Fourier series

Revisiting what you have learned in “Advanced Mathematical Analysis”

Let $f(x)$ be a periodic function of period $2\pi$ and is integrable over a period. $f(x)$ can be represented by a trigonometric series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$$= a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos(2x) + b_2 \sin(2x) + \cdots$$

where the coefficients are computed as

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)dx \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)dx$$

$n = 1, 2, 3, \cdots$
Consider a continuous-time signal $x(t)$ periodic with period $T > 0$: $x(t) = x(t + T)$, for all $t$

- $T$: fundamental period
- $\omega_0 = 2\pi / T$: fundamental frequency (in radians/second)

$x(t)$ can also be represented as complex exponential function

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 t}$$

$$= \cdots + a_{-2} e^{j(-2\omega_0)t} + a_{-1} e^{j(-\omega_0)t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j(2\omega_0)t} + \cdots$$

As $x(t)$ can be complex-valued, all $a_k$’s are generally complex

<table>
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<th>$a_0$</th>
<th>: DC component</th>
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<td>$a_{\pm 1} e^{j(\pm \omega_0)t}$</td>
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<tr>
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<td>: second harmonic components</td>
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<td>$\cdots$</td>
<td>: third harmonic components</td>
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Example 1

Consider a periodic signal \( x(t) \), which is of the form

\[ x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t} \]

where \( a_0 = 1 \), \( a_1 = a_{-1} = 1/4 \), \( a_2 = a_{-2} = 1/2 \), \( a_3 = a_{-3} = 1/3 \).

(1) The fundamental frequency \( \omega_0 = 2\pi \) or \( T = 1 \).

(2) \( x(t) \) contains only (the first) three harmonic components,

\[
\begin{align*}
x(t) &= \frac{1}{3} e^{j(-6\pi)t} + \frac{1}{2} e^{j(-4\pi)t} + \frac{1}{4} e^{j(-2\pi)t} + 1 + \frac{1}{4} e^{j2\pi t} + \frac{1}{2} e^{j4\pi t} + \frac{1}{3} e^{j6\pi t} \\
&= 1 + \frac{1}{4} [e^{j2\pi t} + e^{j(-2\pi)t}] + \frac{1}{2} [e^{j4\pi t} + e^{j(-4\pi)t}] + \frac{1}{3} [e^{j6\pi t} + e^{j(-6\pi)t}] \\
&= 1 + \frac{1}{2} \cos(2\pi t) + \cos(4\pi t) + \frac{2}{3} \cos(6\pi t) \quad \rightarrow \quad \text{a real function}
\end{align*}
\]
1st harmonic component

\[ x_1(t) = \frac{1}{2} \cos 2\pi t \]

2nd harmonic component

\[ x_2(t) = \cos 4\pi t \]

3rd harmonic component

\[ x_3(t) = \frac{2}{3} \cos 6\pi t \]

DC component

\[ x_0(t) = 1 \]

\[ x(t) = x_0(t) + x_1(t) + x_2(t) + x_3(t) \]
As in the above example, for a real periodic signal $x(t)$, its complex conjugate equals itself, i.e. $x^*(t) = x(t)$.

Since the conjugate of $e^{jk\omega_0 t}$ is $e^{-jk\omega_0 t}$, we have

$$x^*(t) = \text{conj}\left( \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right) = \sum_{k=-\infty}^{\infty} a^*_k e^{-jk\omega_0 t} = x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\Rightarrow a_k = a^*_{-k} \text{ or } a^*_k = a_{-k}, \text{ thus}$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a^*_k e^{-jk\omega_0 t}] = a_0 + \sum_{k=1}^{\infty} 2 \text{Re}\{a_k e^{jk\omega_0 t}\}$$

