z-Transform

z-transform is a generalization of Fourier transform for discrete-time signals (DTFT).

Given a discrete-time signal $x[n]$, the z-transform is defined as

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

where $z$ is a complex variable.

Let $z = re^{j\omega}$ ($r =$ magnitude of $z$ and $\omega =$ angle of $z$),

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-jn\omega} = F\{x[n]r^{-n}\}$$

$\Rightarrow$ $X(z)$ is the Fourier transform of $x[n]r^{-n}$.

When $r=1$ or $z = e^{j\omega}$,

$$X(z) = F\{x[n]\} = X(e^{j\omega})$$
Discrete-time Fourier transform is equal to the z-transform evaluated along the unit circle with $\omega$ varying from 0 to $2\pi$. 
Example 32

Determine the z-transform of \( x[n] = a^n u[n] \).

Solution:

\[
X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n
\]

\( X(z) \) converges if \( \sum_{n=0}^{\infty} \left| az^{-1} \right|^n < \infty \). This requires \( \left| az^{-1} \right| < 1 \) or \( |z| > |a| \), and

\[
X(z) = \frac{1}{1 - az^{-1}}
\]

Notice that for another signal \( x[n] = -a^n u[-n-1] \),

\[
X(z) = \sum_{n=-\infty}^{-1} (-a^n) z^{-n} = - \sum_{m=1}^{\infty} a^{-m} z^m = - \sum_{m=1}^{\infty} \left( a^{-1} z \right)^m
\]
In this case, $X(z)$ converges if $\left| a^{-1}z \right| < 1$ or $|z| < |a|$, and

$$X(z) = \frac{1}{1 - az^{-1}}$$
Example 33

Determine the z-transform of \( x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]. \)

Solution:

\[
X(z) = \sum_{n=0}^{\infty} 7\left(\frac{1}{3}\right)^n z^{-n} - \sum_{n=0}^{\infty} 6\left(\frac{1}{2}\right)^n z^{-n}
\]

\[
= \frac{7}{1 - \frac{1}{3}z^{-1}} - \frac{6}{1 - \frac{1}{2}z^{-1}}, \quad \left|\frac{1}{3}z^{-1}\right| < 1, \left|\frac{1}{2}z^{-1}\right| < 1
\]

\[
= \frac{z(z - \frac{3}{2})}{(z - \frac{1}{3})(z - \frac{1}{2})}, \quad \left|z\right| > \frac{1}{2}
\]
Example 34

Determine the z-transform of \( x[n] = \left( \frac{1}{3} \right)^n \sin \left( \frac{n\pi}{4} \right) u[n] \).

Solution:

\[
X(z) = \sum_{n=0}^{\infty} \left( \frac{1}{3} \right)^n \sin \left( \frac{n\pi}{4} \right) z^{-n}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{3} e^{\frac{j\pi}{4}} \right)^n - \left( \frac{1}{3} e^{-\frac{j\pi}{4}} \right)^n z^{-n}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{3} e^{\frac{j\pi}{4} z^{-1}} \right)^n - \left( \frac{1}{3} e^{-\frac{j\pi}{4} z^{-1}} \right)^n \]

\[
= \frac{1}{2j} \left[ \frac{1}{1 - \frac{1}{3} e^{\frac{j\pi}{4} z^{-1}}} - \frac{1}{1 - \frac{1}{3} e^{-\frac{j\pi}{4} z^{-1}}} \right], \quad |z| > \frac{1}{3}
\]
**ROC for z-Transform**

**Property 1:** The ROC of \( X(z) \) consists of a ring centered about the origin in the \( z \)-plane.

\[
\therefore X(z) = F\{x[n]r^{-n}\}, \quad z = re^{j\omega}, \text{ and } \{Fx[n]r^{-n}\} \text{ converges when } \sum_{n=-\infty}^{\infty} |x[n]|r^{-n} < \infty \text{ which depends only on } r = |z| \text{ and not on } \angle z.
\]

That is, if the ROC of \( X(z) \) contains a point \( z_0 \), it must contain the entire circle with \( |z| = |z_0| \).

