has a solution, where

$$\Omega_{11} = -2\widetilde{A}\overline{P} + 2BK\overline{P} + \overline{S} + (\overline{\varepsilon}_1 + \overline{\varepsilon}_2 + \overline{\varepsilon}_3)\begin{pmatrix} MM^{\mathrm{T}} & 0 \\ 0 & MM^{\mathrm{T}} \end{pmatrix}$$
$$+ (C_0\overline{P} + D_0K\overline{P})^{\mathrm{T}}(C_0\overline{P} + D_0K\overline{P}),$$
$$\widetilde{N}_A = \begin{pmatrix} N_A & 0 \\ 0 & N_C \end{pmatrix}^{\mathrm{T}}, \quad \widetilde{N}_D = \begin{pmatrix} 0 & 0 \\ N_D & 0 \end{pmatrix}^{\mathrm{T}},$$
$$\widetilde{N}_W = \begin{pmatrix} 0 & N_W \\ 0 & 0 \end{pmatrix}^{\mathrm{T}}, \quad \overline{\gamma} = \gamma^{-2}$$

and I is the identity matrix.

5 Robust uniformly bounded control

In this section, a sufficient condition for the solvability of robust uniformly bounded control problem is proposed to ensure mean-square robust asymptotical stability of GRN (10). In some real system, the external noise always exists and will not vanish as the time goes, that is, the energy of the external noise intensity function $v(t)$ is infinite. Therefore Assumption A$_4$ is not satisfied. For example, $v(t) = 0.001\overline{I}$, where $\overline{I} \in R^{2n}$ is a vector and each entry in $\overline{I}$ is 1. Note that external noise intensity function $v(t)$ is bounded. Based on Assumption A$_5$, we have the following theorem:

Theorem 3: Under the Assumptions A$_1$–A$_3$ and A$_5$, $u = Kx(t)$ is a robust uniformly bounded controller in system (10) if there are positive-definite diagonal matrix $\overline{\Lambda} = \mathrm{diag}(\overline{\Lambda}_1, \overline{\Lambda}_2, \ldots, \overline{\Lambda}_{2n}) > 0$, positive-definite matrices $\overline{P} = (\overline{p}_{ij})_{2n\times 2n}$, $\overline{\Xi}_1 = (\overline{\Xi}_{1ij})_{2n\times 2n}$, $\overline{\Xi}_2 = (\overline{\Xi}_{2ij})_{2n\times 2n}$, positive constants $\overline{\varepsilon}_1$, $\overline{\varepsilon}_2$, $\overline{\varepsilon}_3$, $\overline{\rho}$, k_1, k_2 and matrix $\overline{K}$, such that

$$\widehat{J}_1 = \begin{pmatrix} \widehat{J}_{11} & \overline{P}P_1^{\mathrm{T}} & \overline{P}\widetilde{N}_A \\ * & -\overline{\rho}I & 0 \\ * & * & -\overline{\varepsilon}_1 I \end{pmatrix} < 0 \tag{52}$$

$$\widehat{J}_2 = \begin{pmatrix} -k_2\overline{P} & \overline{P}P_2^{\mathrm{T}} & \overline{P}\widetilde{N}_D & \overline{P}\widetilde{D}^{\mathrm{T}} & \overline{P}H \\ * & -\overline{\rho}I & 0 & 0 & 0 \\ * & * & -\overline{\varepsilon}_2 I & 0 & 0 \\ * & * & * & -\overline{\Xi}_1 & 0 \\ * & * & * & * & -\overline{\Lambda} \end{pmatrix} < 0 \tag{53}$$

$$\widehat{J}_3 = \begin{pmatrix} -\overline{\Lambda} & \overline{\Lambda}\widetilde{W}^{\mathrm{T}} & \overline{\Lambda}\widetilde{N}_W \\ * & -\overline{\Xi}_2 & 0 \\ * & * & -\overline{\varepsilon}_3 \end{pmatrix} < 0 \tag{54}$$

$$k_1 > k_2 \tag{55}$$

$$\overline{P} \geq \overline{\rho} I \tag{56}$$

where

$$\widehat{J}_{11} = -2(\widetilde{A}\overline{P} - B\overline{K}) + (\overline{\varepsilon}_1 + \overline{\varepsilon}_2 + \overline{\varepsilon}_3)\begin{pmatrix} MM^{\mathrm{T}} & 0 \\ 0 & MM^{\mathrm{T}} \end{pmatrix}$$
$$+ \overline{\Xi}_1 + \overline{\Xi}_2 + k_1\overline{P},$$
$$\widetilde{N}_A = \begin{pmatrix} N_A & 0 \\ 0 & N_C \end{pmatrix}^{\mathrm{T}}, \quad \widetilde{N}_D = \begin{pmatrix} 0 & 0 \\ N_D & 0 \end{pmatrix}^{\mathrm{T}},$$
$$\widetilde{N}_W = \begin{pmatrix} 0 & N_W \\ 0 & 0 \end{pmatrix}^{\mathrm{T}}$$

and I is the identity matrix. In this case, the appropriate state feedback controller can be chosen by

$$u(t) = Kx(t), \quad K = \overline{K}\overline{P}^{-1} \tag{57}$$

Then the output converges exponentially to a small region $\mathcal{D}$ in the mean-square, where

$$\mathcal{D} = \left\{ \mathbf{E}\|y(t)\|^2 \,\middle|\, \mathbf{E}\|y(t)\|^2 \leq \frac{\beta^2\lambda_M^2(C_0 + D_0K)\lambda_M(\overline{P})}{\overline{\rho}r} \right\} \tag{58}$$

in which $\lambda_M(C_0 + D_0K)$ and $\lambda_M(\overline{P})$ are the maximal eigenvalues of $C_0 + D_0K$ and $\overline{P}$, and r is the unique positive solution of $-r = -k_1 + k_2e^{r\tau}$.

