Detailed Proof of Lemmas and Theorems

1 Proof of Lemma 2

Proof Define the formation errors $\tilde{x}_i(t) = x_i(t) - x_0(t) - x^*_i, i = 1, 2, \cdots N$, with $\tilde{x}_0(t) = -x^*_0 = 0$.

The Filippov solution of $\tilde{x}_i(t)$ is defined as the absolutely continuous solution of the differential inclusion

$$\dot{\tilde{x}}_i(t) \in \mathcal{K} \left[ f_i(t, x_i(t)) - f_0(t, x_0(t)) - \alpha \text{sgn} \left\{ \sum_{j \in \mathcal{N}_i} a_{ij} [\tilde{x}_i(t) - \tilde{x}_j(t)] \right\} \right], \forall i = 1, 2, \cdots, N. \quad (1)$$

Based on Assumption 1, one follower must receive information from other followers or the leader, namely, it is connected with other followers or the leader. Define $\tilde{x}_+ (t)$ as the maximal formation error component which is connected with non-maximal error components of the followers or connected with the component of the leader. Similarly, define $\tilde{x}_-(t)$ as the minimal formation error component which is connected with non-minimal error components of the followers or connected with the component of the leader. Suppose that, at any time $t$, $\tilde{x}_+ (t)$ is the $k$th error component of agent $i$, and $\tilde{x}_- (t)$ is the $l$th error component of agent $j$, where $i, j \in \{1, 2, \cdots, N\}, k, l \in \{1, 2, \cdots, n\}$. The Filippov solutions of $\tilde{x}_+ (t)$ and $\tilde{x}_- (t)$ can be described by

$$\dot{\tilde{x}}_+ (t) \in \mathcal{K} \left[ f^k_i(t, x_i(t)) - f^k_0(t, x_0(t)) - \alpha \text{sgn} \left\{ \sum_{r \in \mathcal{N}_i} a_{ir} [\tilde{x}^+_i(t) - \tilde{x}^+_r(t)] \right\} \right],$$

$$\dot{\tilde{x}}_- (t) \in \mathcal{K} \left[ f^l_j(t, x_j(t)) - f^l_0(t, x_0(t)) - \alpha \text{sgn} \left\{ \sum_{s \in \mathcal{N}_j} a_{js} [\tilde{x}^-_j(t) - \tilde{x}^-_s(t)] \right\} \right]. \quad (2)$$

Based on Assumptions 2 and 3, for any $i = 1, 2, \cdots, N$ and each $t \in \mathbb{R}^+$, one has
\[ \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \| = \| f_i(t, x_i(t)) - f_i(t, x_0(t)) + f_i(t, x_0(t)) + f_0(t, x_0(t)) \| \]
\[ \leq \| f_i(t, x_i(t)) - f_i(t, x_0(t)) \| + \| f_i(t, x_0(t)) \| + \| f_0(t, x_0(t)) \| \]
\[ \leq \| f_i(t, x_i(t)) - f_i(t, x_0(t)) \| + \| f_i(t, x_0(t)) - f_i(t, x_0(t)) \| + \| f_0(t, x_0(t)) - f_0(t, x_0(t)) \| \]
\[ \leq L_j^F \| x_i(t) - x_0(t) \| + L_j^F \| x_0(t) - x_i(t) \| + L_j^L \| x_0(t) - x_0(t) \| \]
\[ \leq L_j^F \left( \| x_i(t) - x_0(t) - x_i^* \| + \| x_0(t) - x_i(t) \| \right) + L_j^L \| x_0(t) - x_0(t) \| \]
\[ \leq L_j^F \left( \sqrt{n} \max \{ | \ddot{x}^+(t) |, | \ddot{x}^-(t) | \} + \max_{i=1,2,\ldots,N} \{ \| x_i^* \| + \| x_i^E \| \} + \beta \right) + L_j^L \left( \| x_0^E \| + \beta \right). \quad (3) \]

Let
\[ P(t) = L_j^F \left( \sqrt{n} \max \{ | \ddot{x}^+(t) |, | \ddot{x}^-(t) | \} + \max_{i=1,2,\ldots,N} \{ \| x_i^* \| + \| x_i^E \| \} + \beta \right) + L_j^L \left( \| x_0^E \| + \beta \right). \quad (4) \]

If \( \alpha > P(t), \forall t \in \mathbb{R}^+, \) then \( \alpha > \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in \mathbb{R}^+, \forall i = 1, 2, \ldots, N. \)

Now, it can be proved that if \( \alpha > P(0) \) then \( \alpha > P(t), \forall t \in \mathbb{R}^+. \) Because \( \alpha > P(0) \) and \( P(t) \) are continuously changing, suppose that \( t_1 \in \mathbb{R}^+ \) is the first time at which \( \alpha = P(t) \). Since \( \alpha, \| x_0^E \|, \beta, L_j^F, L_j^L \) and \( \max_{i=1,2,\ldots,N} \{ \| x_i^* \| + \| x_i^E \| \} \) are constants, one has \( \max \{ | \ddot{x}^+(t_1) |, | \ddot{x}^-(t_1) | \} > \max \{ | \ddot{x}^+(0) |, | \ddot{x}^-(0) | \}. \) So, there must exist a \( t_2 \in [0, t_1) \) such that the derivative of \( \max \{ | \ddot{x}^+(t) |, | \ddot{x}^-(t) | \} \) is greater than zero.

Now, consider the following three cases.

- **Case (i):** \( \{ \ddot{x}^+(t) > 0, \ddot{x}^-(t) \geq 0 \}. \)

  In this case, \( \max \{ | \ddot{x}^+(t) |, | \ddot{x}^-(t) | \} = \ddot{x}^+(t) \), and the derivative of \( \max \{ | \ddot{x}^+(t) |, | \ddot{x}^-(t) | \} \) is \( \ddot{x}^+(t) \). Since Assumption 1 holds and \( \ddot{x}^+(t) > 0 \), one has \( \sum_{r \in \mathcal{M}} a_{ir} [\ddot{x}^+(t) - \ddot{x}^k(t)] > 0. \) Thus,

  \[ \ddot{x}^+(t) \in \mathcal{K} \left[ f^k_i(t, x_i(t)) - f^k_0(t, x_0(t)) - \alpha \right] \]

If the derivative of \( \max \{ | \ddot{x}^+(t) |, | \ddot{x}^-(t) | \} \) is greater than zero at \( t_2 \in [0, t_1) \), one has \( \ddot{x}^+(t_2) > 0. \) Then, there must exist \( i \in \{1, 2, \ldots, N\} \) and \( k \in \{1, 2, \ldots, n\} \) such that \( f^k_i(t_2, x_i(t_2)) - f^k_0(t_2, x_0(t_2)) > 0 \) and the positive constant \( \alpha < \| f^k_i(t_2, x_i(t_2)) - f^k_0(t_2, x_0(t_2)) \| \). Since

\[ \| f^k_i(t_2, x_i(t_2)) - f^k_0(t_2, x_0(t_2)) \| \leq \| f_i(t_2, x_i(t_2)) - f_0(t_2, x_0(t_2)) \| \]

one has \( \alpha < \| f_i(t_2, x_i(t_2)) - f_0(t_2, x_0(t_2)) \| \). It follows that \( \alpha > P(t_2) \) based on (3). Because \( \alpha > P(0) \) and \( P(t) \) are continuously changing, there must be a \( t_3 \in [0, t_2) \) such that \( \alpha = P(t_3) \). It contradicts the assumption that \( t_1 \in \mathbb{R}^+ \) is the first time at which \( \alpha = P(t) \).
Then, there must exist

Proof

Six cases are discussed as follows:

2 Proof of Lemma 3

Based on (3).

\[ \alpha > P(0) \]

\[ |\tilde{x}(t)| < \alpha \]

\[ \exists t \]

\[ \tilde{x}(t) \in K \left[ f_j'(t, x_j(t)) - f_0'(t, x_0(t)) + \alpha \right]. \]

If the derivative of \( \max\{ |\tilde{x}^+(t)|, |\tilde{x}^-(t)| \} \) is greater than zero at \( t_2 \in [0, t_1) \), one has \( \dot{\tilde{x}}(t_2) < 0 \).

Then, there must exist \( j \in \{1, 2, \ldots, N \} \) and \( l \in \{1, 2, \ldots, n \} \) such that \( f_j'(t_2, x_j(t_2)) - f_0'(t_2, x_0(t_2)) < 0 \) and the positive constant \( \alpha < |f_j'(t_2, x_j(t_2)) - f_0'(t_2, x_0(t_2))| \). Since \( |f_j'(t_2, x_j(t_2)) - f_0'(t_2, x_0(t_2))| \leq \| f_j(t_2, x_j(t_2)) - f_0(t_2, x_0(t_2)) \| \), one has \( \alpha < \| f_j(t_2, x_j(t_2)) - f_0(t_2, x_0(t_2)) \| \). It follows that \( \alpha < P(t_2) \) based on (3). Because \( \alpha > P(0) \) and \( P(t) \) are continuously changing, there must be a \( t_3 \in [0, t_2) \) such that \( \alpha = P(t_3) \). It contradicts the assumption that \( t_1 \in R^+ \) is the first time at which \( \alpha = P(t) \).

• Case (iii): \( \{ \tilde{x}^+(t) > 0, \tilde{x}^-(t) < 0 \} \).

(i) If \( \{ \tilde{x}^+(t) \geq -\tilde{x}^-(t) \} \), then \( \max\{ |\tilde{x}^+(t)|, |\tilde{x}^-(t)| \} = \tilde{x}^+(t) \). So, the proof is the same as that in Case (i).

(ii) If \( \{ \tilde{x}^+(t) < -\tilde{x}^-(t) \} \), then \( \max\{ |\tilde{x}^+(t)|, |\tilde{x}^-(t)| \} = -\tilde{x}^-(t) \). So, the proof is the same as that in Case (ii).

Combining the above three cases, it can be concluded that the derivative of \( \max\{ |\tilde{x}^+(t)|, |\tilde{x}^-(t)| \} \) will not be greater than zero. Hence, if \( \alpha > P(0) \), i.e., Assumption 4 holds, then \( \alpha > P(t), \forall t \in R^+ \). It follows that \( \alpha > \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in R^+, \forall i = 1, 2, \ldots, N \), based on (3).

The proof is now completed.

2 Proof of Lemma 3

Proof Six cases are discussed as follows:
• **Case (i):** \((\bar{x}^+(t), \bar{x}^-(t)), (\bar{x}^+(t)', \bar{x}^-(t)') \in D_1.\)

\[
\| V(\bar{x}^+(t), \bar{x}^-) - V(\bar{x}^+(t)', \bar{x}^-) \| = \| \bar{x}^+(t) - \bar{x}^+(t)' \| \\
\leq \| \bar{x}^+(t) - \bar{x}^+(t)' \| + \| \bar{x}^-(t) - \bar{x}^-(t)' \| \\
\leq \sqrt{2} \| (\bar{x}^+(t), \bar{x}^-) - (\bar{x}^+(t)', \bar{x}^-) \|^T. 
\]

• **Case (ii):** \((\bar{x}^+(t), \bar{x}^-(t)), (\bar{x}^+(t)', \bar{x}^-(t)') \in D_2.\)

\[
\| V(\bar{x}^+(t), \bar{x}^-) - V(\bar{x}^+(t)', \bar{x}^-) \| = \| (\bar{x}^+(t) - \bar{x}^-(t)) - (\bar{x}^+(t)' - \bar{x}^-(t)') \| \\
\leq \| \bar{x}^+(t) - \bar{x}^+(t)' \| + \| \bar{x}^-(t) - \bar{x}^-(t)' \| \\
\leq \sqrt{2} \| (\bar{x}^+(t), \bar{x}^-) - (\bar{x}^+(t)', \bar{x}^-) \|^T. 
\]

• **Case (iii):** \((\bar{x}^+(t), \bar{x}^-(t)), (\bar{x}^+(t)', \bar{x}^-(t)') \in D_3.\)

\[
\| V(\bar{x}^+(t), \bar{x}^-) - V(\bar{x}^+(t)', \bar{x}^-) \| = \| -\bar{x}^- - (-\bar{x}^-) \| \\
\leq \| \bar{x}^+(t) - \bar{x}^+(t)' \| + \| \bar{x}^-(t) - \bar{x}^-(t)' \| \\
\leq \sqrt{2} \| (\bar{x}^+(t), \bar{x}^-) - (\bar{x}^+(t)', \bar{x}^-) \|^T. 
\]

• **Case (iv):** \((\bar{x}^+(t), \bar{x}^-) \in D_1, (\bar{x}^+(t)', \bar{x}^-) \in D_2.\)

\[
\| V(\bar{x}^+(t), \bar{x}^-) - V(\bar{x}^+(t)', \bar{x}^-) \| = \| \bar{x}^+(t) - (\bar{x}^+(t)' - \bar{x}^-(t)') \| \\
\leq \| \bar{x}^+(t) - \bar{x}^+(t)' \| + \| \bar{x}^- (t)' \|. 
\]

