EVALUATION OF VARIOUS FFT METHODS FOR SINGLE TONE DETECTION AND FREQUENCY ESTIMATION

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ABSTRACT

The periodogram, implemented using the fast Fourier transform (FFT), is widely used for the detection and frequency measurement of single tones. This paper evaluates the detection and frequency estimation performance of the periodogram and its variants, such as the Welch and Bartlett methods and the polyphase-FFT. Performance results for the detection and frequency estimation performance of the periodogram and its variants are presented and compared. The standard periodogram generally gives the best detection performance and the minimum mean square frequency error for a fixed length of signal data. However, if the FFT length is fixed the Welch method gives the best performance.

1. INTRODUCTION

The problems of signal detection and frequency estimation are of fundamental importance in radar, electronic warfare, and sonar systems. The fast Fourier transform (FFT) has been widely used in approaches for detection and parameter estimation which are based on spectral analysis [1], [2]. The periodogram is a common method for using the FFT to calculate the power spectral density (PSD). Its variants include the Bartlett and Welch methods and the polyphase-FFT. The periodogram for the N-point sequence $x(n)$ is given by

$$P(k/N) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \right|^2, k = 0, 1, ..., N-1$$

(1)

where frequency $f_k = k/N$. The FFT is often employed in the computation of this result. The Bartlett and Welch methods divide the input data sequence into $K$ segments, calculate the PSD of each sequence using the FFT, and then average the PSD. The concept of averaging is widely used to reduce the variance of the PSD.

The Welch method differs from the Bartlett method in employing overlapped segments. The segments are often multiplied by a window function to reduce the spectral leakage [3]. However, the use of window functions has several disadvantages, such as the degradation in frequency resolution and increased processing loss. The polyphase-FFT [4] is an alternative approach for reducing spectral leakage without sacrificing frequency resolution. It has also been advocated for signal detection with the argument that the processing loss is less than that of periodogram approaches based on the use of the FFT with a window function [1].

Although these techniques are well known, no comprehensive study of detection and frequency estimation performance attainable with these methods has been reported. This paper presents a comparative study of these methods for the problems of detecting and measuring the frequency of a single tone in additive white noise.

The following sections present the problem description, theoretical analysis, the results of simulation experiments and the conclusions which have been reached.

2. DETECTION

The detection problem of interest can be modeled as a test between two statistical hypotheses, $H_0$ and $H_1$, described by

$$H_0 : x(n) = q(n), n = 1, ..., N - 1$$
$$H_1 : x(n) = A \sin(2\pi f_0 n + \psi) + q(n)$$

(2)

In this model, the signal $x(n)$ has the unknown parameters frequency $f_0$, amplitude $A$ and phase, $\psi$. In order to detect this signal, we calculate the PSD using (1) for $k = 1, 2, ..., N/2 - 1$, then select the peak to compare with a threshold $V_T$. If this peak is exceeds $V_T$, then $H_1$ is accepted, otherwise $H_0$ is chosen. We use the probability of detection $P_d$, and probability of false alarm, $P_{fa}$ to measure the detection performance. In
order to make these performance calculations we need to know the probability density functions (PDFs) of the PSD $P(f_k)$. From (1) we have

$$P(f_k) = X^2 + Y^2$$

where

$$X = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (\sin(2\pi f_0 n + \varphi(n)) + q(n)) \cos(\frac{2\pi n k}{N})$$
$$Y = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (\sin(2\pi f_0 n + \varphi(n)) + q(n)) \sin(\frac{2\pi n k}{N})$$

are real Gaussian variables. It can be shown [5] that $X$ and $Y$ are independent and have identical variances equal to $\sigma^2 = \sigma^2/2$ where $\sigma^2$ represents the variance of $q(n)$. For the periodogram, following [6], we can derive the PDF when the signal is present as

$$p_s(u) = \frac{1}{2\sigma^2 \sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}} I_0(\sqrt{u^2})$$

and the PDF of the noise is

$$p_q(u) = \frac{1}{\sqrt{2\pi} \sigma_q} e^{-\frac{u^2}{2\sigma_q^2}}$$

where $\sigma^2$ is the squared sum of the means of $X$ and $Y$, and $I_0$ is the zero order Bessel function. Now $P_{fa}$ and $P_d$ are readily derived as

$$P_{fa} = 1 - \left[ \int_0^{V_T} p_t(u) du \right]^{\frac{N}{2}-1}$$
$$P_d = 1 - \left[ \int_0^{V_T} p_q(u) du \right]^{\frac{N}{2}-2} - \int_0^{V_T} p_s(u) du$$

For the Bartlett method, the probability density functions of signal and the noise are given as

$$p_s(u) = \frac{1}{2\sigma^2} e^{-\frac{u^2}{2\sigma^2}}$$
$$p_q(u) = \frac{1}{\sigma_q^2} e^{-\frac{u^2}{2\sigma_q^2}}$$

and the probability of false alarm and the detection are derived as

$$P_{fa} = 1 - \left[ \int_0^{V_T} p_q(u) du \right]^{\frac{N}{2}-1}$$
$$P_d = 1 - \left[ \int_0^{V_T} p_q(u) du \right]^{\frac{N}{2}-2} - \int_0^{V_T} p_s(u) du$$

For the Welch method or polyphase FFT, the probabilities of detection and false alarm are more difficult to derive. A theoretical result for the probability of false alarm for a restrictive case of the Welch method has been recently given in [8].