Proof: Consider the Lyapunov candidate

$$V(t) = x^{\mathrm{T}}(t)Px(t) \tag{59}$$

where $P = \overline{P}^{-1}$ is a positive-definite matrix.

Similarly as the discussion in (30) and (45), we obtain

$$\begin{aligned} \mathcal{L}V(t) \leq {} & -2x^{\mathrm{T}}(t)P\widetilde{A}_k x(t) + 2x^{\mathrm{T}}(t)P\widetilde{D}x(t-\tau) \\ & + 2x^{\mathrm{T}}(t)P\widetilde{W}g(x(t-\tau)) + \rho[x^{\mathrm{T}}(t)P_1^{\mathrm{T}}P_1x(t) \\ & + x^{\mathrm{T}}(t-\tau)P_2^{\mathrm{T}}P_2x(t-\tau)] + x^{\mathrm{T}}(t-\tau)H\overline{\Lambda}^{-1}Hx(t-\tau) \\ & - g^{\mathrm{T}}(x(t-\tau))\overline{\Lambda}^{-1}g(x(t-\tau)) \\ & + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_3^{-1})x^{\mathrm{T}}(t)P\begin{pmatrix} MM^{\mathrm{T}} & 0 \\ 0 & MM^{\mathrm{T}} \end{pmatrix}Px(t) \\ & + \varepsilon_1 x^{\mathrm{T}}(t)\begin{pmatrix} N_A^{\mathrm{T}}N_A & 0 \\ 0 & N_C^{\mathrm{T}}N_C \end{pmatrix}x(t) \\ & + \varepsilon_2 x^{\mathrm{T}}(t-\tau)\begin{pmatrix} N_D^{\mathrm{T}}N_D & 0 \\ 0 & 0 \end{pmatrix}x(t-\tau) \\ & + \varepsilon_3 g^{\mathrm{T}}(x(t-\tau))\begin{pmatrix} 0 & 0 \\ 0 & N_W^{\mathrm{T}}N_W \end{pmatrix}g(x(t-\tau)) \\ & + \rho v^{\mathrm{T}}(t)v(t) \end{aligned} \tag{60}$$

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From Lemma 1, we have

$$\begin{aligned} 2x^{\mathrm{T}}(t)P\widetilde{D}x(t-\tau) &\le x^{\mathrm{T}}(t)P\Xi_1 Px(t) \\ &\quad + x^{\mathrm{T}}(t-\tau)\widetilde{D}^{\mathrm{T}}\Xi_1^{-1}\widetilde{D}x(t-\tau) \end{aligned} \tag{61}$$

and

$$\begin{aligned} 2x^{\mathrm{T}}(t)P\widetilde{W}g(x(t-\tau)) &\le x^{\mathrm{T}}(t)P\Xi_2 Px(t) \\ &\quad + g^{\mathrm{T}}(x(t-\tau))\widetilde{W}^{\mathrm{T}}\Xi_2^{-1}\widetilde{W}g(x(t-\tau)) \end{aligned} \tag{62}$$

where Ξ_1 and Ξ_2 are positive-definite matrices.

Let $\overline{\rho} = \rho^{-1}$, $\overline{K} = KP^{-1}$, $\overline{\varepsilon}_1 = \varepsilon_1^{-1}$, $\overline{\varepsilon}_2 = \varepsilon_2^{-1}$, $\overline{\varepsilon}_3 = \varepsilon_3^{-1}$. Combining (60)–(62) and by Assumption A_5, we obtain

$$\begin{aligned} \mathcal{L}V(t) &\le x^{\mathrm{T}}(t)[-2P(\widetilde{A}-BK) + \rho P_1^{\mathrm{T}}P_1 \\ &\quad + (\varepsilon_1^{-1}+\varepsilon_2^{-1}+\varepsilon_3^{-1})P\begin{pmatrix} MM^{\mathrm{T}} & 0 \\ 0 & MM^{\mathrm{T}} \end{pmatrix}P \\ &\quad + \varepsilon_1\begin{pmatrix} N_A^{\mathrm{T}}N_A & 0 \\ 0 & N_C^{\mathrm{T}}N_C \end{pmatrix} + P(\Xi_1+\Xi_2)P]x(t) \\ &\quad + x^{\mathrm{T}}(t-\tau)[\rho P_2^{\mathrm{T}}P_2 + \widetilde{D}^{\mathrm{T}}\Xi_1^{-1}\widetilde{D} \\ &\quad + \varepsilon_2\begin{pmatrix} N_D^{\mathrm{T}}N_D & 0 \\ 0 & 0 \end{pmatrix} + H\overline{\Lambda}^{-1}H]x(t-\tau) \\ &\quad + g^{\mathrm{T}}(x(t-\tau))[-\overline{\Lambda}^{-1} + \varepsilon_3\begin{pmatrix} 0 & 0 \\ 0 & N_W^{\mathrm{T}}N_W \end{pmatrix} \\ &\quad + \widetilde{W}^{\mathrm{T}}\Xi_2^{-1}\widetilde{W}]g(x(t-\tau)) + \rho v^{\mathrm{T}}(t)v(t) \\ &\le x^{\mathrm{T}}(t)(J_1-k_1P)x(t) + x^{\mathrm{T}}(t-\tau)(J_2+k_2P)x(t-\tau) \\ &\quad + g^{\mathrm{T}}(x(t-\tau))J_3g(x(t-\tau)) + \rho\beta^2 \end{aligned} \tag{63}$$