For \((\bar{x}^+(t), \bar{x}^-) \in D_1, (\bar{x}^+(t)', \bar{x}^-) \in D_2\), one has \(\bar{x}^- (t) \geq 0, \bar{x}^- (t)' < 0\), thus

\[
\| \bar{x}^- (t)' \| \leq \| \bar{x}^- (t) - \bar{x}^- (t)' \|. 
\]

Hence,

\[
\| V(\bar{x}^+(t), \bar{x}^-) - V(\bar{x}^+(t)', \bar{x}^-) \| \leq \| \bar{x}^+(t) - \bar{x}^+(t)' \| + \| \bar{x}^- (t)' \| \\
\leq \| \bar{x}^+(t) - \bar{x}^+(t)' \| + \| \bar{x}^- (t) - \bar{x}^- (t)' \| \\
\leq \sqrt{2} \| (\bar{x}^+(t), \bar{x}^-) - (\bar{x}^+(t)', \bar{x}^-) \|^T. 
\]
• Case (v): \((\tilde{x}^+(t), \tilde{x}^-(t)) \in D_1, (\tilde{x}^+(t)', \tilde{x}^-(t)') \in D_3\).

\[
\| V(\tilde{x}^+(t), \tilde{x}^-(t)) - V(\tilde{x}^+(t)', \tilde{x}^-(t)') \| \\
= \| \tilde{x}^+(t) - (-\tilde{x}^-(t)') \| \\
\leq \| \tilde{x}^+(t) \| + \| \tilde{x}^-(t) \| .
\]

For \((\tilde{x}^+(t), \tilde{x}^-(t)) \in D_1, (\tilde{x}^+(t)', \tilde{x}^-(t)') \in D_3\), one has \(\tilde{x}^+(t) \geq 0, \tilde{x}^-(t) \geq 0, \tilde{x}^+(t)' \leq 0, \tilde{x}^-(t)' < 0\), thus

\[
\| \tilde{x}^+(t) \| \leq \| \tilde{x}^+(t) - \tilde{x}^+(t)' \|,
\]

and

\[
\| \tilde{x}^-(t)' \| \leq \| \tilde{x}^-(t) - \tilde{x}^-(t)' \|. \]

Hence,

\[
\| V(\tilde{x}^+(t), \tilde{x}^-(t)) - V(\tilde{x}^+(t)', \tilde{x}^-(t)') \| \\
\leq \| \tilde{x}^+(t) \| + \| \tilde{x}^-(t) \| \\
\leq \| \tilde{x}^+(t) - \tilde{x}^+(t)' \| + \| \tilde{x}^-(t) - \tilde{x}^-(t)' \| \\
\leq \sqrt{2} \| (\tilde{x}^+(t), \tilde{x}^-(t))^T - (\tilde{x}^+(t)', \tilde{x}^-(t)')^T \|. \]

• Case (vi): \((\tilde{x}^+(t), \tilde{x}^-(t)) \in D_2, (\tilde{x}^+(t)', \tilde{x}^-(t)') \in D_3\).

\[
\| V(\tilde{x}^+(t), \tilde{x}^-(t)) - V(\tilde{x}^+(t)', \tilde{x}^-(t)') \| \\
= \| (\tilde{x}^+(t) - \tilde{x}^-(t)) - (-\tilde{x}^-(t)') \| \\
\leq \| \tilde{x}^+(t) \| + \| \tilde{x}^-(t) - \tilde{x}^-(t) \|. \]

For \((\tilde{x}^+(t), \tilde{x}^-(t)) \in D_2, (\tilde{x}^+(t)', \tilde{x}^-(t)') \in D_3\), one has \(\tilde{x}^+(t) > 0, \tilde{x}^+(t)' \leq 0\), thus

\[
\| \tilde{x}^+(t) \| \leq \| \tilde{x}^+(t) - \tilde{x}^+(t)' \|. \]

Hence,

\[
\| V(\tilde{x}^+(t), \tilde{x}^-(t)) - V(\tilde{x}^+(t)', \tilde{x}^-(t)') \| \\
\leq \| \tilde{x}^+(t) \| + \| \tilde{x}^-(t) - \tilde{x}^-(t)' \| \\
\leq \| \tilde{x}^+(t) - \tilde{x}^+(t)' \| + \| \tilde{x}^-(t) - \tilde{x}^-(t)' \| \\
\leq \sqrt{2} \| (\tilde{x}^+(t), \tilde{x}^-(t))^T - (\tilde{x}^+(t)', \tilde{x}^-(t)')^T \|. \]
Combining the above six cases, it can be concluded that, for every \((\tilde{x}^+(t), \tilde{x}^-(t)), (\tilde{x}^+(t)', \tilde{x}^-(t)') \in D\), one has
\[
\| V(\tilde{x}^+(t), \tilde{x}^-(t)) - V(\tilde{x}^+(t)', \tilde{x}^-(t)') \| \\
\leq \sqrt{2} \| (\tilde{x}^+(t), \tilde{x}^-(t))^T - (\tilde{x}^+(t)', \tilde{x}^-(t)')^T \|.
\]
Therefore, \(V\) is a locally Lipschitz function on \(D\).

The proof is now completed.

3 Proof of Lemma 4

**Proof** If a function is continuously differentiable at \(x\), it is regular at \(x\). Since \(V\) is continuously differentiable everywhere except for \(\{\tilde{x}^+(t) > 0, \tilde{x}^-(t) = 0\}, \{\tilde{x}^+(t) = 0, \tilde{x}^-(t) < 0\}\) and \(\{\tilde{x}^+(t) = 0, \tilde{x}^-(t) = 0\}\), it needs to show that \(V\) is regular on these three sets.

Let \(y = (\tilde{x}^+(t), \tilde{x}^-(t))^T\) and \(v = (v_1, v_2)^T\). The right directional derivative of \(V\) at \(y \in \mathbb{R}^2\) in the direction \(v \in \mathbb{R}^2\) is defined as
\[
V'(y; v) = \lim_{h \to 0^+} \frac{V(\tilde{x}^+(t) + hv_1, \tilde{x}^-(t) + hv_2) - V(\tilde{x}^+(t), \tilde{x}^-(t))}{h}.
\]
The general directional derivative of \(V\) at \(y\) in the direction \(v\) is defined as
\[
V^o(y; v) = \lim_{\delta \to 0^+} \sup_{\epsilon \to 0^+, z \in B(y, \delta)} \sup_{h \in [0, \epsilon)} \frac{V(z_1 + hv_1, z_2 + hv_2) - V(z_1, z_2)}{h}.
\]

- **Case (i)**: \(\{\tilde{x}^+(t) > 0, \tilde{x}^-(t) = 0\}\).

If \(v_1 \geq 0, v_2 \geq 0\), then \((\tilde{x}^+(t) + hv_1, hv_2)_h \to 0^+ \in D_1\), hence
\[
V'(y; v) = \lim_{h \to 0^+} \frac{(\tilde{x}^+(t) + hv_1) - \tilde{x}^+(t)}{h} = v_1.
\]

For \(z \in B(y, \delta)\), when \(\delta \to 0^+, z \in D_1\) and \(z \in D_2\) are possible, hence
\[
V^o(y; v) = \lim_{\delta \to 0^+} \sup_{\epsilon \to 0^+, z \in B(y, \delta)} \sup_{h \in [0, \epsilon)} \left\{ \frac{(z_1 + hv_1) - z_1}{h}, \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h} \right\}
= v_1.
\]
So, $V'(y; v) = V^o(y; v)$.

If $v_1 \leq 0, v_2 < 0$, then $(\bar{x}^+(t) + hv_1, hv_2)_{h \to 0^+} \in D_2$, hence
\[
V'(y; v) = \lim_{h \to 0^+} \frac{(\bar{x}^+(t) + hv_1) - hv_2 - \bar{x}^+(t)}{h} = v_1 - v_2.
\]

For $z \in B(y, \delta)$, when $\delta \to 0^+$, $z \in D_1$ and $z \in D_2$ are possible, hence
\[
V^o(y; v) = \lim_{\delta \to 0^+} \sup_{\epsilon \to 0^+} \frac{\left\{ (z_1 + hv_1) - z_1, \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h} \right\}}{h}
\]
\[
= v_1 - v_2.
\]

So, $V'(y; v) = V^o(y; v)$.

If $v_1 < 0, v_2 \geq 0$, then $(\bar{x}^+(t) + hv_1, hv_2)_{h \to 0^+} \in D_1$, hence
\[
V'(y; v) = \lim_{h \to 0^+} \frac{(\bar{x}^+(t) + hv_1) - \bar{x}^+(t)}{h} = v_1.
\]

For $z \in B(y, \delta)$, when $\delta \to 0^+$, $z \in D_1$ and $z \in D_2$ are possible, hence
\[
V^o(y; v) = \lim_{\delta \to 0^+} \sup_{\epsilon \to 0^+} \frac{\left\{ (z_1 + hv_1) - z_1, \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h} \right\}}{h}
\]
\[
= v_1.
\]

So, $V'(y; v) = V^o(y; v)$.

If $v_1 > 0, v_2 < 0$, then $(\bar{x}^+(t) + hv_1, hv_2)_{h \to 0^+} \in D_2$, hence
\[
V'(y; v) = \lim_{h \to 0^+} \frac{((\bar{x}^+(t) + hv_1) - hv_2) - \bar{x}^+(t)}{h} = v_1 - v_2.
\]

For $z \in B(y, \delta)$, when $\delta \to 0^+$, $z \in D_1$ and $z \in D_2$ are possible, hence
\[
V^o(y; v) = \lim_{\delta \to 0^+} \sup_{\epsilon \to 0^+} \frac{\left\{ (z_1 + hv_1) - z_1, \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h} \right\}}{h}
\]
\[
= v_1 - v_2.
\]
So, \( V'(y; v) = V^o(y; v) \).

- **Case (ii):** \( \{\bar{x}^+(t) = 0, \bar{x}^-(t) < 0\} \).

  If \( v_1 > 0, v_2 \geq 0 \), then \( (hv_1, \bar{x}^-(t) + hv_2)_{h \to 0^+} \in D_2 \), hence

  \[
  V'(y; v) = \lim_{h \to 0^+} \frac{(hv_1 - (\bar{x}^-(t) + hv_2)) - (-\bar{x}^-(t))}{h} = v_1 - v_2.
  \]

  For \( z \in B(y, \delta) \), when \( \delta \to 0^+ \), \( z \in D_2 \) and \( z \in D_3 \) are possible, hence

  \[
  V^o(y; v) = \lim_{\delta \to 0^+} \sup_{z \in B(y, \delta)} \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2) - ((z_2 + hv_2)) - (-z_2)}{h} \]

  \[
  = v_1 - v_2.
  \]

  So, \( V'(y; v) = V^o(y; v) \).

  If \( v_1 \leq 0, v_2 < 0 \), then \( (hv_1, \bar{x}^-(t) + hv_2)_{h \to 0^+} \in D_3 \), hence

  \[
  V'(y; v) = \lim_{h \to 0^+} \frac{(-(\bar{x}^-(t) + hv_2)) - (-\bar{x}^-(t))}{h} = -v_2.
  \]

  For \( z \in B(y, \delta) \), when \( \delta \to 0^+ \), \( z \in D_2 \) and \( z \in D_3 \) are possible, hence

  \[
  V^o(y; v) = \lim_{\delta \to 0^+} \sup_{z \in B(y, \delta)} \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2) - ((z_2 + hv_2)) - (-z_2)}{h} \]

  \[
  = -v_2.
  \]

  So, \( V'(y; v) = V^o(y; v) \).

  If \( v_1 \leq 0, v_2 \geq 0 \), then \( (hv_1, \bar{x}^-(t) + hv_2)_{h \to 0^+} \in D_3 \), hence

  \[
  V'(y; v) = \lim_{h \to 0^+} \frac{(-(\bar{x}^-(t) + hv_2)) - (-\bar{x}^-(t))}{h} = -v_2.
  \]

  For \( z \in B(y, \delta) \), when \( \delta \to 0^+ \), \( z \in D_2 \) and \( z \in D_3 \) are possible, hence

  \[
  V^o(y; v) = \lim_{\delta \to 0^+} \sup_{z \in B(y, \delta)} \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2) - ((z_2 + hv_2)) - (-z_2)}{h} \]

  \[
  = -v_2.
  \]
So, $V'(y; v) = V^o(y; v)$.

If $v_1 > 0, v_2 < 0$, then $(hv_1, \bar{x}^{-}(t) + hv_2)_{h \to 0^+} \in D_2$, hence

$$V'(y; v) = \lim_{h \to 0^+} \frac{(hv_1 - (\bar{x}^- + hv_2)) - (-\bar{x}^-)}{h} = v_1 - v_2.$$ 

For $z \in B(y, \delta)$, when $\delta \to 0^+$, $z \in D_2$ and $z \in D_3$ are possible, hence

$$V^o(y; v) = \lim_{\delta \to 0^+} \sup_{z \in B(y, \delta)} \left\{ \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h}, \frac{-(z_2 + hv_2)) - (-z_2)}{h} \right\} = v_1 - v_2.$$ 

So, $V'(y; v) = V^o(y; v)$.

- Case $(iii)$: $\{ \bar{x}^+(t) = 0, \bar{x}^-(t) = 0 \}$.