Writing $a_k$ as polar form, i.e., $a_k = A_k e^{j\theta_k}$, gives

$$x(t) = a_0 + 2 \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$
Alternatively, write \( a_k = B_k + jC_k \),

\[
x(t) = a_0 + 2 \sum_{k=1}^{\infty} [B_k \cos(k\omega_0 t) - C_k \sin(k\omega_0 t)]
\]

To summarize, for a real periodic signal, the following three forms of Fourier series representation are equivalent:

\[
\sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 t} \quad \text{summation of complex exponentials}
\]

\[
a_0 + 2 \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k) \quad \text{summation of cosine functions of non-zero phase}
\]

\[
a_0 + 2 \sum_{k=1}^{\infty} [B_k \cos(k\omega_0 t) - C_k \sin(k\omega_0 t)] \quad \text{summation of cosine and sine functions with zero phase}
\]
Computing Fourier Series Coefficients

Given $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$,

$$x(t) \cdot e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \cdot e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}$$

Integrating both sides from 0 to $T$,

$$\int_{0}^{T} x(t) e^{-jn\omega_0 t} dt = \int_{0}^{T} \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t} dt = \sum_{k=-\infty}^{\infty} a_k \left[ \int_{0}^{T} e^{j(k-n)\omega_0 t} dt \right]$$

Knowing that $\int_{0}^{T} e^{j(k-n)\omega_0 t} dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$,

$$a_n = \frac{1}{T} \int_{0}^{T} x(t) e^{-jn\omega_0 t} dt, \text{ for } n = 0, \pm 1, \pm 2, \ldots$$
Example 2

Find the Fourier series coefficients of \( x(t) = \sin(\omega_0 t) \).

Solution: By Euler’s relation,

\[
x(t) = \sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}
\]

Thus \( a_0 = 0 \), \( a_1 = 1/(2j) \), \( a_{-1} = -1/(2j) \), and \( a_k = 0 \) for \( n = \pm 2, \pm 3, \ldots \)

Example 3

Find the Fourier series coefficients of

\[
x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos(2\omega_0 t + \pi/4).
\]

Solution:

\[
x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \frac{1}{2}e^{\frac{\pi}{4}j2\omega_0 t} + \frac{1}{2}e^{\frac{\pi}{4}j(-2\omega_0 t)}
\]
Thus,

\[ a_0 = 1, \quad a_1 = 1 - \frac{1}{2} j, \quad a_{-1} = 1 + \frac{1}{2} j \]
\[ a_2 = \frac{\sqrt{2}}{4} (1 + j), \quad a_{-2} = \frac{\sqrt{2}}{4} (1 - j) \]
\[ a_k = 0 \text{ for } n = \pm 3, \pm 4, \ldots \]

\[ |a_k| = \sqrt{(\text{Re}\{a_k\})^2 + (\text{Im}\{a_k\})^2} \]

\[ \angle\{a_k\} = \tan^{-1}\left(\frac{\text{Im}\{a_k\}}{\text{Re}\{a_k\}}\right) \]
Example 4

Consider the following periodic square wave

\[ x(t) = \begin{cases} 1, & -T_1 < t < T_1 \\ 0, & \text{otherwise} \end{cases} \]

Find its Fourier series representation.

Solution:
The signal is periodic with period \( T \). Also, it is an even signal.

Over the specific period from \(-T/2\) to \(T/2\), the signal is defined as,

\[ x(t) = \begin{cases} 1, & -T_1 < t < T_1 \\ 0, & \text{otherwise} \end{cases} \]
It is easy to show that

\[ a_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} \, dt = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} \, dt \]

Thus we can do the integration over \([-T/2,T/2]\)

\[ a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} \, dt = \frac{1}{T} \int_{-T_1}^{T_1} e^{-jk\omega_0 t} \, dt \]

For \( k = 0 \),

\[ a_0 = \frac{1}{T} \int_{-T_1}^{T_1} 1 \cdot dt = \frac{2T_1}{T}, \]