where

$$\begin{aligned} J_1 &= -2P(\widetilde{A}-BK) + \rho P_1^{\mathrm{T}}P_1 \\ &\quad + (\varepsilon_1^{-1}+\varepsilon_2^{-1}+\varepsilon_3^{-1})P\begin{pmatrix} MM^{\mathrm{T}} & 0 \\ 0 & MM^{\mathrm{T}} \end{pmatrix}P \\ &\quad + \varepsilon_1\begin{pmatrix} N_A^{\mathrm{T}}N_A & 0 \\ 0 & N_C^{\mathrm{T}}N_C \end{pmatrix} + P(\Xi_1+\Xi_2)P + k_1P \\ &= P[-2(\widetilde{A}\overline{P}-B\overline{K}) + \overline{\rho}^{-1}\overline{P}P_1^{\mathrm{T}}P_1\overline{P} + (\overline{\varepsilon}_1+\overline{\varepsilon}_2+\overline{\varepsilon}_3) \\ &\quad \times\begin{pmatrix} MM^{\mathrm{T}} & 0 \\ 0 & MM^{\mathrm{T}} \end{pmatrix} + \overline{\varepsilon}_1^{-1}\overline{P}\begin{pmatrix} N_A^{\mathrm{T}}N_A & 0 \\ 0 & N_C^{\mathrm{T}}N_C \end{pmatrix}\overline{P} \\ &\quad + \Xi_1 + \Xi_2 + k_1\overline{P}]P \\ &= P\overline{J}_1P \end{aligned} \tag{64}$$

$$\begin{aligned} J_2 &= \rho P_2^{\mathrm{T}}P_2 + \widetilde{D}^{\mathrm{T}}\Xi_1^{-1}\widetilde{D} + \varepsilon_2\begin{pmatrix} N_D^{\mathrm{T}}N_D & 0 \\ 0 & 0 \end{pmatrix} + H\overline{\Lambda}^{-1}H - k_2P \\ &= P[\overline{\rho}^{-1}\overline{P}P_2^{\mathrm{T}}P_2\overline{P} + \overline{P}\widetilde{D}^{\mathrm{T}}\Xi_1^{-1}\widetilde{D}\overline{P} \\ &\quad + \overline{\varepsilon}_2^{-1}\overline{P}\begin{pmatrix} N_D^{\mathrm{T}}N_D & 0 \\ 0 & 0 \end{pmatrix}\overline{P} + \overline{P}H\overline{\Lambda}^{-1}H\overline{P} - k_2\overline{P}]P \\ &= P\overline{J}_2P \end{aligned} \tag{65}$$

and

$$\begin{aligned} J_3 &= -\overline{\Lambda}^{-1} + \varepsilon_3\begin{pmatrix} 0 & 0 \\ 0 & N_W^{\mathrm{T}}N_W \end{pmatrix} + \widetilde{W}^{\mathrm{T}}\Xi_2^{-1}\widetilde{W} \\ &= \overline{\Lambda}^{-1}\left[-\overline{\Lambda} + \overline{\varepsilon}_3^{-1}\overline{\Lambda}\begin{pmatrix} 0 & 0 \\ 0 & N_W^{\mathrm{T}}N_W \end{pmatrix}\overline{\Lambda} + \overline{\Lambda}\widetilde{W}^{\mathrm{T}}\Xi_2^{-1}\widetilde{W}\overline{\Lambda}\right]\overline{\Lambda}^{-1} \\ &= \overline{\Lambda}^{-1}\overline{J}_3\overline{\Lambda}^{-1} \end{aligned} \tag{66}$$

From Lemma 3, it is easy to see that $\overline{J}_1 < 0$, $\overline{J}_2 < 0$ and $\overline{J}_3 < 0$ are equivalent to $\widehat{J}_1 < 0$, $\widehat{J}_2 < 0$ and $\widehat{J}_3 < 0$ in (52)–(54), respectively. Based on (63)–(66), we obtain

$$\mathcal{L}V(t) \le -k_1V(t) + k_2V(t-\tau) + \overline{\rho}^{-1}\beta^2 \tag{67}$$

From Itô formula, it is obvious to see that

$$\mathbf{E}\mathrm{d}V(t) = \mathbf{E}\mathcal{L}V(t) \le -k_1\mathbf{E}V(t) + k_2\mathbf{E}V(t-\tau) + \overline{\rho}^{-1}\beta^2$$

From Lemma 4 and (55), we obtain

$$\mathbf{E}V(t) < \|\mathbf{E}V(0)\|_{\tau}\mathrm{e}^{-r\tau} + \frac{\beta^2}{\overline{\rho}r}, \quad t \ge 0 \tag{68}$$

where $\|V(0)\|_{\tau} = \sup_{-\tau\le s\le 0}|V(s)|$ and r is the unique positive solution of $-r = -k_1 + k_2e^{r\tau}$.

From the definition of $V(t)$ in (59), we have

$$\lambda_m(P)\mathbf{E}\|x(t)\|^2 < \|\mathbf{E}V(0)\|_{\tau}\mathrm{e}^{-r\tau} + \frac{\beta^2}{\overline{\rho}r} \tag{69}$$

where $\lambda_m(P)$ is the minimal eigenvalue of P. Then

$$\begin{aligned} \lim_{t\to\infty}\mathbf{E}\|y(t)\|^2 &\le \lim_{t\to\infty}\mathbf{E}\|x(t)\|^2\|C_0+D_0K\|^2 \\ &\le \frac{\beta^2\lambda_M^2(C_0+D_0K)}{\overline{\rho}r\lambda_m(P)} = \frac{\beta^2\lambda_M^2(C_0+D_0K)\lambda_M(\overline{P})}{\overline{\rho}r} \end{aligned} \tag{70}$$

where $\lambda_M(C_0+D_0K)$ and $\lambda_M(\overline{P})$ are the maximal eigenvalues of C_0+D_0K and $\overline{P}$.