If $v_1 \geq 0, v_2 \geq 0$, then $(hv_1, hv_2)_{h \to 0^+} \in D_1$, hence

$$V'(y; v) = \lim_{h \to 0^+} \frac{hv_1 - 0}{h} = v_1.$$ 

For $z \in B(y, \delta)$, when $\delta \to 0^+$, $z \in D_1, z \in D_2$ and $z \in D_3$ are all possible, hence

$$V^o(y; v) = \lim_{\delta \to 0^+} \sup_{z \in B(y, \delta)} \left\{ \frac{(z_1 + hv_1) - z_1}{h}, \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h}, \frac{-(z_2 + hv_2)) - (-z_2)}{h} \right\} = v_1.$$ 

So, $V'(y; v) = V^o(y; v)$.

If $v_1 \leq 0, v_2 < 0$, then $(hv_1, hv_2)_{h \to 0^+} \in D_3$, hence

$$V'(y; v) = \lim_{h \to 0^+} \frac{-hv_2 - 0}{h} = -v_2.$$ 

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For \( z \in B(y, \delta) \), when \( \delta \to 0^+ \), \( z \in D_1, z \in D_2 \) and \( z \in D_3 \) are all possible, hence

\[
V^o(y; v) = \lim_{\delta \to 0^+} \sup_{\epsilon \to 0^+} \frac{1}{h} \left\{ \frac{(z_1 + hv_1) - z_1}{h}, \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h}, \right. \\
\left. -\frac{((z_2 + hv_2)) - (-z_2))}{h} \right\}
\]

\[= -v_2.\]

So, \( V'(y; v) = V^o(y; v) \).

The case of \( v_1 < 0, v_2 \geq 0 \) is impossible for \( \bar{x}^+(t) > \bar{x}^-(t) \).

If \( v_1 > 0, v_2 < 0 \), then \( (hv_1, hv_2)_{h \to 0^+} \in D_2 \), hence

\[
V'(y; v) = \lim_{h \to 0^+} \frac{hv_1 - hv_2}{h} \\
= v_1 - v_2.
\]

For \( z \in B(y, \delta) \), when \( \delta \to 0^+ \), \( z \in D_1, z \in D_2 \) and \( z \in D_3 \) are all possible, hence

\[
V^o(y; v) = \lim_{\delta \to 0^+} \sup_{\epsilon \to 0^+} \frac{1}{h} \left\{ \frac{(z_1 + hv_1) - z_1}{h}, \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h}, \right. \\
\left. -\frac{((z_2 + hv_2)) - (-z_2))}{h} \right\}
\]

\[= v_1 - v_2.\]

So, \( V'(y; v) = V^o(y; v) \).

For all the cases, the right directional derivative of \( V \) is equal to the generalized directional derivative of \( V \), i.e., \( V'(y; v) = V^o(y; v) \). Therefore, the function \( V \) is regular on \( D \).

The proof is now completed.

4 \hspace{1em} \textbf{Proof of Lemma 5}

\textbf{Proof} If \( \bar{x}^+(t) = 0 \) and \( \bar{x}^-(t) = 0 \), then \( V = 0 \). If \( \bar{x}^+(t) > 0 \) and \( \bar{x}^-(t) \geq 0 \), i.e., \( (\bar{x}^+(t), \bar{x}^-(t)) \in D_1 \setminus \{(0,0)\} \), then \( V = \bar{x}^+(t) > 0 \). If \( \bar{x}^+(t) > 0 \) and \( \bar{x}^-(t) < 0 \), i.e., \( (\bar{x}^+(t), \bar{x}^-(t)) \in D_2 \), then \( V = \bar{x}^+(t) - \bar{x}^-(t) > 0 \). If \( \bar{x}^+(t) \leq 0 \) and \( \bar{x}^-(t) < 0 \), i.e., \( (\bar{x}^+(t), \bar{x}^-(t)) \in D_3 \), then \( V = -\bar{x}^-(t) > 0 \). So, \( V \) is globally positive definite.
If \((\tilde{x}^+(t), \tilde{x}^-(t)) \in D_1\), then as either \(\tilde{x}^+ \to \infty\) or both \(\tilde{x}^+, \tilde{x}^- \to \infty\), one has \(V = \tilde{x}^+(t) \to \infty\).

If \((\tilde{x}^+(t), \tilde{x}^-(t)) \in D_2\), then as either \(\tilde{x}^+ \to \infty\) or \(\tilde{x}^- \to -\infty\), or both, one has \(V = \tilde{x}^+(t) - \tilde{x}^- (t) \to \infty\). If \((\tilde{x}^+(t), \tilde{x}^-(t)) \in D_3\), then as either \(\tilde{x}^+ \to -\infty\) or both \(\tilde{x}^+, \tilde{x}^- \to -\infty\), one has \(V = -\tilde{x}^-(t) \to \infty\). So, \(V\) is radially unbounded.

The proof is now completed.

5 Proof of Lemma 6

Proof If Assumptions 1 - 4 hold, then Lemma 2 holds, i.e., \(\alpha > \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in R^+, \forall i = 1, 2, \cdots, N\). Five cases are discussed as follows:

- Case (i): \(\tilde{x}^+(t) > 0\) and \(\tilde{x}^-(t) > 0\).

Since \(\tilde{x}^+(t) > 0\) and Assumption 1 holds, one has \(\sum_{r \in N_i} a_{ir} [\tilde{x}^+(t) - \tilde{x}^k_r(t)] > 0\), and for

\[
\partial V(\tilde{x}^+(t), \tilde{x}^-(t)) = \{(1, 0)\},
\]

one has

\[
\tilde{L}_x V = K [f^i(t, x_i(t)) - f^k(t, x_0(t)) - \alpha].
\]

Since \(| f^i(t, x_i(t)) - f^k(t, x_0(t)) | \leq \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in R^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n\), it follows from Lemma 2 that

\[
\max \tilde{L}_x V < 0.
\]

- Case (ii): \(\tilde{x}^+(t) > 0\) and \(\tilde{x}^-(t) < 0\).

Since \(\tilde{x}^+(t) > 0, \tilde{x}^-(t) < 0\) and Assumption 1 holds, one has \(\sum_{r \in N_i} a_{ir} [\tilde{x}^+(t) - \tilde{x}^k_r(t)] > 0, \sum_{s \in N_j} a_{js} [\tilde{x}^-(t) - \tilde{x}^k_s(t)] < 0\), and for

\[
\partial V(\tilde{x}^+(t), \tilde{x}^-(t)) = \{(1, -1)\},
\]

one has

\[
\tilde{L}_x V = K \left[ (f^i(t, x_i(t)) - f^k(t, x_0(t)) - \alpha) - (f^j(t, x_j(t)) - f^0(t, x_0(t)) + \alpha) \right].
\]

Since \(| f^i(t, x_i(t)) - f^k(t, x_0(t)) | \leq \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in R^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n\), it follows from Lemma 2 that

\[
\max \tilde{L}_x V < 0.
\]
\textbullet{} Case (iii): $\bar{x}^+(t) < 0$ and $\bar{x}^-(t) < 0$.

Since $\bar{x}^-(t) < 0$ and Assumption 1 holds, one has $\sum_{s \in \mathcal{N}_j} a_{js}[\bar{x}^-(t) - \bar{x}^+_s(t)] < 0$, and for
\[ \partial V(\bar{x}^+(t), \bar{x}^-(t)) = \{(0, -1)\}, \]
one has
\[ \hat{L}_F V = \mathcal{K} \left[ -(f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha) \right]. \]
Since $|f_j^l(t, x_j(t)) - f_0^l(t, x_0(t))| \leq \| f_j(t, x_j(t)) - f_0(t, x_0(t)) \|, \forall t \in \mathbb{R}^+, \forall j = 1, 2, \cdots, N, \forall l = 1, 2, \cdots, n,$ it follows from Lemma 2 that
\[ \max \hat{L}_F V < 0. \]

\textbullet{} Case (iv): $\bar{x}^+(t) > 0$ and $\bar{x}^-(t) = 0$.

Since $\bar{x}^+(t) > 0, \bar{x}^-(t) = 0$ and Assumption 1 holds, one has $\sum_{r \in \mathcal{N}_i} a_{ir}[\bar{x}^+(t) - \bar{x}^+_r(t)] > 0, \sum_{s \in \mathcal{N}_j} a_{js}[\bar{x}^-(t) - \bar{x}^+_s(t)] \leq 0$. So, if $v \in \mathcal{F}(\bar{x}^+(t), \bar{x}^-(t))$, then $v^T = (v_1, v_2)$ with $v_1 \in \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha]$ and $v_2 \in \mathcal{K}[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha] \cup \mathcal{K}[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t))]$. For
\[ \partial V(\bar{x}^+(t), \bar{x}^-(t)) = \{1\} \times [-1, 0], \]
if $\zeta \in \partial V(\bar{x}^+(t), \bar{x}^-(t))$, then $\zeta^T = (1, y)$ with $y \in [-1, 0]$. Therefore,
\[ \zeta^T v = v_1 + y v_2. \]
If there exists an element $a$ satisfying that $\zeta^T v = a$ for all $y \in [-1, 0]$, then $v_2 = 0$. So, if $v_2 \neq 0$, one has $\hat{L}_F V = \emptyset$; if $v_2 = 0$, one has $\hat{L}_F V = \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha]$, and then it follows from Lemma 2 that $\max \hat{L}_F V < 0$. Thus, $\max \hat{L}_F V < 0$ or $\hat{L}_F V = \emptyset$ in this case.

\textbullet{} Case (v): $\bar{x}^+(t) = 0$ and $\bar{x}^-(t) < 0$.

Since $\bar{x}^+(t) = 0, \bar{x}^-(t) < 0$ and Assumption 1 holds, one has $\sum_{r \in \mathcal{N}_i} a_{ir}[\bar{x}^+(t) - \bar{x}^+_r(t)] \geq 0, \sum_{s \in \mathcal{N}_j} a_{js}[\bar{x}^-(t) - \bar{x}^+_s(t)] < 0$. So, if $v \in \mathcal{F}(\bar{x}^+(t), \bar{x}^-(t))$, then $v^T = (v_1, v_2)$ with $v_1 \in \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha] \cup \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t))]$ and $v_2 \in \mathcal{K}[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha]$. For
\[ \partial V(\bar{x}^+(t), \bar{x}^-(t)) = [0, 1] \times \{-1\}, \]
Therefore, the converging time satisfies
\[ \zeta^T v = yv_1 - v_2. \]

If there exists an element \( a \) satisfying that \( \zeta^T v = a \) for all \( y \in [0, 1] \), then \( v_1 = 0 \). So, if \( v_1 \neq 0 \), one has \( \hat{\mathcal{L}}_F V = 0 \); if \( v_1 = 0 \), one has \( \hat{\mathcal{L}}_F V = -\mathcal{K}[f^k_i(t, x_i(t)) - f^k_0(t, x_0(t)) + \alpha] \), and then it follows from Lemma 2 that \( \max \hat{\mathcal{L}}_F V < 0 \). Thus, \( \max \hat{\mathcal{L}}_F V < 0 \) or \( \hat{\mathcal{L}}_F V = 0 \) in this case.

Combining the above five cases, it can be concluded that \( \max \hat{\mathcal{L}}_F V < 0 \) for all \( (\bar{x}^+(t), \bar{x}^-(t)) \in \mathcal{D} \setminus \{(0, 0)\} \).

The proof is now completed.

### 6 Proof of Theorem 1

**Proof** The nonsmooth function \( V \), which was given by (6) in the manuscript, is chosen as the Lyapunov function. If Assumptions 1 - 4 hold, then Lemma 6 holds. By using Lemma 1, it follows from Lemmas 3 - 6 that \( (\bar{x}^+(t), \bar{x}^-(t)) = (0, 0) \) is a globally stable equilibrium point for system (2).

Next, the maximal converging time is considered.

- **Case (i):** \( \bar{x}^+(t) > 0 \) and \( \bar{x}^-(t) \geq 0 \).

  In this case, \( V = \bar{x}^+(t) \) and \( \hat{\mathcal{L}}_F V = \mathcal{K}[f^k_i(t, x_i(t)) - f^k_0(t, x_0(t)) - \alpha] \). By the proof of Lemma 2, one has \( \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \| \leq P(t), P(t) \leq P(0), \forall t \in \mathbb{R}^+, \forall i = 1, 2, \ldots, N \).

  Since \( | f^k_i(t, x_i(t)) - f^k_0(t, x_0(t)) | \leq \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in \mathbb{R}^+, \forall i = 1, 2, \ldots, N, \forall k = 1, 2, \ldots, n \), one has

  \[
  \max \hat{\mathcal{L}}_F V \leq -(\alpha - P(t)) \leq -(\alpha - P(0)).
  \]

  Therefore, the converging time satisfies

  \[
  T_1 \leq \frac{1}{\alpha - P(0)} \bar{x}^+(0).
  \]

- **Case (ii):** \( \bar{x}^+(t) > 0 \) and \( \bar{x}^-(t) < 0 \).