For \( k \neq 0 \),

\[ a_k = \frac{1}{T} \int_{-T_1}^{T_1} e^{-jk\omega_0 t} \, dt = -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t} \bigg|_{-T_1}^{T_1} = \frac{\sin(k\omega_0 T_1)}{k\pi} \]
\[ T = 4T_1 \]

\[ T = 8T_1 \]

\[ T = 16T_1 \]
Convergence Problems of Fourier series

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \]
\[ a_k = \frac{1}{T} \int_{0}^{T} x(t)e^{-jk\omega_0 t} dt \]

The problems:
1) The integral \( \frac{1}{T} \int_{0}^{T} x(t)e^{-jk\omega_0 t} dt \) may not converge, i.e. \( a_k \rightarrow \infty \)

2) Even if all \( a_k \)'s are finite, the summation \( \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \) may not be equal to the original signal \( x(t) \).

Virtually all periodic signals arising in engineering do have a Fourier series representation, without convergence problems.
Dirichlet Conditions of Convergence

For a periodic signal $x(t)$ to have converged Fourier series,

1) $x(t)$ must be **absolutely integrable** over any period, i.e.

$$\int_{t_0}^{t_0+T} |x(t)| \, dt < \infty \text{ for any } t_0$$

2) $x(t)$ has a **finite** number of maxima & minima over any period

3) $x(t)$ has a **finite** number of discontinuous over any period
Convergence at Discontinuities

1) If a periodic signal has no discontinuities, its Fourier series representation
   - converges, and
   - equals the original signal at every value of $t$.

2) If the signal has a finite number of discontinuities in each period, its Fourier series representation
   - equals the original signal everywhere except at the discontinuities
   - converges to the midpoint of $x(t)$ at each discontinuity
Example 5

\[ x(t) = \begin{cases} 
1, & -T_1 < t < T_1 \\
0, & \text{otherwise} 
\end{cases} \]

Expanding \( x(t) \),

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \text{ where } a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} \text{ and } a_0 = 1 \]

Let \( x_N(t) \) denote the Fourier series truncated at the \( N \)th harmonics, i.e.

\[ x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \]

\( x_N(t) \) becomes a good approximation of \( x(t) \) if \( N \) is large enough.
Properties of Fourier series

**Notations:** The relation that $a_k$’s are Fourier series coefficients of $x(t)$ is represented by $x(t) \xrightarrow{FS} a_k$.

Assume $x(t) \xrightarrow{FS} a_k$ and $y(t) \xrightarrow{FS} b_k$ and $x(t)$ and $y(t)$ have the same fundamental period $T$ (or $\frac{2\pi}{\omega_0}$).

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<td>Time shifting</td>
<td>$x(t - t_0)$</td>
<td>$a_k e^{-j\omega_0 t_0}$</td>
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<tr>
<td>Frequency shifting</td>
<td>$e^{jM\omega_0 t} x(t)$</td>
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<td>Periodic convolution</td>
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<tr>
<td>Multiplication</td>
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<td>Differentiation</td>
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<td>$jk \omega_0 a_k$</td>
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<tr>
<td>Integration</td>
<td>$\int_{-\infty}^{t} x(t) dt$</td>
<td>$\frac{1}{jk \omega_0} a_k$</td>
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Example 6
Derive the Fourier series representation of the following signal.

Solution:
Define $g(t)$ over a period, $g(t) = \begin{cases} 1/2 & 0 < t \leq 2 \\ -1/2 & 2 < t \leq 4 \end{cases}$