Therefore from (70) and the discussion in [36], we know that the equilibrium of (10) is mean-square robustly asymptotically stabilisable ($\beta = 0$), and $u(t)$ in (57) is a robust uniformly bounded controller. This completes the proof. □

Note that the terms $k_1\overline{P}$ and $k_2\overline{P}$ in (52) and (53) are not LMI. In the following, we propose the LMI method to design a robust uniformly bounded controller.

Corollary 4: Under Assumptions A_1–A_3 and A_5, $u = Kx(t)$ is a robust uniformly bounded controller in system (10) if there are positive-definite diagonal matrix $\overline{\Lambda} = \text{diag}(\overline{\Lambda}_1, \overline{\Lambda}_2, \ldots, \overline{\Lambda}_{2n}) > 0$, positive-definite matrices $\Xi_1 = (\overline{\Xi}_{1ij})_{2n\times 2n}$, $\Xi_2 = (\overline{\Xi}_{2ij})_{2n\times 2n}$, and positive constants $\overline{\varepsilon}_1, \overline{\varepsilon}_2, \overline{\varepsilon}_3, \overline{\rho}, \overline{k}_1, \overline{k}_2, \overline{\eta}$ and matrix $\overline{K}$, such that

$$\widetilde{J}_1 = \begin{pmatrix} \widetilde{J}_{11} & \overline{\rho}P_1^{\mathrm{T}} & \overline{\rho}\widetilde{N}_A \\ * & -\overline{\rho}I & 0 \\ * & * & -\overline{\varepsilon}_1 I \end{pmatrix} < 0 \qquad (71)$$

$$\widetilde{J}_2 = \begin{pmatrix} -\overline{k}_2 & \overline{\rho}P_2^{\mathrm{T}} & \overline{\rho}\widetilde{N}_D & \overline{\rho}\widetilde{D}^{\mathrm{T}} & \overline{\rho}H \\ * & -\overline{\rho}I & 0 & 0 & 0 \\ * & * & -\overline{\varepsilon}_2 I & 0 & 0 \\ * & * & * & -\Xi_1 & 0 \\ * & * & * & * & -\overline{\Lambda} \end{pmatrix} < 0 \qquad (72)$$

$$\widetilde{J}_3 = \begin{pmatrix} -\overline{\Lambda} & \overline{\Lambda}\widetilde{W}^{\mathrm{T}} & \overline{\Lambda}\widetilde{N}_W \\ * & -\Xi_2 & 0 \\ * & * & -\overline{\varepsilon}_3 I \end{pmatrix} < 0 \qquad (73)$$

$$\widetilde{J}_4 = \begin{pmatrix} -\overline{\eta}I & (C_0\overline{\rho} + D_0\overline{K})^{\mathrm{T}} \\ * & -I \end{pmatrix} < 0 \qquad (74)$$

$$\overline{k}_1 > \overline{k}_2 \qquad (75)$$

where

$$\widetilde{J}_{11} = -2(\overline{\rho}\widetilde{A} - B\overline{K}) + (\overline{\varepsilon}_1 + \overline{\varepsilon}_2 + \overline{\varepsilon}_3)\begin{pmatrix} MM^{\mathrm{T}} & 0 \\ 0 & MM^{\mathrm{T}} \end{pmatrix} + \Xi_1 + \Xi_2 + \overline{k}_1,$$

$$\widetilde{N}_A = \begin{pmatrix} N_A & 0 \\ 0 & N_C \end{pmatrix}^{\mathrm{T}}, \quad \widetilde{N}_D = \begin{pmatrix} 0 & 0 \\ N_D & 0 \end{pmatrix}^{\mathrm{T}},$$

$$\widetilde{N}_W = \begin{pmatrix} 0 & N_W \\ 0 & 0 \end{pmatrix}^{\mathrm{T}}$$

and I is the identity matrix. In this case, the appropriate state feedback controller can be chosen by

$$u(t) = Kx(t), K = \overline{\rho}^{-1}\overline{K} \qquad (76)$$

Then the output converges exponentially to a small region $\mathcal{D}$ in the mean-square, where

$$\mathcal{D} = \left\{ \mathbf{E}\|y(t)\|^2 \middle| \mathbf{E}\|y(t)\|^2 \leq \frac{\overline{\eta}\beta^2}{\overline{\rho}^2 r} \right\} \qquad (77)$$

in which $\lambda_M(C_0 + D_0K)$ and $\lambda_M(\overline{P})$ are the maximal eigenvalues of $C_0 + D_0K$ and $\overline{P}$, r is the unique positive solution of $-r = -k_1 + k_2 e^{r\tau}$, $k_1 = \overline{k}_1/\overline{\rho}$ and $k_2 = \overline{k}_2/\overline{\rho}$.