  In this case, \( V = \bar{x}^+(t) - \bar{x}^-(t) \) and \( \hat{\mathcal{L}}_F V = \mathcal{K}[(f^k_i(t, x_i(t)) - f^k_0(t, x_0(t)) - \alpha) - (f_j^k(t, x_j(t)) - f^k_0(t, x_0(t)) + \alpha)] \). By the proof of Lemma 2, one has \( \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \| \leq P(t), P(t) \leq P(0), \forall t \in \mathbb{R}^+, \forall i = 1, 2, \ldots, N, \forall k = 1, 2, \ldots, n \), one has

  \[
  \max \hat{\mathcal{L}}_F V \leq -(\alpha - P(t)) \leq -(\alpha - P(0)).
  \]

  Therefore, the converging time satisfies

  \[
  T_1 \leq \frac{1}{\alpha - P(0)} \bar{x}^+(0).
  \]
\(P(0), \forall t \in \mathbb{R}^+, \forall i = 1, 2, \cdots, N.\) Since \(|f_i^k(t, x_i(t)) - f_0^k(t, x_0(t))| \leq \|f_i(t, x_i(t)) - f_0(t, x_0(t))\|, \forall t \in \mathbb{R}^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n,\) one has
\[
\max \tilde{L}_FV \leq -2(\alpha - P(t)) \\
\leq -2(\alpha - P(0)).
\]

Therefore, the converging time satisfies
\[
T_2 \leq \frac{1}{2(\alpha - P(0))}(\bar{x}^+(0) - \bar{x}^-(0)) \\
\leq \frac{1}{\alpha - P(0)} \max \{\bar{x}^+(0), -\bar{x}^-(0)\}.
\]

• Case (iii): \(\bar{x}^+(t) \leq 0\) and \(\bar{x}^-(t) < 0.\)

In this case, \(V = -\bar{x}^-(t).\) Since \(\bar{x}^-(t) < 0\) and Assumption 1 holds, one has \(\sum_{s \in \mathcal{N}_j} a_{js} [\bar{x}^-(t) - \bar{x}^+(t)] < 0,\) then \(\tilde{L}_FV = K[\alpha - (f_j^l(t, x_j(t)) - f_{0j}^l(t, x_{0j}(t)) + \alpha)].\) By the proof of Lemma 2, one has \(\|f_j(t, x_j(t)) - f_0(t, x_0(t))\| \leq P(t), P(t) \leq P(0), \forall t \in \mathbb{R}^+, \forall j = 1, 2, \cdots, N.\) Since \(|f_j^l(t, x_j(t)) - f_{0j}^l(t, x_{0j}(t))| \leq \|f_j(t, x_j(t)) - f_0(t, x_0(t))\|, \forall t \in \mathbb{R}^+, \forall j = 1, 2, \cdots, N, \forall l = 1, 2, \cdots, n,\) one has
\[
\max \tilde{L}_FV \leq -(\alpha - P(t)) \\
\leq -(\alpha - P(0)).
\]

Therefore, the converging time satisfies
\[
T_3 \leq -\frac{1}{\alpha - P(0)} \bar{x}^-(0).
\]

Combining the above three cases, the maximal converging time is obtained as
\[
T = \frac{1}{\alpha - P(0)} \max_{k=1,2,\cdots,n} \{\max_{i=1,2,\cdots,N} \left| x_i^k(0) - x_{0i}^k(0) - x_i^*(k) \right|\}.
\]

The proof is now completed.

7 Supplementary Lemma i

**Supplementary Lemma i** If Assumptions 1, 5 and 6 hold, then \(\alpha > \|f_i(t, x_i(t)) - f_0(t, x_0(t))\|, \forall t \in \mathbb{R}^+, \forall i = 1, 2, \cdots, N.\)
Proof Based on Assumption 5, for any $i = 1, 2, \cdots, N$ and each $t \in \mathbb{R}^+$, one has

$$
\parallel f_i(t, x_i(t)) - f_0(t, x_0(t)) \parallel \\
= \parallel f_0(t, x_i(t)) - f_0(t, x_0(t)) \parallel \\
\leq L_f^i ( \parallel x_i(t) - x_0(t) \parallel ) \\
\leq L_f^i ( \parallel x_i(t) - x_0(t) - x_i^* \parallel + \parallel x_i^* \parallel ) \\
\leq L_f^i ( \sqrt{n} \max \{ \parallel \dot{x}^+(t) \parallel, \parallel \dot{x}^-(t) \parallel \} + \max_{i = 1, 2, \cdots, N} \{ \parallel x_i^* \parallel \} ) .
$$

(5)

Let

$$
Q(t) = L_f^i ( \sqrt{n} \max \{ \parallel \dot{x}^+(t) \parallel, \parallel \dot{x}^-(t) \parallel \} + \max_{i = 1, 2, \cdots, N} \{ \parallel x_i^* \parallel \} ) .
$$

(6)

If $\alpha > Q(t), \forall t \in \mathbb{R}^+$, then $\alpha > \parallel f_i(t, x_i(t)) - f_0(t, x_0(t)) \parallel, \forall t \in \mathbb{R}^+, \forall i = 1, 2, \cdots, N$.

Now, it can be proved that if $\alpha > Q(0)$ then $\alpha > Q(t), \forall t \in \mathbb{R}^+$. Because $\alpha > Q(0)$ and $Q(t)$ are continuously changing, suppose that $t_1 \in \mathbb{R}^+$ is the first time at which $\alpha = Q(t)$. Since $\alpha$, $L_f^i$ and $\max_{i = 1, 2, \cdots, N} \{ \parallel x_i^* \parallel \}$ are constants, one has $\max \{ \parallel \dot{x}^+(t_1) \parallel, \parallel \dot{x}^-(t_1) \parallel \} > \max \{ \parallel \dot{x}^+(0) \parallel, \parallel \dot{x}^-(0) \parallel \}$. So, there must exist a $t_2 \in [0, t_1)$ such that the derivative of $\max \{ \parallel \dot{x}^+(t) \parallel, \parallel \dot{x}^-(t) \parallel \}$ is greater than zero.

Now, consider the following three cases.

- **Case (i):** $\{ \dot{x}^+(t) > 0, \dot{x}^-(t) \geq 0 \}$.

  In this case, $\max \{ \parallel \dot{x}^+(t) \parallel, \parallel \dot{x}^-(t) \parallel \} = \dot{x}^+(t)$, and the derivative of $\max \{ \parallel \dot{x}^+(t) \parallel, \parallel \dot{x}^-(t) \parallel \}$ is $\dot{x}^+(t)$. Since Assumption 1 holds and $\dot{x}^+(t) > 0$, one has $\sum_{r \in \mathbb{N}} a_{ir} [\dot{x}^+(t) - \dot{x}^k_r(t)] > 0$. Thus,

$$
\dot{x}^+(t) \in K \left[ f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha \right].
$$

If the derivative of $\max \{ \parallel \dot{x}^+(t) \parallel, \parallel \dot{x}^-(t) \parallel \}$ is greater than zero at $t_2 \in [0, t_1)$, one has $\dot{x}^+(t_2) > 0$. Then, there must exist $i \in \{1, 2, \cdots, N\}$ and $k \in \{1, 2, \cdots, n\}$ such that $f_i^k(t_2, x_i(t_2)) - f_0^k(t_2, x_0(t_2)) > 0$ and the positive constant $\alpha < \parallel f_i^k(t_2, x_i(t_2)) - f_0^k(t_2, x_0(t_2)) \parallel$. Since $f_i^k(t_2, x_i(t_2)) - f_0^k(t_2, x_0(t_2)) \leq \parallel f_i(t_2, x_i(t_2)) - f_0(t_2, x_0(t_2)) \parallel$, one has $\alpha < \parallel f_i(t_2, x_i(t_2)) - f_0(t_2, x_0(t_2)) \parallel$. It follows that $\alpha < Q(t_2)$ based on (5). Because $\alpha > Q(0)$ and $Q(t)$ are continuously changing, there must be a $t_3 \in [0, t_2)$ such that $\alpha = Q(t_3)$. It contradicts the assumption that $t_1 \in \mathbb{R}^+$ is the first time at which $\alpha = Q(t)$.

- **Case (ii):** $\{ \dot{x}^+(t) \leq 0, \dot{x}^-(t) < 0 \}$.
In this case, \( \max\{|\bar{x}^+(t)|, |\bar{x}^-(t)|\} = -\bar{x}^-(t) \), and the derivative of \( \max\{|\bar{x}^+(t)|, |\bar{x}^-(t)|\} \) is \(-\dot{\bar{x}}^-(t)\). Since Assumption 1 holds and \( \bar{x}^-(t) < 0 \), one has \( \sum_{s \in N_j} a_{js} [\bar{x}^-(t) - \bar{x}^+_s(t)] < 0 \). Thus,

\[
\dot{\bar{x}}^-(t) \in \mathcal{K} \left[ f^l_j(t, x_j(t)) - f^l_0(t, x_0(t)) + \alpha \right].
\]

If the derivative of \( \max\{|\bar{x}^+(t)|, |\bar{x}^-(t)|\} \) is greater than zero at \( t_2 \in [0, t_1) \), one has \( \dot{\bar{x}}^-(t_2) < 0 \). Then, there must exist \( j \in \{1, 2, \ldots, N\} \) and \( l \in \{1, 2, \ldots, n\} \) such that \( f^l_j(t_2, x_j(t_2)) - f^l_0(t_2, x_0(t_2)) < 0 \) and the positive constant \( \alpha < |f^l_j(t_2, x_j(t_2)) - f^l_0(t_2, x_0(t_2))| \). Since \( |f^l_j(t_2, x_j(t_2)) - f^l_0(t_2, x_0(t_2))| \leq ||f^l_j(t_2, x_j(t_2)) - f_0(t_2, x_0(t_2))|| \), one has \( \alpha < ||f^l_j(t_2, x_j(t_2)) - f_0(t_2, x_0(t_2))|| \). It follows that \( \alpha < Q(t_2) \) based on (5). Because \( \alpha > Q(0) \) and \( Q(t) \) are continuously changing, there must be a \( t_3 \in [0, t_2) \) such that \( \alpha = Q(t_3) \). It contradicts the assumption that \( t_1 \in R^+ \) is the first time at which \( \alpha = Q(t) \).

- Case (iii): \( \{\bar{x}^+(t) > 0, \bar{x}^-(t) < 0\} \).
  
  (i) If \( \{\bar{x}^+(t) \geq -\bar{x}^-(t)\} \), then \( \max\{|\bar{x}^+(t)|, |\bar{x}^-(t)|\} = \bar{x}^+(t) \). So, the proof is the same as that in Case (i).

  (ii) If \( \{\bar{x}^+(t) < -\bar{x}^-(t)\} \), then \( \max\{|\bar{x}^+(t)|, |\bar{x}^-(t)|\} = -\bar{x}^-(t) \). So, the proof is the same as that in Case (ii).

Combining the above three cases, it can be concluded that the derivative of \( \max\{|\bar{x}^+(t)|, |\bar{x}^-(t)|\} \) will not be greater than zero. Hence, if \( \alpha > Q(0) \), i.e., Assumption 6 holds, then \( \alpha > Q(t), \forall t \in R^+ \). It follows that \( \alpha > ||f_i(t, x_i(t)) - f_0(t, x_0(t))||, \forall t \in R^+, \forall i = 1, 2, \ldots, N, \) based on (5).

The proof is now completed.

8 Supplementary Lemma ii

**Supplementary Lemma ii** Let \( \mathcal{F} \) denote the set-valued map. If Assumptions 1, 5 and 6 hold, then the set-valued Lie derivative \( \tilde{\mathcal{L}}_\mathcal{F} V \) of \( V \) with respect to \( \mathcal{F} \) satisfies that \( \max \tilde{\mathcal{L}}_\mathcal{F} V < 0 \) for all \( (\bar{x}^+(t), \bar{x}^-(t)) \in \mathcal{D} \setminus \{(0,0)\} \).

**Proof** If Assumptions 1, 5 and 6 hold, then Supplementary Lemma i holds, i.e., \( \alpha > ||f_i(t, x_i(t)) - f_0(t, x_0(t))||, \forall t \in R^+, \forall i = 1, 2, \ldots, N \). Five cases are discussed as follows:

- **Case (i):** \( \bar{x}^+(t) > 0 \) and \( \bar{x}^-(t) > 0 \).
Since $\bar{x}^+(t) > 0$ and Assumption 1 holds, one has $\sum_{r \in N_i} a_{ir} [\bar{x}^+(t) - \bar{x}^+_r(t)] > 0$, and for

$$\partial V(\bar{x}^+(t), \bar{x}^-(t)) = \{(1, 0)\},$$

one has

$$\tilde{L}_F V = \mathcal{K} [f^k_i(t, x_i(t)) - f^k_0(t, x_0(t)) - \alpha].$$

Since $| f^k_i(t, x_i(t)) - f^k_0(t, x_0(t)) | \leq \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in R^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, it follows from Supplementary Lemma i that

$$\max \tilde{L}_F V < 0.$$

- **Case (ii):** $\bar{x}^+(t) > 0$ and $\bar{x}^-(t) < 0$.