Let $g(t) \overset{FS}{\rightarrow} \rho_k$, for $k = 0$, $\rho_0 = \frac{1}{4} \left[ \int_0^2 1 \, dt + \int_2^4 \left(-\frac{1}{2}\right) \, dt \right] = 0$

$$\rho_k = \frac{1}{4} \int_0^2 e^{-jk\left(\frac{2\pi}{4}\right)t} \, dt + \frac{1}{4} \int_2^4 (-\frac{1}{2})e^{-jk\left(\frac{2\pi}{4}\right)t} \, dt = \frac{\sin(k\pi/2)}{k\pi} e^{-jk\pi/2}, k \neq 0$$
Alternatively, we can use properties of Fourier series to solve it. Recalling the signal \( x(t) \) in Example 4:

\[
\text{By the property of time-shift,}
\begin{align*}
\mathcal{F}_S \{ x(t-1) \} &= a_k e^{-j k \omega_0} \\
\Rightarrow \quad x(t-1) &\mathcal{F}_S a_k e^{-j k \pi/2}
\end{align*}
\]

Subtracting the DC offset of \( 1/2 \) from \( a_0 \),

\[
\rho_k = \begin{cases} 
  a_k e^{-j k \pi/2}, & k \neq 0 \\
  a_0 - 1/2, & k = 0
\end{cases} = \begin{cases} 
  \frac{\sin(k \pi/2)}{k \pi} e^{-j k \pi/2}, & k \neq 0 \\
  0, & k = 0
\end{cases}
\]

\( a_k = \frac{\sin(k \omega_0 T_1)}{k \pi} \),

\( a_0 = 2T_1 / T = 1/2 \)

\( g(t) \) can be expressed as \( g(t) = x(t-1) - \frac{1}{2} \), with \( T = 4 \) & \( T_1 = 1 \)
Note that when we chose the interval \([0,4)\):

\[
a_k = \frac{1}{4} \int_0^2 \left( \frac{1}{2} \right) e^{-jk\left(\frac{\pi}{2}\right)t} \, dt + \frac{1}{4} \int_0^4 \left( -\frac{1}{2} \right) e^{-jk\left(\frac{\pi}{2}\right)t} \, dt
\]

\[
= \frac{1}{8} \cdot e^{-jk\left(\frac{\pi}{2}\right)t} \bigg|_0^2 - \frac{1}{8} \cdot e^{-jk\left(\frac{\pi}{2}\right)t} \bigg|_0^4
\]

\[
= -\frac{1}{4jk\pi} \left[ e^{-jk\left(\frac{\pi}{2}\right)t} \bigg|_0^2 - e^{-jk\left(\frac{\pi}{2}\right)t} \bigg|_0^4 \right]
\]

\[
= -\frac{1}{4jk\pi} \left[ e^{-jk\pi} - 1 - (e^{-j2k\pi} - e^{-jk\pi}) \right]
\]
\[ = -\frac{1}{2jk\pi} \left[ e^{-jk\pi} - 1 \right] \quad \therefore e^{-j2k\pi} = 1 \text{ for integer } k \]

\[ = -\frac{1}{2jk\pi} \cdot e^{-jk\pi/2} \left[ e^{-jk\pi/2} - e^{jk\pi/2} \right] \]

\[ = -\frac{1}{2jk\pi} \cdot e^{-jk\pi/2} \cdot -2j\sin(k\pi/2) \]

\[ = \frac{\sin(k\pi/2)}{k\pi} \cdot e^{-jk\pi/2} \]

On the other hand, if we choose \([-2,2)\)

\[ a_k = \frac{1}{4} \int_{-2}^{0} \left(-\frac{1}{2}\right) e^{-jk\left(\frac{\pi}{2}\right)t} dt + \frac{1}{4} \int_{0}^{2} \left(\frac{1}{2}\right) e^{-jk\left(\frac{\pi}{2}\right)t} dt \]
\[
\begin{align*}
-\frac{1}{8} - \frac{1}{jk \pi / 2} e^{-jk(\pi/2)t} & \bigg|_0^0 + \frac{1}{8} - \frac{1}{jk \pi / 2} e^{-jk(\pi/2)t} \bigg|_0^2 \\
= \frac{1}{4jk \pi} \begin{bmatrix}
- \frac{1}{8} - \frac{1}{jk \pi / 2} e^{-jk(\pi/2)t} & 0 \\
- e^{-jk(\pi/2)t} & - e^{-jk(\pi/2)t}
\end{bmatrix} \bigg|_{-2}^0 \\
= -\frac{1}{4jk \pi} \left[(1 - e^{-jk \pi}) - (e^{-jk \pi} - 1)\right] \\
= \frac{1}{2jk \pi} \left[1 - e^{-jk \pi}\right] = \frac{\sin(k \pi / 2)}{k \pi} \cdot e^{-jk \pi / 2}
\end{align*}
\]