Proof: Let $\overline{P} = \overline{\rho}I$, and use the same steps as in (68) Theorem 3, we have

$$\lim_{t\to\infty} \mathbf{E}[x^{\mathrm{T}}(t)x(t)] \leq \frac{\beta^2}{r}, \quad t \geq 0 \qquad (78)$$

$$\begin{aligned} \lim_{t\to\infty} \mathbf{E}\|y(t)\|^2 &\leq \lim_{t\to\infty} \mathbf{E}\{x^{\mathrm{T}}(t)[(C_0 + D_0K)^{\mathrm{T}}(C_0 + D_0K) \\ &\quad - \eta P]x(t)\} + \eta \lim_{t\to\infty} \mathbf{E}[x^{\mathrm{T}}(t)Px(t)] \\ &= \lim_{t\to\infty} \mathbf{E}[x^{\mathrm{T}}(t)PJ_4Px(t)] + \frac{\eta}{\overline{\rho}} \lim_{t\to\infty} \mathbf{E}[x^{\mathrm{T}}(t)x(t)] \\ &\leq \frac{\overline{\eta}\beta^2}{\overline{\rho}^2 r} \end{aligned} \qquad (79)$$

where η is a positive constant, $\overline{\eta} = \eta\overline{\rho}$ and $J_4 = (C_0\overline{P} + D_0\overline{K})^{\mathrm{T}}(C_0\overline{P} + D_0\overline{K}) - \overline{\eta}I$. From (74) and Lemma 3, we know that $J_4 < 0$ in (79) is equivalent to $\widetilde{J} < 0$ in (74). Therefore we have

$$\lim_{t\to\infty} \mathbf{E}\|y(t)\|^2 \leq \frac{\overline{\eta}\beta^2}{\overline{\rho}^2 r} \qquad (80)$$

The controller in (76) is a robust uniformly bounded controller. This completes the proof. □

Remark 2: Robust H_∞ control has been intensively investigated in recent years [37–40]. However, the robust H_∞ control requires that $v(t) \in L_2[0, \infty)$ under zero initial conditions. In the real situation, it is very natural to see that the noise will not vanish as the time goes to infinity and the initial conditions are not zero. Therefore, in this paper, when Assumption A_4 used in robust H_∞ control is not satisfied, we proposed robust uniformly bounded control to let the output converge to a small region. To the best of our knowledge, this kind of control for the uncertain stochastic delayed system is rarely studied anywhere.

6 Numerical examples

In this section, some numerical examples are given to show the effectiveness and correctness of the proposed theoretical results.

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The dynamics of the repressilator has been theoretically predicted and experimentally investigated in *Escherichia coli* [1]. Three repressor-protein concentrations p_i, and their mRNA concentrations m_i (where i is lacI, tetR or cI) were treated as continuous dynamical variables. The repressilator is a cyclic negative-feedback loop composed of three genes and their corresponding promoters. The kinetics of the system are determined by six coupled first-order differential equations

$$\begin{aligned}\frac{\mathrm{d}m_i(t)}{\mathrm{d}t} &= -m_i(t) + \frac{\alpha}{1+p_j^n}\\ \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} &= -\zeta(p_i(t) - m_i(t))\\ & i = \text{lacI, tetR, cI}; j = \text{cI, lacI, tetR}\end{aligned} \tag{81}$$

where ζ denotes the ratio of the protein decay rate to the mRNA decay rate, and n is a Hill coefficient. Construction, design and simulation of the repressilator is shown in Fig. 1 [1]. The preceding analysis neglects the stochastic and delayed characters of their interactions; however, such effects are believed to be important in biochemical and genetic networks. Therefore we consider the following delayed stochastic GRN with parameter uncertainties

$$\begin{aligned}\mathrm{d}x(t) &= [-(\tilde{A}+\Delta\tilde{A})x(t) + (\tilde{D}+\Delta\tilde{D})x(t-\tau)\\ &\quad + (\tilde{W}+\Delta\tilde{W})g(x(t-\tau))]\,\mathrm{d}t + \sigma(x(t), x(t-\tau))\,\mathrm{d}\omega\\ &\quad + v(t)\,\mathrm{d}v + Bu(t)\,\mathrm{d}t\\ y(t) &= C_o x(t) + D_o u(t)\end{aligned} \tag{82}$$

where

$$\tilde{A} = \begin{pmatrix} A & 0\\ 0 & C\end{pmatrix},\quad \Delta\tilde{A} = \begin{pmatrix} \Delta A & 0\\ 0 & \Delta C\end{pmatrix},\quad \tilde{D} = \begin{pmatrix} 0 & 0\\ D & 0\end{pmatrix},$$

$$\Delta\tilde{D} = \begin{pmatrix} 0 & 0\\ \Delta D & 0\end{pmatrix},\quad \tilde{W} = \begin{pmatrix} 0 & W\\ 0 & 0\end{pmatrix},\quad \Delta\tilde{W} = \begin{pmatrix} 0 & \Delta W\\ 0 & 0\end{pmatrix},$$

$$\begin{aligned}&\begin{pmatrix}\Delta A & \Delta W & \Delta C & \Delta D\end{pmatrix}\\ &\quad = MF(t)\begin{pmatrix}N_A & N_W & N_C & N_D\end{pmatrix}, u(t) = Kx(t)\end{aligned}$$

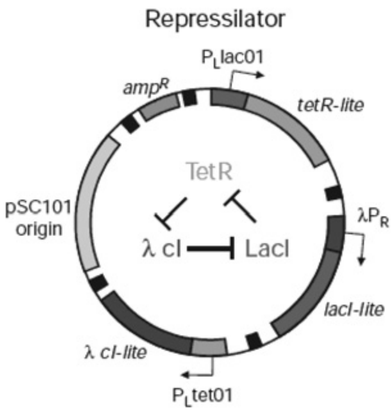

Figure 1 *Construction, design and simulation of the repressilator* [1]