Since $\bar{x}^+(t) > 0, \bar{x}^-(t) < 0$ and Assumption 1 holds, one has $\sum_{r \in N_i} a_{ir} [\bar{x}^+(t) - \bar{x}^+_r(t)] > 0, \sum_{s \in N_j} a_{js} [\bar{x}^-(t) - \bar{x}^-_s(t)] < 0$, and for

$$\partial V(\bar{x}^+(t), \bar{x}^-(t)) = \{(1, -1)\},$$

one has

$$\tilde{L}_F V = \mathcal{K} \left[ (f^k_i(t, x_i(t)) - f^k_0(t, x_0(t)) - \alpha) - (f^l_j(t, x_j(t)) - f^l_0(t, x_0(t)) + \alpha) \right].$$

Since $| f^k_i(t, x_i(t)) - f^k_0(t, x_0(t)) | \leq \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in R^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, it follows from Supplementary Lemma i that

$$\max \tilde{L}_F V < 0.$$

- **Case (iii):** $\bar{x}^+(t) < 0$ and $\bar{x}^-(t) < 0$.

Since $\bar{x}^-(t) < 0$ and Assumption 1 holds, one has $\sum_{s \in N_j} a_{js} [\bar{x}^-(t) - \bar{x}^-_s(t)] < 0$, and for

$$\partial V(\bar{x}^+(t), \bar{x}^-(t)) = \{(0, -1)\},$$

one has

$$\tilde{L}_F V = \mathcal{K} \left[ -(f^l_j(t, x_j(t)) - f^l_0(t, x_0(t)) + \alpha) \right].$$

Since $| f^l_j(t, x_j(t)) - f^l_0(t, x_0(t)) | \leq \| f_j(t, x_j(t)) - f_0(t, x_0(t)) \|, \forall t \in R^+, \forall j = 1, 2, \cdots, N, \forall l = 1, 2, \cdots, n$, it follows from Supplementary Lemma i that

$$\max \tilde{L}_F V < 0.$$
• **Case (iv):** $\bar{x}^+(t) > 0$ and $\bar{x}^-(t) = 0$.

Since $\bar{x}^+(t) > 0, \bar{x}^-(t) = 0$ and Assumption 1 holds, one has $\sum_{r \in \mathcal{N}_i} a_{ir} [\bar{x}^+(t) - \bar{x}_i^k(t)] > 0, \sum_{s \in \mathcal{N}_j} a_{js} [\bar{x}^-(t) - \bar{x}_s^0(t)] \leq 0$. So, if $v \in \mathcal{F}(\bar{x}^+(t), \bar{x}^-(t))$, then $v^T = (v_1, v_2)$ with $v_1 \in \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha]$ and $v_2 \in \mathcal{K}[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha] \cup \mathcal{K}[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t))].$ For

$$\partial V(\bar{x}^+(t), \bar{x}^-(t)) = \{1\} \times [-1, 0],$$

if $\zeta \in \partial V(\bar{x}^+(t), \bar{x}^-(t))$, then $\zeta^T = (1, y)$ with $y \in [-1, 0]$. Therefore,

$$\zeta^T v = v_1 + y v_2.$$

If there exists an element $a$ satisfying that $\zeta^T v = a$ for all $y \in [-1, 0]$, then $v_2 = 0$. So, if $v_2 \neq 0$, one has $\hat{L}_\mathcal{F} V = \emptyset$; if $v_2 = 0$, one has $\hat{L}_\mathcal{F} V = \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha]$, and then it follows from Supplementary Lemma i that $\max \hat{L}_\mathcal{F} V < 0$. Thus, $\max \hat{L}_\mathcal{F} V < 0$ or $\hat{L}_\mathcal{F} V = \emptyset$ in this case.

• **Case (v):** $\bar{x}^+(t) = 0$ and $\bar{x}^-(t) < 0$.

Since $\bar{x}^+(t) = 0, \bar{x}^-(t) < 0$ and Assumption 1 holds, one has $\sum_{r \in \mathcal{N}_i} a_{ir} [\bar{x}^+(t) - \bar{x}_i^k(t)] \geq 0, \sum_{s \in \mathcal{N}_j} a_{js} [\bar{x}^-(t) - \bar{x}_s^0(t)] < 0$. So, if $v \in \mathcal{F}(\bar{x}^+(t), \bar{x}^-(t))$, then $v^T = (v_1, v_2)$ with $v_1 \in \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha] \cup \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t))]$ and $v_2 \in \mathcal{K}[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha].$ For

$$\partial V(\bar{x}^+(t), \bar{x}^-(t)) = [0, 1] \times \{-1\},$$

if $\zeta \in \partial V(\bar{x}^+(t), \bar{x}^-(t))$, then $\zeta^T = (y, -1)$ with $y \in [0, 1]$. Therefore,

$$\zeta^T v = y v_1 - v_2.$$

If there exists an element $a$ satisfying that $\zeta^T v = a$ for all $y \in [0, 1]$, then $v_1 = 0$. So, if $v_1 \neq 0$, one has $\hat{L}_\mathcal{F} V = \emptyset$; if $v_1 = 0$, one has $\hat{L}_\mathcal{F} V = -\mathcal{K}[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha]$, and then it follows from Supplementary Lemma i that $\max \hat{L}_\mathcal{F} V < 0$. Thus, $\max \hat{L}_\mathcal{F} V < 0$ or $\hat{L}_\mathcal{F} V = \emptyset$ in this case.

Combining the above five cases, it can be concluded that $\max \hat{L}_\mathcal{F} V < 0$ for all $(\bar{x}^+(t), \bar{x}^-(t)) \in \mathcal{D} \setminus \{(0,0)\}$.

The proof is now completed.
9 Proof of Corollary 1

Proof The nonsmooth function $V$, which was given by (6) in the manuscript, is chosen as the Lyapunov function. If Assumptions 1, 5 and 6 hold, then Supplementary Lemma ii holds. By using Lemma 1, it follows from Lemmas 3 - 5 and Supplementary Lemma ii that $(\tilde{x}^+(t), \tilde{x}^-(t)) = (0, 0)$ is a globally stable equilibrium point for system (2).

Next, the maximal converging time is considered.

- **Case (i):** $\tilde{x}^+(t) > 0$ and $\tilde{x}^-(t) \geq 0$.

In this case, $V = \tilde{x}^+(t)$ and $\tilde{L}_x V = K \left[ f^k_i(t, x_i(t)) - f^k_0(t, x_0(t)) - \alpha \right]$. By the proof of Supplementary Lemma i, one has $\| f_i(t, x_i(t)) - f_0(t, x_0(t)) \| \leq Q(t), Q(t) \leq Q(0), \forall t \in R^+, \forall i = 1, 2, \cdots, N$. Since $| f^k_i(t, x_i(t)) - f^k_0(t, x_0(t)) | \leq \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in R^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, one has

$$\max \tilde{L}_x V \leq -(\alpha - Q(t))$$

Therefore, the converging time satisfies

$$T_1 \leq \frac{1}{\alpha - Q(0)} \tilde{x}^+(0).$$

- **Case (ii):** $\tilde{x}^+(t) > 0$ and $\tilde{x}^-(t) < 0$.

In this case, $V = \tilde{x}^+(t) - \tilde{x}^-(t)$ and $\tilde{L}_x V = K \left[ (f^k_i(t, x_i(t)) - f^k_0(t, x_0(t)) - \alpha - (f^k_j(t, x_j(t)) - f_0^j(t, x_0(t)) + \alpha) \right]$. By the proof of Supplementary Lemma i, one has $\| f_i(t, x_i(t)) - f_0(t, x_0(t)) \| \leq Q(t), Q(t) \leq Q(0), \forall t \in R^+, \forall i = 1, 2, \cdots, N$. Since $| f^k_i(t, x_i(t)) - f^k_0(t, x_0(t)) | \leq \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in R^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, one has

$$\max \tilde{L}_x V \leq -2(\alpha - Q(t))$$

Therefore, the converging time satisfies

$$T_2 \leq \frac{1}{2(\alpha - Q(0))}(\tilde{x}^+(0) - \tilde{x}^-(0))$$

$$\leq \frac{1}{\alpha - Q(0)} \max \{\tilde{x}^+(0), -\tilde{x}^-(0)\}.$$
In this case, \( V = -\ddot{x} - (t) \) and \( L_{\nu} V = K \left[ -(f_j(t, x_j(t)) - f_0(t, x_0(t)) + \alpha) \right] \). By the proof of Supplementary Lemma i, one has \( \| f_j(t, x_j(t)) - f_0(t, x_0(t)) \| \leq Q(t), Q(t) \leq Q(0), \forall t \in R^+, \forall j = 1, 2, \cdots, N \). Since \( \| f_j(t, x_j(t)) - f_0(t, x_0(t)) \| \leq \| f_j(t, x_j(t)) - f_0(t, x_0(t)) \|, \forall t \in R^+, \forall j = 1, 2, \cdots, N, \forall l = 1, 2, \cdots, n \), one has

\[
\max L_{\nu} V \leq -(\alpha - Q(t)) \leq -(\alpha - Q(0)).
\]

Therefore, the converging time satisfies

\[
T_3 \leq -\frac{1}{\alpha - Q(0)} \ddot{x}(0).
\]

Combining the above three cases, the maximal converging time is obtained as

\[
T = -\frac{1}{\alpha - Q(0)} \max_{i \in 1, 2, \cdots, N} \left\{ | x_i^k(0) - x_i^k(0) - x_i^k | \right\}.
\]

The proof is now completed.

10 Proof of Lemma 7

**Proof** Define the formation position errors \( \tilde{r}_i(t) = r_i(t) - r_0(t) - r_i^* \) and the velocity errors \( \tilde{v}_i(t) = v_i(t) - v_0(t), i = 1, 2, \cdots, N \), with \( \tilde{r}_0(t) = 0 \) and \( \tilde{v}_0(t) = 0 \). Sliding mode is designed as \( S_i(t) = \tilde{r}_i(t) + \tilde{v}_i(t) \). The Filippov solution of \( S_i(t) \) is defined as the absolutely continuous solution of the differential inclusion

\[
\dot{S}_i(t) \in K \left[ F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) - \alpha \sgn \left\{ \sum_{j \in N_i} a_{ij} [S_i(t) - S_j(t)] \right\} \right],
\]

\( \forall i = 1, 2, \cdots, N \).

Based on Assumption 1, one follower must receive information from other followers or the leader, in other words, it is connected with other followers or the leader. Define \( S^+(t) \) as the maximal error component which is connected with non-maximal error components of the followers or connected with the component of the leader. Similarly, define \( S^-(t) \) as the minimal error component which is connected with non-minimal error components of the followers or connected with the component of the leader. Suppose that, at any time \( t \), \( S^+(t) \) is the \( k \)th error component of agent
Based on Assumptions 7 and 8, for any $i, j \in \{1, 2, \ldots, N\}$, $k, l \in \{1, 2, \ldots, n\}$.

The Filippov solutions of $S^+(t)$ and $S^-(t)$ can be described by

$$
\begin{align*}
\dot{S}^+(t) & \in \mathcal{K} \left[ F^k_i(t, r_i(t), v_i(t)) - F^k_0(t, r_0(t), v_0(t)) - \alpha \text{ sgn} \left\{ \sum_{r \in \mathcal{N}_i} a_{ir}[S^+(t) - S^k_r(t)] \right\} \right], \\
\dot{S}^-(t) & \in \mathcal{K} \left[ F^l_j(t, r_j(t), v_j(t)) - F^l_0(t, r_0(t), v_0(t)) - \alpha \text{ sgn} \left\{ \sum_{s \in \mathcal{N}_j} a_{js}[S^-(t) - S^l_s(t)] \right\} \right].
\end{align*}
$$

(7)

Based on Assumptions 7 and 8, for any $i = 1, 2, \ldots, N$ and each $t \in \mathbb{R}^+$, one has

$$
\begin{align*}
& \| F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \| \\
= & \| f_i(t, r_i(t), v_i(t)) + v_i(t) - f_0(t, r_0(t), v_0(t)) - v_0(t) \| \\
= & \| f_i(t, r_i(t), v_i(t)) - f_i(t, r_0(t), v_0(t)) + f_i(t, r_0(t), v_0(t)) - f_0(t, r_0(t), v_0(t)) + v_i(t) - v_0(t) \| \\
\leq & \| f_i(t, r_i(t), v_i(t)) - f_i(t, r_0(t), v_0(t)) \| + \| f_i(t, r_0(t), v_0(t)) \| \\
& + \| f_0(t, r_0(t), v_0(t)) - f_0(t, r_i^E, v_i^E) \| \\
& + \| f_0(t, r_0(t), v_0(t)) - f_0(t, r_0^E, v_0^E) \| + \| v_i(t) - v_0(t) \| \\
\leq & L_j^F (\| r_i(t) - r_0(t) \| + \| v_i(t) - v_0(t) \|) + L_j^F (\| r_0(t) - r_i^E \| + \| v_0(t) - v_i^E \|) \\
& + L_j^L (\| r_0(t) - r_0^E \| + \| v_0(t) - v_0^E \|) + (\| v_i(t) - v_0(t) \|) \\
\leq & L_j^F (\| r_i(t) - r_0(t) - r_i^* \| + \| v_i(t) - v_0(t) \|) + L_j^F (\| r_0(t) - r_i^E \| + \| v_0(t) - v_i^E \|) \\
& + L_j^L (\| r_0(t) - r_0^E \| + \| v_0(t) - v_0^E \|) + (\| v_i(t) - v_0(t) \|) \\
\leq & L_j^F \| \tilde{r}_i(t) \| + (L_j^F + 1) \| \tilde{v}_i(t) \| + L_j^F (\| r_i^* \| + \| v_i^E \| + \| v_0^E \| + \beta_r + \beta_v) \\
& + L_j^L (\| r_0^E \| + \| v_0^E \| + \beta_r + \beta_v). \\
\end{align*}
$$

Let

$$
G = L_j^F \left( \max_{i=1,2,\ldots,N} \left\{ \| r_i^* \| + \| v_i^E \| + \| v_0^E \| \right\} + \beta_r + \beta_v \right) + L_j^L (\| r_0^E \| + \| v_0^E \| + \beta_r + \beta_v).
$$

Clearly, $G$ is a constant. Since $\tilde{v}_i(t) = S_i(t) - \tilde{r}_i(t)$, one has $\| \tilde{v}_i(t) \| \leq \| S_i(t) \| + \| \tilde{r}_i(t) \|$.