**Property 2:** The ROC of \( X(z) \) contains no poles.

As with the Laplace transform, it is a consequence of the fact that at a pole \( X(z) \) is infinite which means that \( X(z) \) does not converge.
Property 3: If $x[n]$ has finite duration, the ROC of $X(z)$ is the entire $z$-plane, except possibly $z = 0$ and/or $z = \infty$.

Property 4: If $x[n]$ is right-sided, and if the circle $|z| = r_0$ is in ROC of $X(z)$, then all finite values of $z$ for which $|z| > r_0$ must be in ROC.

Property 5: If $x[n]$ is left-sided, and if the circle $|z| = r_0$ is in ROC of $X(z)$, then all values of $z$ for which $0 < |z| < r_0$ must be in ROC.

Property 6: If $x[n]$ is two-sided, and if the circle $|z| = r_0$ is in ROC of $X(z)$, then the ROC will consist of a ring that includes $|z| = r_0$.

<table>
<thead>
<tr>
<th>Signal type</th>
<th>Laplace transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite duration</td>
<td>Entire s-plane</td>
<td>Entire z-plane</td>
</tr>
<tr>
<td>Right-sided</td>
<td>The right of a line</td>
<td>Outside a circle</td>
</tr>
<tr>
<td>Left-sided</td>
<td>The left of a line</td>
<td>Inside a circle</td>
</tr>
<tr>
<td>Two-sided</td>
<td>A strip</td>
<td>A ring</td>
</tr>
</tbody>
</table>
**z-Plane and s-Plane**

Consider a continuous-time signal $x(t)$ and its sampled version:

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

The Laplace transform of $x_p(t)$ is,

$$X_p(s) = \sum_{n=-\infty}^{\infty} x(nT)e^{-snT} = \sum_{n=-\infty}^{\infty} x[n]e^{-snT}$$

where $x[n] = x(nT)$ is a discrete-time signal.

Comparing $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ and $X_p(s) = \sum_{n=-\infty}^{\infty} x[n]e^{-snT}$,

and since $X(z)$ & $X_p(s)$ are representing the same spectrum (except for a scaling of time axis), we have a mapping between $z$ and $s$:

$$z = e^{sT}$$
z-transform can be viewed as the Laplace transform of a sampled continuous-time signal with the change of variable $z = e^{sT}$.

- **$s$-plane**
  - $j\omega$-axis (e.g., $s = j\omega_1$)
  - Left-half (e.g., $s = \sigma_2 + j\omega_2, \sigma_2 > 0$)
  - Right-half

- **$z$-plane**
  - Unit circle ($z = e^{j\omega_1}$)
  - Inside unit circle ($z = e^{\sigma_2 T} e^{j\omega_2 T}$)
  - Outside unit circle
Property 7: If $X(z)$ is rational, then its ROC is bounded by poles or extends to infinity. This is because that ROC cannot contain any poles.

Property 8: If $X(z)$ is rational and $x[n]$ is right-sided, then its ROC is the region outside the origin-centered circle containing the outermost pole. Furthermore, if $x[n] = 0$ for all $n < 0$, the ROC contains $z = \infty$.

Property 9: If $X(z)$ is rational and $x[n]$ is left-sided, then its ROC is the region inside the origin-centered circle containing the innermost pole. If $x[n] = 0$ for all $n > 0$, the ROC contains $z = 0$. 
Example 35

Find out all of the possible ROCs for \( X(z) = \frac{1}{(1 - \frac{1}{3} z^{-1})(1 - 2z^{-1})} \).