Example 1: Robust asymptotical stabilisation

Let

$$A = 2\begin{pmatrix}0.4 & 0 & 0\\ 0 & 0.2 & 0\\ 0 & 0 & 0.1\end{pmatrix},\quad C = 0.9\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$W = 2.5\begin{pmatrix}0 & 0 & -1\\ -1 & 0 & 0\\ 0 & -1 & 0\end{pmatrix},\quad D = 0.8\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$M = 0.1\begin{pmatrix}0.1 & 0.2 & 0.3\\ 0.2 & 0.2 & 0.1\\ 0.3 & 0.1 & 0.1\end{pmatrix},\quad N_A = \begin{pmatrix}0.02 & 0.01 & 0.03\\ 0.05 & 0.01 & 0.02\\ 0.03 & 0.02 & 0.03\end{pmatrix},$$

$$N_C = \begin{pmatrix}0.04 & 0.03 & 0.02\\ 0.02 & 0.02 & 0.01\\ 0.04 & 0.02 & 0.01\end{pmatrix},\quad N_W = \begin{pmatrix}0.01 & 0 & 0\\ 0 & 0.02 & 0\\ 0 & 0 & 0.03\end{pmatrix},$$

$$N_D = \begin{pmatrix}0.02 & 0.02 & 0.01\\ 0.02 & 0.03 & 0.04\\ 0.04 & 0.01 & 0\end{pmatrix},\quad B = 8\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\end{pmatrix}^{\mathrm{T}},$$

$$\begin{aligned}F(t) &= \mathrm{diag}(\sin(t), \cos(t), -\sin(t)), \sigma(x(t), x(t-\tau))\\ &= 0.5x(t), g(z) = \frac{z^2}{1+z^2}, \tau = 1\end{aligned}$$

The states of the system (82) with $u(t) = 0$ and $v(t) = 0$ is shown in Fig. 2, so it is not mean-square asymptotically stable.

By Theorem 1 and using the Matlab LMI toolbox, the feasible solutions of (19) and (20) can be obtained. Here, we have

$$K = \begin{pmatrix}-0.5862 & -0.0018 & -0.0015 & 0 & 0 & 0\\ -0.0005 & -0.6107 & -0.0012 & 0 & 0 & 0\\ -0.0005 & -0.0008 & -0.6231 & 0 & 0 & 0\end{pmatrix}$$

From Theorem 1, we know that the system (82) with $v(t) = 0$ is mean-square robustly asymptotically stabilisable. The states of the system (82) with $u(t) = Kx(t)$ and $v(t) = 0$ is illustrated in Fig. 3.

Example 2: Robust H_∞ control

Consider the same parameters and functions in (82) as in Example 1. Also, let $v(t) = 0.1\mathrm{e}^{-0.1t}\bar{I}$, where $\bar{I} = \begin{pmatrix}1 & 1 & 1 & 1 & 1 & 1\end{pmatrix}^{\mathrm{T}}$, $C_0 = 0.2\bar{I}^{\mathrm{T}}$ and $D_0 = 0.02$

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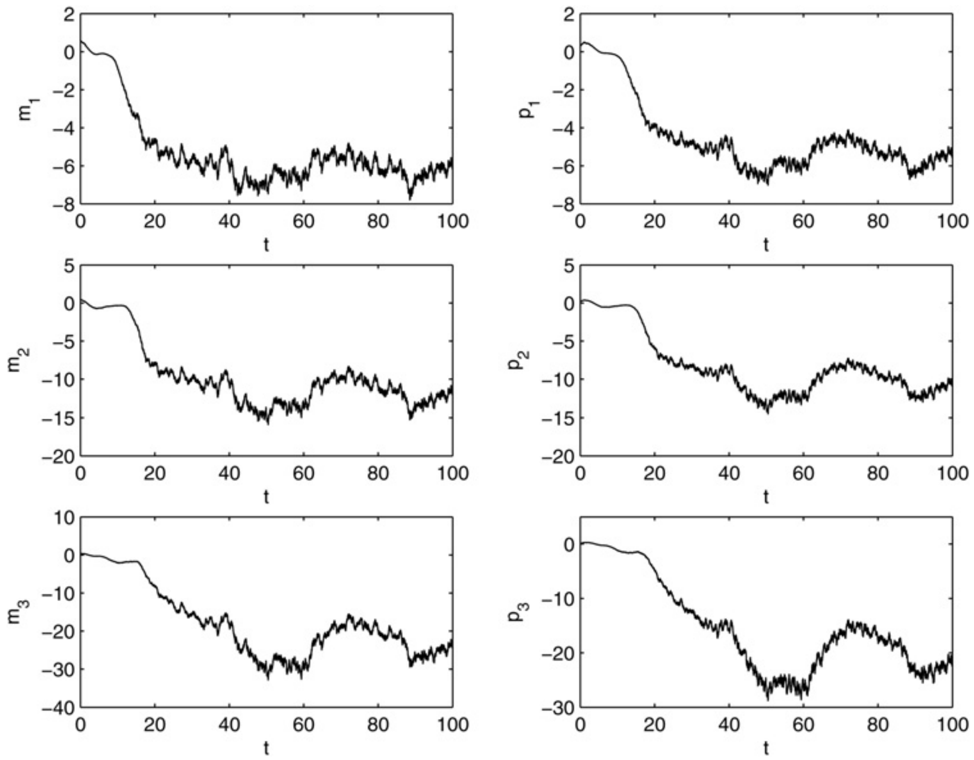

Figure 2 *Trajectories of states in system (82) with u(t) = 0 and v(t) = 0*

$\begin{pmatrix}1 & 1 & 1\end{pmatrix}$. By Corollary 2, the feasible robust H_∞ controller (see equation at the bottom of the page)

and minimal $\gamma = 0.0282$ are obtained. In order to improve the above high feedback gain matrix, we use the controller

$$K = \begin{pmatrix} -0.5862 & -0.0018 & -0.0015 & 0 & 0 & 0 \\ -0.0005 & -0.6107 & -0.0012 & 0 & 0 & 0 \\ -0.0005 & -0.0008 & -0.6231 & 0 & 0 & 0 \end{pmatrix}$$

in Example 1 and by Corollary 3, we can obtain the minimal $\gamma = 1/\sqrt{\gamma} = 1.9185$. Under zero initial conditions, the states of the system (82) with $u(t) = Kx(t)$ and $v(t) = 0.1\mathrm{e}^{-0.1t}\bar{I}$ is shown in Fig. 4.