Thus,
\[ \| F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \| \]
\[ \leq L_{ij}^F \| \tilde{r}_i(t) \| + (L_{ij}^F + 1) \| (S_i(t) - \tilde{r}_i(t)) \| + G \]
\[ \leq (2L_{ij}^F + 1) \| \tilde{r}_i(t) \| + (L_{ij}^F + 1) \| S_i(t) \| + G \]
\[ \leq (2L_{ij}^F + 1) \sqrt{n} \max_{i=1,2,\ldots,N} \{ |\tilde{r}_i^k(t) |, |S_i^k(t) | \} + (L_{ij}^F + 1) \sqrt{n} \max_{i=1,2,\ldots,N} \{ |S_i^k(t) | \} + G \quad (8) \]

Let
\[ M(t) = (2L_{ij}^F + 1) \sqrt{n} \max_{i=1,2,\ldots,N} \{ |\tilde{r}_i^k(t) |, |S_i^k(t) | \} + (L_{ij}^F + 1) \sqrt{n} \max_{i=1,2,\ldots,N} \{ |S_i^k(t) | \} + G. \]

If \( \alpha > M(t), \forall t \in \mathbb{R}^+ \), then \( \alpha > \| F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \|, \forall t \in \mathbb{R}^+, \forall i = 1, 2, \ldots, N. \)

Now, it can be proved that if \( \alpha > M(0) \) then \( \alpha > M(t), \forall t \in \mathbb{R}^+ \). Because \( \alpha > M(0) \) and \( M(t) \) are continuously changing, suppose that \( t_1 \in \mathbb{R}^+ \) is the first time at which \( \alpha = M(t) \). Thus, \( M(t_1) > M(0) \).

Now, consider the following two cases.

- **Case (i):** The signs of \( \tilde{r}_i^k(t) \) and \( \tilde{v}_i^k(t) \) are the same. In this case, \( |\tilde{r}_i^k(t) | \) will increase. Since \( |\tilde{r}_i^k(t) | + |\tilde{v}_i^k(t) | = |\tilde{r}_i^k(t) + \tilde{v}_i^k(t) | = |S_i^k(t) | \), it follows that \( |\tilde{r}_i^k(t) | \leq |S_i^k(t) | \).

- **Case (ii):** The signs of \( \tilde{r}_i^k(t) \) and \( \tilde{v}_i^k(t) \) are opposite. In this case, one has \( |\tilde{r}_i^k(t) | + |\tilde{v}_i^k(t) | = |\tilde{r}_i^k(t) - \tilde{v}_i^k(t) | \geq |S_i^k(t) | \). Both \( |\tilde{r}_i^k(t) | \leq |S_i^k(t) | \) and \( |\tilde{r}_i^k(t) | \geq |S_i^k(t) | \) are possible. For \( \tilde{v}_i^k(t) = \tilde{r}_i^k(t) \) and their signs are opposite, \( |\tilde{r}_i^k(t) | \) must decrease.

Combining the above two cases, it can be concluded that \( |\tilde{r}_i^k(t) | \) must be decreasing when \( |\tilde{r}_i^k(t) | \geq |S_i^k(t) | \). Since \( \alpha, L_{ij}^F \) and \( G \) are constants, if \( M(t_1) > M(0) \), then \( \max_{i=1,2,\ldots,N} \{ |S_i^k(t_1) | \} \) must be larger than \( \max_{i=1,2,\ldots,N} \{ |S_i^k(0) | \} \). So, there must exist a \( t_2 \in [0, t_1) \) such that the derivative of \( \max\{|S^+(t) |, |S^-(t) | \} \) is greater than zero.

Now, consider the following three cases.

- **Case (i):** \( \{S^+(t) > 0, S^-(t) \geq 0\} \).

  In this case, \( \max\{|S^+(t) |, |S^-(t) | \} = S^+(t) \), and the derivative of \( \max\{|S^+(t) |, |S^-(t) | \} \) is \( \dot{S}^+(t) \). Since Assumption 1 holds and \( S^+(t) > 0 \), one has \( \sum_{r \in \mathcal{M}_l} a_{ir}[S^+(t) - S^k_i] > 0 \). Thus,
\[
\dot{S}^+(t) = \mathcal{K} \left[ F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) - \alpha \right].
\]

If the derivative of \( \max\{|S^+(t) |, |S^-(t) | \} \) is greater than zero at \( t_2 \in [0, t_1) \), one has \( \dot{S}^+(t_2) > 0 \).

Then, there must exist \( i \in \{1, 2, \ldots, N\} \) and \( k \in \{1, 2, \ldots, n\} \) such that \( F_i^k(t_2, r_i(t_2), v_i(t_2)) - F_0^k(t_2, r_0(t_2), v_0(t_2)) - \alpha \)
$F^k_0(t_2, r_0(t_2), v_0(t_2)) > 0$ and the positive constant $\alpha < \mid F^k_i(t_2, r_i(t_2), v_i(t_2)) - F^k_0(t_2, r_0(t_2), v_0(t_2)) \mid$. Since $\mid F^k_i(t_2, r_i(t_2), v_i(t_2)) - F^k_0(t_2, r_0(t_2), v_0(t_2)) \mid \leq \| F_i(t_2, r_i(t_2), v_i(t_2)) - F_0(t_2, r_0(t_2), v_0(t_2)) \|$, one has $\alpha < \| F_i(t_2, r_i(t_2), v_i(t_2)) - F_0(t_2, r_0(t_2), v_0(t_2)) \|$. It follows that $\alpha < M(t_2)$ based on (8). Because $\alpha > M(0)$ and $M(t)$ are continuously changing, there must be a $t_3 \in [0, t_2)$ such that $\alpha = M(t_3)$. It contradicts the assumption that $t_1 \in R^+$ is the first time at which $\alpha = M(t)$.

- **Case (ii):** $\{S^+(t) \leq 0, S^-(t) < 0\}$.

In this case, $\max\{|S^+(t)|, |S^-(t)|\} = -S^-(t)$, and the derivative of $\max\{|S^+(t)|, |S^-(t)|\}$ is $-\dot{S}^-(t)$. Since Assumption 1 holds and $S^-(t) < 0$, one has $\sum_{s \in \mathcal{N}_j} a_{js} [S^-(t) - S^+_j(t)] < 0$. Thus,

$$\dot{S}^-(t) \in \mathcal{K} \left[ F^l_j(t, r_j(t), v_j(t)) - F^l_0(t, r_0(t), v_0(t)) + \alpha \right].$$

If the derivative of $\max\{|S^+(t)|, |S^-(t)|\}$ is greater than zero at $t_2 \in [0, t_1)$, one has $\dot{S}^-(t_2) < 0$. Then, there must exist $j \in \{1, 2, \cdots, N\}$ and $l \in \{1, 2, \cdots, n\}$ such that $F^l_j(t_2, r_j(t_2), v_j(t_2)) - F^l_0(t_2, r_0(t_2), v_0(t_2)) < 0$ and the positive constant $\alpha < \mid F^l_j(t_2, r_j(t_2), v_j(t_2)) - F^l_0(t_2, r_0(t_2), v_0(t_2)) \mid$. Since $\mid F^l_j(t_2, r_j(t_2), v_j(t_2)) - F^l_0(t_2, r_0(t_2), v_0(t_2)) \mid \leq \| F_j(t_2, r_j(t_2), v_j(t_2)) - F_0(t_2, r_0(t_2), v_0(t_2)) \|$, one has $\alpha < \| F_j(t_2, r_j(t_2), v_j(t_2)) - F_0(t_2, r_0(t_2), v_0(t_2)) \|$. It follows that $\alpha < M(t_2)$ based on (8). Because $\alpha > M(0)$ and $M(t)$ are continuously changing, there must be a $t_3 \in [0, t_2)$ such that $\alpha = M(t_3)$. It contradicts the assumption that $t_1 \in R^+$ is the first time at which $\alpha = M(t)$.

- **Case (iii):** $\{S^+(t) > 0, S^-(t) < 0\}$.

(a) If $\{S^+(t) \geq -S^-(t)\}$, then $\max\{|S^+(t)|, |S^-(t)|\} = S^+(t)$. So, the proof is the same as that in Case (i).

(b) If $\{S^+(t) < -S^-(t)\}$, then $\max\{|S^+(t)|, |S^-(t)|\} = -S^-(t)$. So, the proof is the same as that in Case (ii).

Combining the above three cases, it can be concluded that the derivative of $\max\{|S^+(t)|, |S^-(t)|\}$ will not be greater than zero. Hence, if $\alpha > M(0)$, i.e., Assumption 9 holds, then $\alpha > M(t)$, $\forall t \in R^+$. It follows that $\alpha > \| F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \|$, $\forall t \in R^+, \forall i = 1, 2, \cdots, N$, based on (8).

The proof is now completed.
11 Proof of Lemma 8

Proof If Assumptions 1 and 7 - 9 hold, then Lemma 7 holds, i.e., \( \alpha > \| F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \| , \forall t \in \mathbb{R}^+ , \forall i = 1, 2, \cdots , N \). Based on (6) in the manuscript, the nonsmooth function \( V(S^+(t), S^-(t)) : \mathbb{R}^2 \rightarrow \mathbb{R} \) is

\[
V(S^+(t), S^-(t)) = \begin{cases}
S^+(t) & S^+(t) \geq 0, S^-(t) \geq 0 \\
S^+(t) - S^-(t) & S^+(t) > 0, S^-(t) < 0 \\
-S^-(t) & S^+(t) \leq 0, S^-(t) < 0.
\end{cases}
\] (9)

Five cases are discussed as follows:

- **Case (i):** \( S^+(t) > 0 \) and \( S^-(t) > 0 \).

Since Assumption 1 holds and \( S^+(t) > 0 \), one has \( \sum_{r \in \mathcal{N}_i} a_{ir}[S^+(t) - S^k_r(t)] > 0 \), and for

\[
\partial V(S^+(t), S^-(t)) = \{(1,0)\},
\]

one has

\[
\tilde{L}_F V = K \left[ F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) - \alpha \right].
\]

Since \( | F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) | \leq | F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) |, \forall t \in \mathbb{R}^+ , \forall i = 1, 2, \cdots , N, \forall k = 1, 2, \cdots , n \), it follows from Lemma 7 that

\[
\max \tilde{L}_F V < 0.
\]

- **Case (ii):** \( S^+(t) > 0 \) and \( S^-(t) < 0 \).

Since \( S^+(t) > 0, S^-(t) < 0 \) and Assumption 1 holds, one has \( \sum_{r \in \mathcal{N}_i} a_{ir}[S^+(t) - S^k_r(t)] > 0, \sum_{s \in \mathcal{N}_j} a_{js}[S^-(t) - S^k_s(t)] < 0 \), and for

\[
\partial V(S^+(t), S^-(t)) = \{(1,-1)\},
\]

one has

\[
\tilde{L}_F V = K \left[ (F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) - \alpha) \\
- (F_j^l(t, r_j(t), v_j(t)) - F_0^l(t, r_0(t), v_0(t)) + \alpha) \right].
\]

Since \( | F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) | \leq | F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) |, \forall t \in \mathbb{R}^+ , \forall i = 1, 2, \cdots , N, \forall k = 1, 2, \cdots , n \), it follows from Lemma 7 that

\[
\max \tilde{L}_F V < 0.
\]
• Case (iii): $S^+(t) < 0$ and $S^-(t) < 0$.