You can choose intervals \([3,7), [4,8), \text{ etc.} \) Make use of \( e^{j2k \pi} = 1 \) you can get the same answer.
Fourier series of even & odd signals

Real even signals

By the property of time reversal, we have \( x(-t) \xrightarrow{\text{FS}} a_{-k} \).

\( x(t) \) is even \( \Rightarrow x(-t) = x(t) \) \( \Rightarrow x(t) \xrightarrow{\text{FS}} a_{-k} \)

\( x(t) \) is real \( \Rightarrow a_{-k} = a_{k} \) \( \Rightarrow x(t) \xrightarrow{\text{FS}} a_{-k}^{*} \)

Therefore, we have \( a_{k} = a_{k}^{*} \) for all \(-\infty < k < \infty\).

For a real even signal

1) \( \{a_{k}\} \) are real: \( a_{k} = a_{k}^{*} \)
2) \( \{a_{k}\} \) are even: \( a_{k} = a_{-k} \)

\[
\begin{align*}
x(t) &= a_{0} + \sum_{k=1}^{\infty} a_{k} (e^{jk\omega_{0}t} + e^{-jk\omega_{0}t}) = a_{0} + \sum_{k=1}^{\infty} 2a_{k} \cos(k\omega_{0}t)
\end{align*}
\]
Real odd signals
Similar to the above derivation, we have $a_k = -a_k^*$.

For a real odd signal

1) \{a_k\} are purely imaginary: $a_k = -a_k^*$
2) \{a_k\} are odd: $a_k = -a_{-k}$
3) $a_0 = 0$;

Let $a_k = jC_k$, where $C_k$ is real and $C_k = -C_{-k}$

$$x(t) = \sum_{k=1}^{\infty} jC_k (e^{jk\omega_0 t} - e^{-jk\omega_0 t}) = -\sum_{k=1}^{\infty} 2C_k \sin(k\omega_0 t)$$
**Parseval’s Theorem**

Average power over a period

\[
\frac{1}{T} \int_T |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2
\]

Sum of squared magnitudes of all harmonics

Recall \( x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \)

Average power of the \( k \)th harmonic

\[
\frac{1}{T} \int_T |a_k e^{jk\omega_0 t}|^2 \, dt = |a_k|^2
\]

The total average power over a period equals the sum of the average powers in all of the harmonic components.
Proof:

\[
\frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T} \int_{-T/2}^{T/2} \left( \sum_{m=-\infty}^{\infty} a_m e^{jm\omega_0 t} \right) \left( \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t} \right)^* dt \\
= \frac{1}{T} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_m a_n^* \int_{-T/2}^{T/2} e^{j(m-n)\omega_0 t} dt \\
= \sum_{m=-\infty}^{\infty} |a_m|^2
\]

**Example 7**

Suppose we want to use \( g_N(t) \) to approximate \( g(t) \) of Example 6 where \( g_N(t) \) denotes the Fourier series truncated at the \( N \)th harmonics, i.e. \( g_N(t) = \sum_{k=-N}^{N} \rho_k e^{jk\omega_0 t} \). What is the minimum value of \( N \) so that \( g_N(t) \) contains at least 90% power of \( g(t) \)?
Solution:

Average power of \( g(t) = \frac{1}{T} \int_0^T |g(t)|^2 \, dt = \frac{1}{4} \int_0^4 \left(\frac{1}{2}\right)^2 \, dt = \frac{1}{4} \)