Example 3: Robust uniformly bounded control

Using the same system as in Example 2 except $v(t) = 0.01\bar{I}$, where $\bar{I} = \begin{pmatrix}1 & 1 & 1 & 1 & 1 & 1\end{pmatrix}^{\mathrm{T}}$. It is easy to see that $v^{\mathrm{T}}v \le 6 \times 10^{-3} = \beta^2$. By Corollary 4, we can get the following feasible solutions: $r = 0.2464$, $\bar{\rho} = 5.3367$, $\bar{\eta} = 7.8345$

$$K = \begin{pmatrix} -0.3229 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.3477 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.3603 & 0 & 0 & 0 \end{pmatrix}$$

Therefore by (78) and (80) in Corollary 4, we have

$$\lim_{t\to\infty} \mathbf{E}[x^{\mathrm{T}}(t)x(t)] \le \frac{\beta^2}{r} = 0.0024$$

and

$$\lim_{t\to\infty} \mathbf{E}\|y(t)\|^2 \le \frac{\bar{\eta}\beta^2}{\bar{\rho}^2 r} = 6.6976 \times 10^{-4}$$

The trajectories of $\|x(t)\|^2$ and $\|y(t)\|^2$ in system (82) with $u(t) = Kx(t)$ and $v(t) = 0.01\bar{I}$ are illustrated in Fig. 5. Here, we simulated 20 times to obtain this approximated expectation. It is easy to see that the obtained controller is a robust uniformly bounded controller and

$$K = \begin{pmatrix} -665.9790 & 163.3515 & 167.0200 & 3.7951 & 2.6664 & 2.5358 \\ 356.4580 & -371.3710 & 256.6930 & -3.2065 & -2.8213 & -2.4338 \\ 308.5207 & 207.0193 & -424.7132 & -1.5884 & -0.8449 & -1.1019 \end{pmatrix}$$

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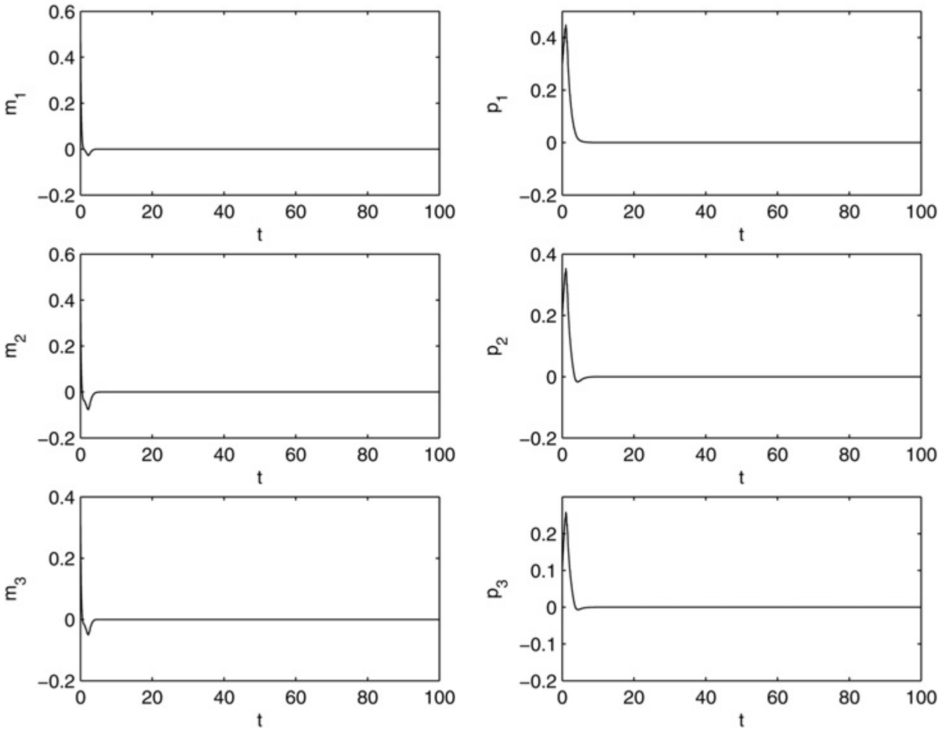