Solution:

\( X(z) \) has two poles at \( z = \frac{1}{3} \) and \( z = 2 \). According to Properties 4 – 9, its ROCs can be:

1) Inside the circle \( |z| = \frac{1}{3} \), for which \( x[n] \) is left-sided.
2) Outside the circle \( |z| = 2 \), for which \( x[n] \) is right-sided.
3) The ring between \( |z| = \frac{1}{3} \) and \( |z| = 2 \), for which \( x[n] \) is two-sided.
Inverse z-Transform

The formal expression of inverse z-transform is given by

\[ x[n] = Z^{-1}\{X(z)\} = \frac{1}{2\pi j} \oint X(z)z^{n-1} dz \]

where \( \oint \) represents integration around a counterclockwise closed circular contour located in the ROC of \( X(z) \). Alternatively,

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega})(re^{j\omega})^n d\omega \]

Two useful methods of evaluating inverse z-transform are:

1) partial fraction
2) power-series expansion
Example 36

Find the inverse z-transform of \( X(z) = \frac{3 - \frac{5}{6} z^{-1}}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{3} z^{-1})} \), \(|z| > \frac{1}{3}\)

Solution:

\[
x(n) = \frac{1}{4} + \frac{2}{3} u(n)
\]

For \(|z| > \frac{1}{3}\), both of the two terms above result in right-sided sequence. That is,

\[
x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]
\]
If the ROC is $|z| < \frac{1}{4}$, we have two left-sided sequences,

$$x[n] = -\left(\frac{1}{4}\right)^n u[-n-1] - 2\left(\frac{1}{3}\right)^n u[-n-1]$$

If the ROC is $\frac{1}{4} < |z| < \frac{1}{3}$,

$$x[n] = \left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[-n-1]$$
Example 37

Find the inverse z-transform of \( X(z) = \frac{1}{1 - az^{-1}} \), \(|z| > |a|\), by power series expansion.

Solution:

\[
X(z) = \frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \cdots = \sum_{n=0}^{\infty} a^n z^{-n}, \quad |z| > |a|.
\]

By the definition of z-transform,

\[
x[n] = \begin{cases} 
a^n, & n \geq 0 \\
0, & n < 0
\end{cases} = a^n u[n]
\]

If the ROC is \(|z| < |a|\), the converging power series is

\[
X(z) = \frac{1}{1 - az^{-1}} = -a^{-1} z - a^{-2} z^2 - \cdots = \sum_{n=-\infty}^{1} (-a^n) z^{-n}
\]

Then its inverse z-transform is \( x[n] = -a^n u[-n-1] \)
Example 38
Find the inverse z-transform of \( X(z) = \log(1 + az^{-1}) \), \(|z| > |a|\), by power series expansion.

Solution:
Taylor’s series expansion,

\[
\log(1 + v) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} v^n}{n}, \quad |v| < 1
\]

For \(|z| > |a|\) or \(|az^{-1}| < 1\),

\[
X(z) = \log(1 + az^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (az^{-1})^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n}{n} z^{-n}
\]

Thus,

\[
x[n] = \begin{cases} 
\frac{-(-a)^n}{n}, & n \geq 1 \\
0, & n < 1 
\end{cases} = \frac{-(-a)^n}{n} u[n-1]
\]
# Properties of z-Transform