Try \( N = 1 \), we have

\[
\frac{1}{\pi} \sum_{k=-1}^{1} |\rho_k|^2 = |\rho_{-1}|^2 + |\rho_1|^2 = 2 \frac{\sin^2(\pi/2)}{\pi^2} = \frac{2}{\pi^2} \approx 0.2026
\]

Try \( N = 2 \),

\[
\sum_{k=-2}^{2} |\rho_k|^2 = \frac{2}{\pi^2} + |\rho_{-2}|^2 + |\rho_2|^2 = \frac{2}{\pi^2} + 2 \frac{\sin^2(\pi)}{\pi^2} = \frac{2}{\pi^2}
\]

\( \Rightarrow \rho_k = 0, \quad k \text{ is odd} \)

Try \( N = 3 \), we have

\[
\sum_{k=-3}^{3} |\rho_k|^2 = \frac{2}{\pi^2} + |\rho_{-3}|^2 + |\rho_3|^2 = \frac{2}{\pi^2} + 2 \frac{\sin^2(3\pi/2)}{(3\pi)^2} = \frac{20}{9\pi^2} > 0.9 \times \frac{1}{4}
\]

Hence the minimum value of \( N \) is 3.
Discrete-time Fourier Series

- Fourier Series is a frequency analysis tool for continuous-time periodic signals while Discrete-Time Fourier Series (DTFS) is used for analyzing discrete-time periodic signals.

- In fact, DTFS can be derived from the Fourier Series (I can show you if you are interested on this).

- Similar to continuous-time case, a discrete-time signal $x[n]$ is periodic with fundamental period $N$ if
  $$x[n] = x[n + N]$$

  where $N$ is the smallest integer for which the equation holds. The fundamental frequency is defined as $\omega_0 = 2\pi / N$.

- However, notice that sampling a continuous-time periodic signal does not necessarily give a discrete-time periodic sequence.
Example 8

Consider a continuous-time periodic signal \( x(t) = \cos(0.5t) \). Sampling \( x(t) \) with a sampling period of 1s gives
\[
\{x[n]\} = \{\cdots \cos(-0.5), \cos(0), \cos(0.5), \cos(1), \cdots\}
\]
\( \Rightarrow \) \( x[n] \) is not periodic

Consider another continuous-time periodic signal \( y(t) = \cos(0.5\pi t) \). Sampling \( y(t) \) with a sampling period of 1s yields
\[
\{y[n]\} = \{\cdots \cos(-0.5\pi), \cos(0), \cos(0.5\pi), \cos(\pi), \cdots\}
\]
\[= \{\cdots 0,1,0,1,0,\cdots\}\]
\( \Rightarrow \) \( y[n] \) is periodic and the fundamental period is \( N = 2 \)

We can use DTFS to analyze \( y[n] \) but not \( x[n] \). In fact, we can use discrete-time Fourier transform (DTFT), which will be discussed later, to analyze \( x[n] \).
Fourier Series & LTI Systems

The output of a continuous-time periodic signal $x(t)$ to a LTI system with impulse response $h(t)$ is given by

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

where $y(t)$ is also periodic with the same fundamental frequency as $x(t)$. That is, if

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t} \Rightarrow y(t) = \sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 t}$$

⇒ only phases and magnitudes are changed
Proof:

\[ y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau \]

Consider the \( k \)th component of \( x(t) \), i.e., \( a_k e^{jk\omega_0 t} \). The system output due to this component is:

\[
y_k(t) = \int_{-\infty}^{\infty} h(\tau) \cdot a_k e^{jk\omega_0(t-\tau)}d\tau = e^{jk\omega_0 t} \cdot a_k \cdot \int_{-\infty}^{\infty} h(\tau) \cdot e^{-jk\omega_0 \tau}d\tau
\]

where we can see that

\[
b_k = a_k \cdot \int_{-\infty}^{\infty} h(\tau) \cdot e^{-jk\omega_0 \tau}d\tau
\]

Combining all harmonic components, we have

\[
y(t) = \sum_{k=-\infty}^{\infty} y_k(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}
\]
Example 9

Given a LTI system with impulse response \( h(t) = e^{-t}u(t) \) and an input signal \( x(t) = 0.5\cos(2\pi t) \). Find the system output \( y(t) \).