Since $S^-(t) < 0$ and Assumption 1 holds, one has $\sum_{a_{js}[S^-(t) - S^t_s(t)] < 0$, and for

$$\partial V(S^+(t), S^-(t)) = \{(0, -1)\},$$

one has

$$\hat{\mathcal{L}}_F V = \mathcal{K}[-(F^t_j(t, r_j(t), v_j(t)) - F^0_0(t, r_0(t), v_0(t)) + \alpha)].$$

Since $| F^t_j(t, r_j(t), v_j(t)) - F^0_0(t, r_0(t), v_0(t)) | \leq \| F_j(t, r_j(t), v_j(t)) - F^0_0(t, r_0(t), v_0(t)) \|, \forall t \in R^+, \forall j = 1, 2, \cdots, N, \forall l = 1, 2, \cdots, n$, it follows from Lemma 7 that

$$\max \hat{\mathcal{L}}_F V < 0.$$

• Case (iv): $S^+(t) > 0$ and $S^-(t) = 0$.

Since $S^+(t) > 0, S^-(t) = 0$ and Assumption 1 holds, one has $\sum_{r \in N_i} a_{ir}[S^+(t) - S^t_i(t)] > 0, \sum_{s \in N_i} a_{js}[S^-(t) - S^t_s(t)] \leq 0$. So, if $v \in \mathcal{F}(S^+(t), S^-(t))$, then $v^T = (v_1, v_2)$ with $v_1 \in \mathcal{K}[F^t_i(t, r_i(t), v_i(t)) - F^0_0(t, r_0(t), v_0(t)) - \alpha]$ and $v_2 \in \mathcal{K}[F^t_j(t, r_j(t), v_j(t)) - F^0_0(t, r_0(t), v_0(t)) + \alpha] \cup \mathcal{K}[F^t_j(t, r_j(t), v_j(t)) - F^0_0(t, r_0(t), v_0(t))].$ For

$$\partial V(S^+(t), S^-(t)) = \{1\} \times [-1, 0],$$

if $\zeta \in \partial V(S^+(t), S^-(t))$, then $\zeta^T = (1, y)$ with $y \in [-1, 0]$. Therefore,

$$\zeta^T v = v_1 + yv_2.$$

If there exists an element $a$ satisfying that $\zeta^T v = a$ for all $y \in [-1, 0]$, then $v_2 = 0$. So, if $v_2 \neq 0$, one has $\hat{\mathcal{L}}_F V = \emptyset$; if $v_2 = 0$, one has $\hat{\mathcal{L}}_F V = \mathcal{K}[F^t_i(t, r_i(t), v_i(t)) - F^0_0(t, r_0(t), v_0(t)) - \alpha]$ and then it follows from Lemma 7 that $\max \hat{\mathcal{L}}_F V < 0$. Thus, $\max \hat{\mathcal{L}}_F V < 0$ or $\hat{\mathcal{L}}_F V = \emptyset$ in this case.

• Case (v): $S^+(t) = 0$ and $S^-(t) < 0$.

Since $S^+(t) = 0, S^-(t) < 0$ and Assumption 1 holds, one has $\sum_{r \in N_i} a_{ir}[S^+(t) - S^t_i(t)] \geq 0, \sum_{s \in N_i} a_{js}[S^-(t) - S^t_s(t)] < 0$. So, if $v \in \mathcal{F}(S^+(t), S^-(t))$, then $v^T = (v_1, v_2)$ with $v_1 \in \mathcal{K}[F^t_i(t, r_i(t), v_i(t)) - F^0_0(t, r_0(t), v_0(t)) - \alpha] \cup \mathcal{K}[F^t_j(t, r_j(t), v_j(t)) - F^0_0(t, r_0(t), v_0(t))]$ and $v_2 \in \mathcal{K}[F^t_j(t, r_j(t), v_j(t)) - F^0_0(t, r_0(t), v_0(t)) + \alpha].$ For

$$\partial V(S^+(t), S^-(t)) = [0, 1] \times \{-1\},$$

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if \( \zeta \in \partial V(S^+(t), S^-(t)) \), then \( \zeta^T = (y, -1) \) with \( y \in [0, 1] \). Therefore,

\[
\zeta^T v = yv_1 - v_2.
\]

If there exists an element \( a \) satisfying that \( \zeta^T v = a \) for all \( y \in [0, 1] \), then \( v_1 = 0 \). So, if \( v_1 \neq 0 \), one has \( \tilde{L}_F V = \emptyset \); if \( v_1 = 0 \), one has \( \tilde{L}_F V = -K[L_f(t, r_j(t), v_j(t)) - L_0(t, r_0(t), v_0(t)) + \alpha] \), and then it follows from Lemma 7 that \( \max \tilde{L}_F V < 0 \). Thus, \( \max \tilde{L}_F V < 0 \) or \( \tilde{L}_F V = \emptyset \) in this case.

Combining the above five cases, it can be concluded that \( \max \tilde{L}_F V < 0 \) for all \( (S^+(t), S^-(t)) \in D \setminus \{(0, 0)\} \).

The proof is now completed.

12 Proof of Theorem 2

**Proof** The nonsmooth function \( V(S^+(t), S^-(t)) \) in (9) is chosen as the Lyapunov function. If Assumptions 1 and 7 - 9 hold, then Lemma 8 holds. By Lemma 1, it follows from Lemmas 3 - 5 and 8 that \( (S^+(t), S^-(t)) = (0, 0) \) is a globally stable equilibrium point for system (7).

Solving

\[
S^+_i(t) = \tilde{r}_i^k(t) + \tilde{v}_i^k(t) = 0,
\]

one has

\[
\tilde{r}_i^k(t) = c e^{-t}, \tilde{v}_i^k(t) = -c e^{-t},
\]

where \( c \) is a constant determined by the initial conditions. Therefore, the errors \( \tilde{r}_i(t) \) and \( \tilde{v}_i(t) \) converge to zero exponentially; that is, the second-order multi-agent system achieves the desired formation asymptotically.

The proof is now completed.

13 Supplementary Lemma iii

**Supplementary Lemma iii** If Assumptions 1, 10 and 11 hold, then \( \alpha > \| F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \|, \forall t \in R^+, \forall i = 1, 2, \cdots, N \), where \( F_i(t, r_i(t), v_i(t)) = v_i(t) + f_i(t, r_i(t), v_i(t)) \)
and $F_0(t, r_0(t), v_0(t)) = v_0(t) + f_0(t, r_0(t), v_0(t))$.

**Proof** Based on Assumption 10, for any $i = 1, 2, \ldots, N$ and each $t \in R^+$, one has

$$\| F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \|$$

$$= \| f_i(t, r_i(t), v_i(t)) + v_i(t) - f_0(t, r_0(t), v_0(t)) - v_0(t) \|$$

$$= \| f_0(t, r_i(t), v_i(t)) - f_0(t, r_0(t), v_0(t)) + v_i(t) - v_0(t) \|$$

$$\leq \| f_0(t, r_i(t), v_i(t)) - f_0(t, r_0(t), v_0(t)) \| + \| v_i(t) - v_0(t) \|$$

$$\leq L_f^L(\| r_i(t) - r_0(t) \| + \| v_i(t) - v_0(t) \|) + \| v_i(t) - v_0(t) \|$$

$$\leq L_f^L(\| r_i(t) - r_0(t) - r_i^* \| + \| r_i^* \| + \| v_i(t) - v_0(t) \|) + \| v_i(t) - v_0(t) \|$$

$$\leq L_f^L(\| \tilde{v}_i(t) \| + \| r_i^* \| + \| \tilde{v}_i(t) \|) + \| \tilde{v}_i(t) \|$$

Since $\tilde{v}_i(t) = S_i(t) - \tilde{r}_i(t)$, one has $\| \tilde{v}_i(t) \| \leq \| S_i(t) \| + \| \tilde{r}_i(t) \|$. Thus,

$$\| F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \|$$

$$\leq (2L_f^L + 1) \| \tilde{v}_i(t) \| + (L_f^L + 1) \| S_i(t) \| + L_f^L \| r_i^* \|$$

$$\leq (2L_f^L + 1)\sqrt{n} \max_{i=1,2,\ldots,N} \max_{k=1,2,\ldots,n} \{ \| \tilde{r}_i^k(t) \|, \| S_i^k(t) \| \}$$

$$+ (L_f^L + 1)\sqrt{n} \max_{i=1,2,\ldots,N} \max_{k=1,2,\ldots,n} \{ \| S_i^k(t) \| \} + L_f^L \max_{i=1,2,\ldots,N} \{ \| r_i^* \| \}.$$  \hspace{1cm} (10)

Let

$$W(t) = (2L_f^L + 1)\sqrt{n} \max_{i=1,2,\ldots,N} \max_{k=1,2,\ldots,n} \{ \| \tilde{r}_i^k(t) \|, \| S_i^k(t) \| \}$$

$$+ \frac{(L_f^L + 1)\sqrt{n} \max_{i=1,2,\ldots,N} \max_{k=1,2,\ldots,n} \{ \| S_i^k(t) \| \} + L_f^L \max_{i=1,2,\ldots,N} \{ \| r_i^* \| \}.}{}$$

If $\alpha > W(t), \forall t \in R^+$, then $\alpha > \| F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \|, \forall t \in R^+, \forall i = 1, 2, \ldots, N$.

Now, it can be proved that if $\alpha > W(0)$ then $\alpha > W(t), \forall t \in R^+$. Because $\alpha > W(0)$ and $W(t)$ are continuously changing, suppose that $t_1 \in R^+$ is the first time at which $\alpha = W(t)$. Thus, $W(t_1) > W(0)$.

Now, consider the following two cases.

- **Case (i):** The signs of $\tilde{r}_i^k(t)$ and $\tilde{v}_i^k(t)$ are the same. In this case, $| \tilde{r}_i^k(t) |$ will increase. Since $| \tilde{r}_i^k(t) | + | \tilde{v}_i^k(t) | = | \tilde{r}_i^k(t) + \tilde{v}_i^k(t) | = | S_i^k(t) |$, it follows that $| \tilde{r}_i^k(t) | \leq | S_i^k(t) |$. 

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• Case (ii): The signs of \( \tilde{r}_i^k(t) \) and \( \tilde{v}_i^k(t) \) are opposite. In this case, one has \( |\tilde{r}_i^k(t)| + |\tilde{v}_i^k(t)| = |\tilde{r}_i^k(t) - \tilde{v}_i^k(t)| \geq |S_i^k(t)| \). Both \( |\tilde{r}_i^k(t)| \leq |S_i^k(t)| \) and \( |\tilde{r}_i^k(t)| \geq |S_i^k(t)| \) are possible. For \( \tilde{v}_i^k(t) = \tilde{r}_i^k(t) \) and their signs are opposite, \( |\tilde{r}_i^k(t)| \) must decrease.

Combining the above two cases, it can be concluded that \( |\tilde{r}_i^k(t)| \) must be decreasing when \( |\tilde{r}_i^k(t)| \geq |S_i^k(t)| \). Since \( \alpha, L_f^i \) and \( \max_{i=1,2,\cdots,N} \{ ||r_i^*|| \} \) are constants, if \( W(t_1) > W(0) \), then \( \max_{i=1,2,\cdots,N} \{|S_i^k(t_1)|\} \) must be larger than \( \max_{i=1,2,\cdots,N} \{|S_i^k(0)|\} \). So, there must exist a \( t_2 \in [0, t_1] \) such that the derivative of \( \max\{|S^+(t)|, |S^-(t)|\} \) is greater than zero.

Now, consider the following three cases.

• Case (i): \( \{S^+(t) > 0, S^-(t) \geq 0\} \).

In this case, \( \max\{|S^+(t)|, |S^-(t)|\} = S^+(t) \), and the derivative of \( \max\{|S^+(t)|, |S^-(t)|\} \) is \( \tilde{S}^+(t) \). Since Assumption 1 holds and \( S^+(t) > 0 \), one has \( \sum_{r \in N_j} a_{ir} [S^+(t) - S_i^k(t)] > 0 \). Thus,

\[
\tilde{S}^+(t) \in K \left[ F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) - \alpha \right].
\]

If the derivative of \( \max\{|S^+(t)|, |S^-(t)|\} \) is greater than zero at \( t_2 \in [0, t_1] \), one has \( \tilde{S}^+(t_2) > 0 \). Then, there must exist \( i \in \{1, 2, \cdots, N\} \) and \( k \in \{1, 2, \cdots, n\} \) such that \( F_i^k(t_2, r_i(t_2), v_i(t_2)) - F_0^k(t_2, r_0(t_2), v_0(t_2)) > 0 \) and the positive constant \( \alpha < |F_i^k(t_2, r_i(t_2), v_i(t_2)) - F_0^k(t_2, r_0(t_2), v_0(t_2))| \).

Since \( |F_i^k(t_2, r_i(t_2), v_i(t_2)) - F_0^k(t_2, r_0(t_2), v_0(t_2))| \leq \| F_i(t_2, r_i(t_2), v_i(t_2)) - F_0(t_2, r_0(t_2), v_0(t_2)) \| \), one has \( \alpha < \| F_i(t_2, r_i(t_2), v_i(t_2)) - F_0(t_2, r_0(t_2), v_0(t_2)) \| \). It follows that \( \alpha < W(t_2) \) based on (10). Because \( \alpha > W(0) \) and \( W(t) \) are continuously changing, there must be a \( t_3 \in [0, t_2] \) such that \( \alpha = W(t_3) \). It contradicts the assumption that \( t_1 \in R^+ \) is the first time at which \( \alpha = W(t) \).