<table>
<thead>
<tr>
<th>Property</th>
<th>Signal</th>
<th>z-Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x[n]$</td>
<td>$X(z)$</td>
<td>$R$</td>
<td></td>
</tr>
<tr>
<td>$x_1[n]$</td>
<td>$X_1(z)$</td>
<td>$R_1$</td>
<td></td>
</tr>
<tr>
<td>$x_2[n]$</td>
<td>$X_2(z)$</td>
<td>$R_2$</td>
<td></td>
</tr>
<tr>
<td><strong>Linearity</strong></td>
<td>$ax_1[n] + bx_2[n]$</td>
<td>$aX_1(z) + bX_2(z)$</td>
<td>At least the intersection of $R_1$ and $R_2$</td>
</tr>
<tr>
<td><strong>Time shifting</strong></td>
<td>$x[n - n_0]$</td>
<td>$z^{-n_0}X(z)$</td>
<td>$R$, except for the possible addition or deletion of the origin</td>
</tr>
<tr>
<td><strong>Scaling in the z-domain</strong></td>
<td>$e^{j\omega_0 n}x[n]$</td>
<td>$X(e^{-j\omega_0 z})$</td>
<td>$R$</td>
</tr>
<tr>
<td></td>
<td>$z_0^n x[n]$</td>
<td>$X\left(\frac{z}{z_0}\right)$</td>
<td>$z_0 R$</td>
</tr>
<tr>
<td></td>
<td>$a^n x[n]$</td>
<td>$X(a^{-1}z)$</td>
<td>Scaled version of $R$ (i.e., $</td>
</tr>
<tr>
<td><strong>Time reversal</strong></td>
<td>$x[-n]$</td>
<td>$X(z^{-1})$</td>
<td>Inverted $R$ (i.e., $R^{-1} =$ the set of points $z^{-1}$, where $z$ is in $R$)</td>
</tr>
<tr>
<td><strong>Time expansion</strong></td>
<td>$x_{r_0}[n] = \begin{cases} x[r], &amp; n = rk \ 0, &amp; n \neq rk \end{cases}$ for some integer $r$</td>
<td>$X(z^r)$</td>
<td>$R^{1/r}$ (i.e., the set of points $z^{1/r}$, where $z$ is in $R$)</td>
</tr>
<tr>
<td><strong>Conjugation</strong></td>
<td>$x^*[n]$</td>
<td>$X^<em>(z^</em>)$</td>
<td>$R$</td>
</tr>
<tr>
<td><strong>Convolution</strong></td>
<td>$x_1[n] * x_2[n]$</td>
<td>$X_1(z)X_2(z)$</td>
<td>At least the intersection of $R_1$ and $R_2$</td>
</tr>
<tr>
<td><strong>First difference</strong></td>
<td>$x[n] - x[n-1]$</td>
<td>$(1-z^{-1})X(z)$</td>
<td>At least the intersection of $R$ and $</td>
</tr>
<tr>
<td><strong>Accumulation</strong></td>
<td>$\sum_{k=-\infty}^{n} x[k]$</td>
<td>$\frac{1}{1-z^{-1}}X(z)$</td>
<td>At least the intersection of $R$ and $</td>
</tr>
<tr>
<td><strong>Differentiation in the z-domain</strong></td>
<td>$nx[n]$</td>
<td>$-\frac{dX(z)}{dz}$</td>
<td>$R$</td>
</tr>
</tbody>
</table>

**Initial Value Theorem**

If $x[n] = 0$ for $n < 0$, then

$$x[0] = \lim_{z \to \infty} X(z)$$

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Example 39

Find out the inverse z-transform of \( X(z) = \log(1 + az^{-1}) \), \(|z| > |a|\), using the properties of z-transform.

Solution:

\[
\frac{dX(z)}{dz} = \frac{1}{1 + az^{-1}} \cdot (-az^{-2}) = -\frac{az^{-2}}{1 + az^{-1}}
\]

\[
- \cdot z \frac{dX(z)}{dz} = \frac{az^{-1}}{1 + az^{-1}} \quad \cdot \frac{d \log(f(x))}{dx} = \frac{d \log(f(x))}{d(f(x))} \cdot \frac{d(f(x))}{dx}
\]

By the property of differentiation in z-domain,

\[
x[n] \leftarrow Z \rightarrow \frac{az^{-1}}{1 + az^{-1}}
\]
Since $Z^{-1}\left\{\frac{1}{1+az^{-1}}\right\} = (-a)^n u[n]$, by the time-shift property,