Figure 3 *Trajectories of states in system (82) with u(t) = Kx(t) and v(t) = 0*

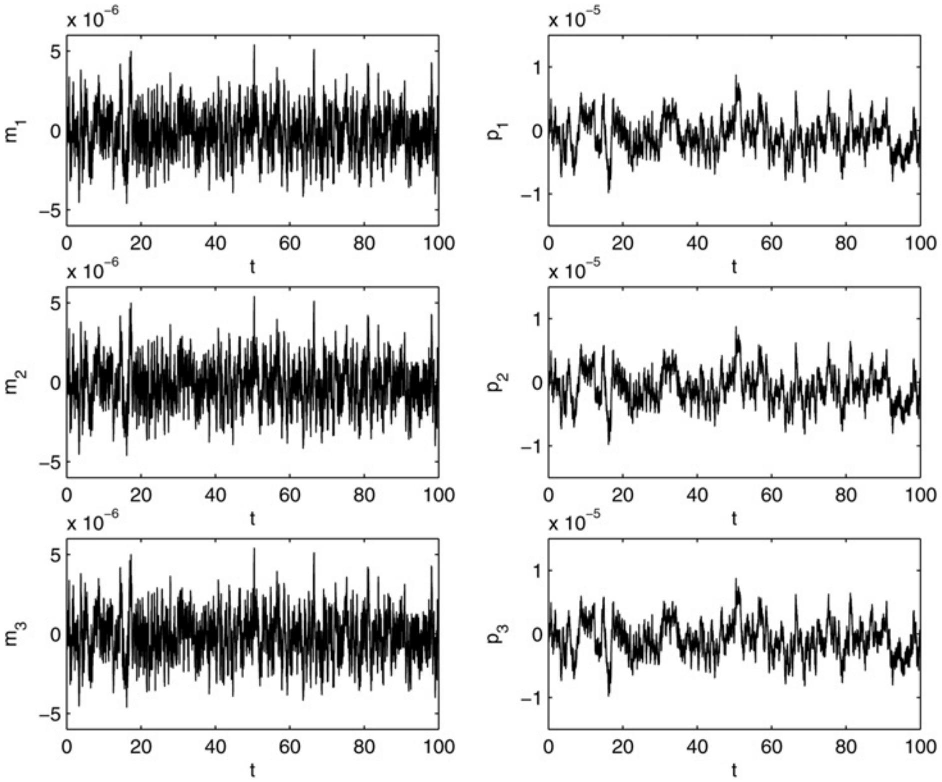

Figure 4 *Trajectories of states in system (82) with u(t) = Kx(t) and v(t) =* $0.1e^{-0.1t}\bar{l}$

the output converges to the region defined in $\mathcal{D} = \{\mathbf{E}\|y(t)\|^2 | \mathbf{E}\|y(t)\|^2 \leq 6.6976 \times 10^{-4}\}$ in (77).

Remark 3: The simulation shows further stabilisation of the mRNA and protein levels in this system by an artificial controller, which translates to the function of further restricting the range of target mRNA and protein levels. This is conceivable to be done by adding some small molecules or drugs to selectively inhibit the target mRNA and proteins to some extent. According to the simulation examples, the proposed controllers are very effective.

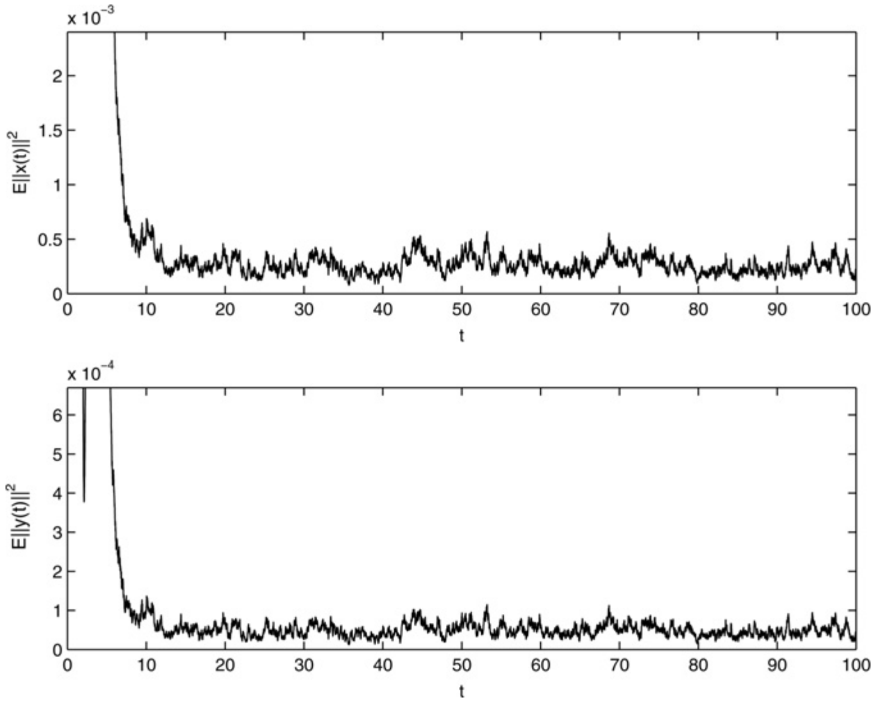

Figure 5 *Trajectories of* $\mathbf{E}\|x(t)\|^2$ *and* $\mathbf{E}\|y(t)\|^2$ *in system (82) with* $u(t) = Kx(t)$ *and* $v(t) = 0.01\bar{l}$

7 Conclusions

In this paper, the mean-square robust stability of delayed GRN with stochastic disturbance is considered, and then robust H_∞ control and uniformly bounded control for the delayed GRN are further investigated. The robust uniform bounded control is rarely investigated where the energy of the noise is infinite. However, it is very easy to see that noise exists anytime, and will not vanish as the time goes to infinity and initial conditions are not zero. Finally, numerical examples are given to verify the effectiveness and correctness of the theoretical results in this paper.

The preceding analysis neglects the stochastic and uncertain characters of the interactions between the molecular components. In this paper, we are interested in the robust control of GRN with stochastic disturbance and parameter uncertainties, which may lead to a hot study about GRN in the biological and control fields. In addition, the SUM logic is not good enough to accurately exhibit the real dynamics of the organism. Usually, there is only one transcription factor for one gene, and there are also other proteins when a gene is transcribed, for example, the transcriptional co-factors, the basic transcriptional machinery and some other enhancers (other proteins). Therefore some other sophisticated regulation functions will be studied in the near future.

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