• Case (ii): \( \{S^+(t) \leq 0, S^-(t) < 0\} \).

In this case, \( \max\{|S^+(t)|, |S^-(t)|\} = -S^-(t) \), and the derivative of \( \max\{|S^+(t)|, |S^-(t)|\} \) is \( -\tilde{S}^-(t) \). Since Assumption 1 holds and \( S^-(t) < 0 \), one has \( \sum_{s \in N_j} a_{js} [S^-(t) - S_j^s(t)] < 0 \). Thus,

\[
\tilde{S}^-(t) \in K \left[ F_j^s(t, r_j(t), v_j(t)) - F_0^s(t, r_0(t), v_0(t)) + \alpha \right].
\]

If the derivative of \( \max\{|S^+(t)|, |S^-(t)|\} \) is greater than zero at \( t_2 \in [0, t_1] \), one has \( \tilde{S}^-(t_2) < 0 \). Then, there must exist \( i \in \{1, 2, \cdots, N\} \) and \( k \in \{1, 2, \cdots, n\} \) such that \( F_i^k(t_2, r_i(t_2), v_i(t_2)) - F_0^k(t_2, r_0(t_2), v_0(t_2)) < 0 \) and the positive constant \( \alpha < |F_i^k(t_2, r_i(t_2), v_i(t_2)) - F_0^k(t_2, r_0(t_2), v_0(t_2))| \).

Since \( |F_i^k(t_2, r_i(t_2), v_i(t_2)) - F_0^k(t_2, r_0(t_2), v_0(t_2))| \leq \| F_i(t_2, r_i(t_2), v_i(t_2)) - F_0(t_2, r_0(t_2), v_0(t_2)) \| \), one has \( \alpha < \| F_i(t_2, r_i(t_2), v_i(t_2)) - F_0(t_2, r_0(t_2), v_0(t_2)) \| \). It follows that \( \alpha < W(t_2) \) based on
Since Assumption 1 holds and \( S^+(0) = \max_{t \in [0, 20]} S^-(t) > 0 \), one has
\( S^+(0) > W(0) \). Because \( \alpha > W(0) \) and \( W(t) \) are continuously changing, there must be a \( t_3 \in [0, t_2) \) such that \( \alpha = W(t_3) \). It contradicts the assumption that \( t_1 \in R^+ \) is the first time at which \( \alpha = W(t) \).

- **Case (iii):** \( \{S^+(t) > 0, S^-(t) < 0\} \).
  - (a) If \( \{S^+(t) \geq -S^-(t)\} \), then max\( \{|S^+(t)|, |S^-(t)|\} = S^+(t) \). So, the proof is the same as that in Case (i).
  - (b) If \( \{S^+(t) < -S^-(t)\} \), then max\( \{|S^+(t)|, |S^-(t)|\} = -S^-(t) \). So, the proof is the same as that in Case (ii).

Combining the above three cases, it can be concluded that the derivative of max\( \{|S^+(t)|, |S^-(t)|\} \) will not be greater than zero. Hence, if \( \alpha > W(0) \), i.e., Assumption 11 holds, then \( \alpha > W(t) \), \( \forall t \in R^+ \). It follows that \( \alpha > \| F_i(t,r_i(t),v_i(t)) - F_0(t,r_0(t),v_0(t)) \| \), \( \forall t \in R^+ \), \( \forall i = 1,2,\cdots,N \), based on (10).

The proof is now completed.

## 14 Supplementary Lemma iv

**Supplementary Lemma iv** Let \( \mathcal{F} \) denote the set-valued map. If Assumptions 1, 10 and 11 hold, then the set-valued Lie derivative \( \hat{\mathcal{L}}_{\mathcal{F}}V \) of \( V \) with respect to \( \mathcal{F} \) satisfies that max \( \hat{\mathcal{L}}_{\mathcal{F}}V < 0 \) for all \( (S^+(t), S^-(t)) \in D \setminus \{(0,0)\} \).

**Proof** If Assumptions 1, 10 and 11 hold, then Supplementary Lemma iii holds, i.e., \( \alpha > \| F_i(t,r_i(t),v_i(t)) - F_0(t,r_0(t),v_0(t)) \| \), \( \forall t \in R^+ \), \( \forall i = 1,2,\cdots,N \). The nonsmooth function \( V(S^+(t),S^-(t)) : R^2 \to R \) was given by (9).

Five cases are discussed as follows:

- **Case (i):** \( S^+(t) > 0 \) and \( S^-(t) > 0 \).

Since Assumption 1 holds and \( S^+(t) > 0 \), one has \( \sum_{r \in N_i} a_{ir}[S^+(t) - S^k_r(t)] > 0 \), and for
\[
\partial V(S^+(t), S^-(t)) = \{(1,0)\},
\]
one has
\[
\hat{\mathcal{L}}_{\mathcal{F}}V = \mathcal{K} \left[ F^k_i(t,r_i(t),v_i(t)) - F^k_0(t,r_0(t),v_0(t)) - \alpha \right].
\]

Since \( | F^k_i(t,r_i(t),v_i(t)) - F^k_0(t,r_0(t),v_0(t)) | \leq | F_i(t,r_i(t),v_i(t)) - F_0(t,r_0(t),v_0(t)) |, \forall t \in \)
Since $S^+(t) > 0, S^-(t) < 0$ and Assumption 1 holds, one has $\sum_{r \in \mathcal{N}} a_{ir}[S^+(t) - S^r_k(t)] > 0, \sum_{s \in \mathcal{N}_j} a_{js}[S^-(t) - S^s_l(t)] < 0$, and for

$$\partial V(S^+(t), S^-(t)) = \{(1, -1)\},$$
on one has

$$\tilde{L}_F V = \mathcal{K} \left[ (F^k_i(t, r_i(t), v_i(t)) - F^k_0(t, r_0(t), v_0(t)) - \alpha) 
- (F^l_j(t, r_j(t), v_j(t)) - F^l_0(t, r_0(t), v_0(t)) + \alpha) \right].$$

Since $| F^k_i(t, r_i(t), v_i(t)) - F^k_0(t, r_0(t), v_0(t)) | \leq \| F^k_i(t, r_i(t), v_i(t)) - F^k_0(t, r_0(t), v_0(t)) \|, \forall t \in R^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, it follows from Supplementary Lemma iii that

$$\max \tilde{L}_F V < 0.$$

• **Case (iii)**: $S^+(t) < 0$ and $S^-(t) < 0$.

Since $S^-(t) < 0$ and Assumption 1 holds, one has $\sum_{s \in \mathcal{N}_j} a_{js}[S^-(t) - S^s_l(t)] < 0$, and for

$$\partial V(S^+(t), S^-(t)) = \{(0, -1)\},$$
on one has

$$\tilde{L}_F V = \mathcal{K} \left[ -(F^l_j(t, r_j(t), v_j(t)) - F^l_0(t, r_0(t), v_0(t)) + \alpha) \right].$$

Since $| F^l_j(t, r_j(t), v_j(t)) - F^l_0(t, r_0(t), v_0(t)) | \leq \| F^l_j(t, r_j(t), v_j(t)) - F^l_0(t, r_0(t), v_0(t)) \|, \forall t \in R^+, \forall j = 1, 2, \cdots, N, \forall l = 1, 2, \cdots, n$, it follows from Supplementary Lemma iii that

$$\max \tilde{L}_F V < 0.$$

• **Case (iv)**: $S^+(t) > 0$ and $S^-(t) = 0$.

Since $S^+(t) > 0, S^-(t) = 0$ and Assumption 1 holds, one has $\sum_{r \in \mathcal{N}_i} a_{ir}[S^+(t) - S^r_k(t)] > 0, \sum_{s \in \mathcal{N}_j} a_{js}[S^-(t) - S^s_l(t)] \leq 0$. So, if $v \in \mathcal{F}(S^+(t), S^-(t))$, then $v^T = (v_1, v_2)$ with $v_1 \in R^+$, $v_2 \in R^{-}$, and for

$$\partial V(S^+(t), S^-(t)) = \{(1, 1)\},$$
on one has

$$\tilde{L}_F V = \mathcal{K} \left[ (F^k_i(t, r_i(t), v_i(t)) - F^k_0(t, r_0(t), v_0(t)) - \alpha) 
- (F^l_j(t, r_j(t), v_j(t)) - F^l_0(t, r_0(t), v_0(t)) + \alpha) \right].$$

Since $| F^k_i(t, r_i(t), v_i(t)) - F^k_0(t, r_0(t), v_0(t)) | \leq \| F^k_i(t, r_i(t), v_i(t)) - F^k_0(t, r_0(t), v_0(t)) \|, \forall t \in R^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, it follows from Supplementary Lemma iii that

$$\max \tilde{L}_F V < 0.$$
If there exists an element \( L \) then it follows from Supplementary Lemma iii that \( \max \tilde{\zeta} \) one has \( \tilde{\zeta} \) if \( \zeta \in \partial V(S^+(t), S^-(t)) \), then \( \zeta^T = (1, y) \) with \( y \in [-1, 0] \). Therefore,

\[
\zeta^T v = v_1 + yv_2.
\]

If there exists an element \( a \) satisfying that \( \zeta^T v = a \) for all \( y \in [-1, 0] \), then \( v_2 = 0 \). So, if \( v_2 \neq 0 \), one has \( \tilde{\mathcal{L}}_F V = \emptyset \); if \( v_2 = 0 \), one has \( \tilde{\mathcal{L}}_F V = K[F_i^k(t, r_i(t), v_i(t)) - F_i^k(t, r_0(t), v_0(t)) - \alpha] \cup K[F_j^k(t, r_j(t), v_j(t)) - F_j^k(t, r_0(t), v_0(t))] \) and then it follows from Supplementary Lemma iii that \( \max \tilde{\mathcal{L}}_F V < 0 \). Thus, \( \max \tilde{\mathcal{L}}_F V < 0 \) or \( \tilde{\mathcal{L}}_F V = \emptyset \) in this case.

- **Case (v):** \( S^+(t) = 0 \) and \( S^-(t) < 0 \).

Since \( S^+(t) = 0, S^-(t) < 0 \) and Assumption 1 holds, one has \( \sum_{i \in N^*_t} a_{ir} [S^+(t) - S^i(t)] \geq 0, \sum_{s \in \mathcal{N}^*_s} a_{sj} [S^-(t) - S^j(t)] \leq 0 \). So, if \( v \in \mathcal{F}(S^+(t), S^-(t)), \) then \( v^T = (v_1, v_2) \) with \( v_1 \in K[F_i^k(t, r_i(t), v_i(t)) - F_i^k(t, r_0(t), v_0(t)) - \alpha] \cup K[F_i^k(t, r_i(t), v_i(t)) - F_i^k(t, r_0(t), v_0(t))] \) and \( v_2 \in K[F_j^k(t, r_j(t), v_j(t)) - F_j^k(t, r_0(t), v_0(t)) + \alpha] \). For

\[
\partial V(S^+(t), S^-(t)) = [0, 1] \times \{-1\},
\]

if \( \zeta \in \partial V(S^+(t), S^-(t)) \), then \( \zeta^T = (y, -1) \) with \( y \in [0, 1] \). Therefore,

\[
\zeta^T v = yv_1 - v_2.
\]

If there exists an element \( a \) satisfying that \( \zeta^T v = a \) for all \( y \in [0, 1] \), then \( v_1 = 0 \). So, if \( v_1 \neq 0 \), one has \( \tilde{\mathcal{L}}_F V = \emptyset \); if \( v_1 = 0 \), one has \( \tilde{\mathcal{L}}_F V = -K[F_j^k(t, r_j(t), v_j(t)) - F_j^k(t, r_0(t), v_0(t)) + \alpha] \), and then it follows from Supplementary Lemma iii that \( \max \tilde{\mathcal{L}}_F V < 0 \). Thus, \( \max \tilde{\mathcal{L}}_F V < 0 \) or \( \tilde{\mathcal{L}}_F V = \emptyset \) in this case.

Combining the above five cases, it can be concluded that \( \max \tilde{\mathcal{L}}_F V < 0 \) for all \( (S^+(t), S^-(t)) \in \mathcal{D} \setminus \{(0, 0)\} \).

The proof is now completed.

### 15 Proof of Corollary 2

**Proof** The nonsmooth function \( V(S^+(t), S^-(t)) \) in (9) is chosen as the Lyapunov function. If Assumptions 1, 10 and 11 hold, then Supplementary Lemma iv holds. By Lemma 1, it follows
from Lemmas 3 - 5 and Supplementary Lemma iv that \((S^+(t), S^-(t)) = (0, 0)\) is a globally stable equilibrium point for system (7).

Solving \(S^k_i(t) = \tilde{r}^k_i(t) + \dot{\tilde{r}}^k_i(t) = 0\), one has

\[
\tilde{r}^k_i(t) = ce^{-t}, \quad \dot{\tilde{r}}^k_i(t) = -ce^{-t},
\]

where \(c\) is a constant determined by the initial conditions. Therefore, the errors \(\tilde{r}_i(t)\) and \(\tilde{v}_i(t)\) converge to zero exponentially; that is, the second-order multi-agent system achieves the desired formation asymptotically.

The proof is now completed.