### 3. FREQUENCY ESTIMATION

The frequency estimate for the detected signal is obtained from the position of the corresponding peak in the PSD. The dependence of the mean square frequency error on the signal-to-noise ratio (SNR) for periodogram and Bartlett methods are given by

$$msfe = (1 - q) var_f(N) + q \int_0^{0.5} 2(u - f_0)^2 du$$
$$msfe_b = \frac{1 - q_b}{K} var_f(N) + q_b \int_0^{0.5} 2(u - f_0)^2 du$$

where

$$var_f(N) = \frac{6\sigma_q^2}{K^2 \pi^2 N (N^2 - 1)}$$

is the Cramer-Rao lower bound (CRLB) which can be found in [7]. The $q$ in the $msfe_b$ equation is the probability of occurrence of an anomaly [9] caused by excessively low SNR, and is derived by

$$q = P\{P(k_0/N) \leq \text{at least one of the remaining } P(k/N)\}$$
$$= \int_0^\infty P\{P(k_0/N) = u\} \cdot \left[ 1 - \prod_{k=1, k \neq k_0}^{N-1} P(k/N < u) \right] du$$
$$= \int_0^\infty p_s(u) \left[ 1 - \prod_{k=1, k \neq k_0}^{N-1} \int_0^\infty p_q(v) dv \right] du$$

using $p_s$ and $p_q$, $q_b$ can be calculated.

### 4. SIMULATION RESULTS AND CONCLUSION

Computer simulation experiments were used to obtain detection and frequency estimation performance results. Performance results were also predicted for the periodogram and its Bartlett method variant from the theoretical analysis in sections 2 and 3. The performance results were compared and some general conclusions reached.

The simulation methodology used follows that in [10]. The Hanning window was employed in all the simulations. To obtain the measures of detection performance, $P_d$ and $P_{fa}$, we carried out 6000 trials for each set of conditions. To obtain the mean square frequency estimation error, we ran 500 independent trials for each set of conditions. The golden section method was used for the fine frequency search. The signal frequency was 16/64 and 16.5/64 for the exact and half bin cases, respectively.
The performance results for the fixed data length case were obtained for sets of signal data, each of which consisted of 256 samples. The periodogram method and its polyphase-FFT variant was carried out using single segments of 256 samples. For the Bartlett and Welch methods, 4 contiguous and 7 overlapping (50\% overlap) segments of 64 samples, respectively, were constructed from each set of 256 samples. Except for the periodogram method, which used a 256 point FFT, 64 point FFTs were used.

Figure 1 shows the theoretical results for the receiver operating characteristics defined by the relationships between the probabilities of detection and false alarm obtained using equations (8) and (9), and the simulation results for the periodogram. Figure 2 shows the corresponding results for the Bartlett method. In both cases, results are shown for signal-to-noise ratios of -9 dB, -12 dB, -15 dB and -18 dB, and for the exact bin case. The theoretical and simulation results are in close agreement and show that the periodogram performs better than the Bartlett method for a fixed data length. Other results show that the periodogram is also preferable to the Welch and polyphase methods.

Simulation experiments were performed to obtain probability of detection results where the FFT length is fixed at 64 points for the periodogram and its variants. This situation holds when a specific FFT bin size is required, for example, to correspond to the bandwidth of a modulated signal. Figures 3 and 4 present results for the exact and half bin cases, respectively, and the threshold is attained by setting a constant false alarm probability at 0.01. For the latter, the magnitude of the spectral peak and its position is estimated using parabolic interpolation. The Welch method is the best, followed, in order of preference, by the Bartlett method, polyphase-FFT and periodogram. The reason for this is that Welch and Bartlett methods reduce the variance of the PSDs. We also note that the exact bin case is slightly better than the half bin case.

Figure 5 shows that the theoretical and simulation results for frequency estimation errors of the periodogram and Bartlett method. Similar to figures 1 and 2, the signal data set and segment lengths were fixed at 256 and 64 samples, respectively. The Cramer-Rao lower bound is also plotted. The theoretical and simulation results are in close agreement. Figure 6 presents the frequency estimation error obtained from simulation results for the periodogram and its variants when FFT length is fixed. The frequency estimation error performance is best for the Welch method, followed in order, by Bartlett method, polyphase-FFT and periodogram.

To conclude, the theoretical and simulation results show that for a given data length, the periodogram generally provides the best performance for single tone detection and frequency estimation in the presence of white noise. However, if the FFT length is fixed, the Welch method gives the best performance for both detection and frequency estimation. This last result is noteworthy in that it shows that the smaller noise bandwidth and better spectral leakage performance obtained with the polyphase-FFT, does not provide an advantage for single tone detection and frequency estimation. This last observation is not discussed in [1].

5. REFERENCES


Figure 1: ROCs for tone detection using periodogram.

Figure 2: ROCs for tone detection using Bartlett method.

Figure 3: Probability of detection vs SNR, exact bin frequency.

Figure 4: Probability of detection vs SNR, half bin frequency.

Figure 5: Mean square frequency estimation error for exact bin frequency.

Figure 6: Mean square frequency estimation error for half bin frequency.