$$Z^{-1}\{z^{-1} \cdot \frac{1}{1+az^{-1}}\} = (-a)^{n-1} u[n-1]$$

$$\Rightarrow Z^{-1}\{\frac{az^{-1}}{1+az^{-1}}\} = a(-a)^{n-1} u[n-1]$$

$$\therefore nx[n] = a(-a)^{n-1} u[n-1]$$

$$\Rightarrow x[n] = \frac{-(a)^n}{n} u[n-1]$$
## Common z-transform pairs

<table>
<thead>
<tr>
<th>Signal</th>
<th>Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta[n] )</td>
<td>1</td>
<td>All ( z )</td>
</tr>
<tr>
<td>( u[n] )</td>
<td>( \frac{1}{1 - z^{-1}} )</td>
<td>(</td>
</tr>
<tr>
<td>( -u[-n-1] )</td>
<td>( \frac{1}{1 - z^{-1}} )</td>
<td>(</td>
</tr>
<tr>
<td>( \delta[n-m] )</td>
<td>( z^{-m} )</td>
<td>All ( z ), except ( 0 ) (if ( m &gt; 0 )) or ( \infty ) (if ( m &lt; 0 ))</td>
</tr>
<tr>
<td>( \alpha^n u[n] )</td>
<td>( \frac{1}{1 - \alpha z^{-1}} )</td>
<td>(</td>
</tr>
<tr>
<td>( -\alpha^n u[-n-1] )</td>
<td>( \frac{1}{1 - \alpha z^{-1}} )</td>
<td>(</td>
</tr>
<tr>
<td>( n\alpha^n u[n] )</td>
<td>( \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2} )</td>
<td>(</td>
</tr>
<tr>
<td>( -n\alpha^n u[-n-1] )</td>
<td>( \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2} )</td>
<td>(</td>
</tr>
<tr>
<td>( [\cos \omega_0 n] u[n] )</td>
<td>( \frac{1 - [\cos \omega_0] z^{-1}}{1 - [2 \cos \omega_0] z^{-1} + z^{-2}} )</td>
<td>(</td>
</tr>
<tr>
<td>( [\sin \omega_0 n] u[n] )</td>
<td>( \frac{[\sin \omega_0] z^{-1}}{1 - [2 \cos \omega_0] z^{-1} + z^{-2}} )</td>
<td>(</td>
</tr>
<tr>
<td>( [r^n \cos \omega_0 n] u[n] )</td>
<td>( \frac{1 - [r \cos \omega_0] z^{-1}}{1 - [2r \cos \omega_0] z^{-1} + r^2 z^{-2}} )</td>
<td>(</td>
</tr>
<tr>
<td>( [r^n \sin \omega_0 n] u[n] )</td>
<td>( \frac{[r \sin \omega_0] z^{-1}}{1 - [2r \cos \omega_0] z^{-1} + r^2 z^{-2}} )</td>
<td>(</td>
</tr>
</tbody>
</table>
**z-Transform and LTI systems**

In $z$-domain, a discrete-time LTI system is described as

$$Y(z) = H(z)X(z)$$

where $Y(z)$ and $X(z)$ are the $z$-transforms of output $y[n]$ and input $x[n]$, respectively, while $H(z)$ is the $z$-transform of impulse response $h[n]$.

$H(z)$ is called **system function** or **transfer function**.

For $z = e^{j\omega}$, $H(e^{j\omega})$ becomes the **frequency response**.
Causality and ROC

Causality condition: \( h[n] = 0 \) for all \( n < 0 \),

\( h[n] \) is right-sided

The ROC for \( H(z) \) is

the exterior of an origin-centered circle (including \( z = \infty \))

If \( H(z) \) is rational, the ROC for \( H(z) \) is

the exterior outside the outermost pole.

And, the order of numerator polynomial is not larger than that of the denominator.
Stability and ROC

Stability condition: \[ \sum_{n=-\infty}^{\infty} |h[n]| < \infty \]

\[ H(e^{j\omega}) \], i.e., the Fourier transform of \( h[n] \), converges

The ROC for \( H(z) \) **includes the unit circle** \( |z| = 1 \)

A discrete-time **causal** LTI system with rational system function \( H(z) \) is **stable** if and only if all of the poles of \( H(z) \) lie inside the unit circle.
**Example 40**

Consider a LTI system with $H(z) = \frac{1 + \frac{1}{2} z^{-1}}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 - 2 z^{-1}}$, $|z| > 2$. Is it causal? Is it stable? Why?