Solution:

\[
y(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau = \int_{-\infty}^{\infty} e^{-\tau}u(\tau) \cdot \frac{1}{2} \cos(2\pi(t - \tau)) d\tau
\]

\[
= \frac{1}{2} \int_{0}^{\infty} e^{-\tau} \cos(2\pi(t - \tau)) d\tau = \frac{1}{4} \int_{0}^{\infty} e^{-\tau} (e^{j2\pi(t - \tau)} + e^{-j2\pi(t - \tau)}) d\tau
\]

\[
= \frac{1}{4} e^{2\pi t} \int_{0}^{\infty} e^{-\tau} e^{-j2\pi \tau} d\tau + \frac{1}{4} e^{-2\pi t} \int_{0}^{\infty} e^{-\tau} e^{j2\pi \tau} d\tau
\]

\[
= \frac{1}{4} e^{2\pi t} \cdot \frac{1}{-(1 + j2\pi)} e^{-\tau(1+2\pi)} \bigg|_{0}^{\infty} + \frac{1}{4} e^{-2\pi t} \cdot \frac{1}{(-1 + j2\pi)} e^{-\tau(-1+2\pi)} \bigg|_{0}^{\infty}
\]
\[
y(t) = \frac{1}{4} \left( \frac{e^{2\pi t}}{1 + j2\pi} + \frac{e^{-2\pi t}}{1 - j2\pi} \right) = \frac{1}{4} \left( \frac{e^{2\pi t}}{1 + j2\pi} + \left( \frac{e^{2\pi t}}{1 + j2\pi} \right)^* \right)
\]
\[
= \frac{2}{4} \text{Re} \left\{ \frac{e^{2\pi t}}{1 + j2\pi} \right\} = \frac{1}{2} \text{Re} \left\{ \frac{\cos(2\pi t) + j \sin(2\pi t)}{1 + j2\pi} \right\}
\]
\[
= \frac{1}{2} \text{Re} \left\{ \frac{\cos(2\pi t) + j \sin(2\pi t) \cdot 1 - j2\pi}{1 + j2\pi \cdot 1 - j2\pi} \right\}
\]
\[
= \frac{1}{2(1 + 4\pi^2)} (\cos(2\pi t) + 2\pi \sin(2\pi t))
\]
\[
= \frac{1}{2(1 + 4\pi^2)} \cos(2\pi t) + \frac{2\pi}{2(1 + 4\pi^2)} \sin(2\pi t)
\]
Express $y(t)$ as a function of $\cos(2\pi t)$ only, we get

\[
y(t) = \frac{1}{2\sqrt{1+4\pi^2}} \left( \frac{1}{\sqrt{1+4\pi^2}} \cos(2\pi t) + \frac{2\pi}{\sqrt{1+4\pi^2}} \sin(2\pi t) \right)
\]

\[
= \frac{1}{2\sqrt{1+4\pi^2}} \left( \cos(\tan^{-1}(2\pi)) \cos(2\pi t) + \sin(\tan^{-1}(2\pi)) \sin(2\pi t) \right)
\]

\[
= \frac{1}{2\sqrt{1+4\pi^2}} \cos(2\pi t - \tan^{-1}(2\pi))
\]

Comparing $x(t)$ ($0.5\cos(2\pi t)$) and $y(t)$, we see that they are of the same frequency but their phases and magnitudes are different.

Nevertheless, we notice that computing the convolution is tedious even for simple signal and system. We will solve this problem again with the use of Fourier transform, which is shown to be easier.