**Solution:**

The ROC of $H(z)$ is $|z| > 2$, outside the outermost pole. Also,

$$H(z) = \frac{2 - \frac{5}{2} z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 - 2 z^{-1})} = \frac{2 z^2 - \frac{5}{2} z}{z^2 - \frac{5}{2} z + 1}$$

The numerator’s order is equal to the denominator’s

Thus, the system is causal. Its impulse response is given as

$$h[n] = \left[ \left( \frac{1}{2} \right)^n + 2^n \right] u[n]$$
Since the ROC of $H(z)$ does not include the unit circle, the system is not stable. This can also be seen from

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \left| \left( \frac{1}{2} \right)^n + 2^n \right| \to \infty$$

For the system to be stable, $H(z)$ should be $\frac{1}{2} < |z| < 2$, i.e., including the unit circle.
**Example 41**

Can the following system be causal and stable?

\[ H(z) = \frac{1}{1 - az^{-1}} \]

**Solution:**

The system function \( \frac{1}{1 - az^{-1}} \) has two possible ROCs: \( |z| > |a| \) and \( |z| < |a| \).

For the system to be causal, the ROC of \( H(z) \) must be the exterior of some origin-centered circle. Hence, the ROC should be \( |z| > |a| \).

And the impulse response is \( h[n] = a^n u[n] \).

For the causal system to be stable, its ROC must include the unit circle \( |z| = 1 \) as well. This requires \( |a| < 1 \).
Example 42
Discuss about the causality and stability of a system with

\[ H(z) = \frac{1}{1 - (2r \cos \theta)z^{-1} + r^2 z^{-2}} \]

Solution:
\( H(z) \) has a pair of conjugate poles \( z_1 = re^{j\theta} \) and \( z_2 = re^{-j\theta} \).

For system to be causal, the ROC of \( H(z) \) should be \(|z| > |r|\).

For system to be stable, we need \(|r| < 1\), i.e., pole inside unit circle.
Example 43

Consider a LTI system for which the input $x[n]$ and output $y[n]$ satisfy the linear constant-coefficient difference equation,

$$y[n] - \frac{1}{2} y[n-1] = x[n] + \frac{1}{3} x[n-1]$$

Find the system function and comment on its causality and stability.

Solution:
Taking z-transform on both sides,

$$Y(z) - \frac{1}{2} z^{-1} Y(z) = X(z) + \frac{1}{3} z^{-1} X(z)$$

Thus,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{1}{2} z^{-1}}$$
For the system to be causal, the ROC of $H(z)$ should be $|z| > \frac{1}{2}$.

And in this case, the system is stable since the ROC includes the unit circle. Expressing $H(z)$ as,

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{3} \cdot \frac{z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

Then, by the property of time-shift, for $|z| > 1/2$

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3} \left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \Rightarrow H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$
Block Diagram & Transfer Function

\[
y[n] + ay[n-1] = bx[n]
\]

\[
\frac{dy(t)}{dt} + ay(t) = bx(t)
\]

\[
y(t) = y(t_0) + \int_{t_0}^{t} [bx(\tau) - ay(\tau)]d\tau
\]
$$H(z) = \frac{1 - 2z^{-1}}{1 - \frac{1}{4}z^{-1}}$$

using less delay units
\[ H(z) = \frac{1}{(1 + \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})} = \frac{1}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} \]

Equivalently, \( y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] \)
Alternatively, \( H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} \cdot \frac{1}{1 - \frac{1}{4}z^{-1}} \)

The system is resulted from the cascade of two systems:
\[
w[n] + \frac{1}{2}w[n-1] = x[n] \quad \text{and} \quad y[n] - \frac{1}{4}y[n-1] = w[n]
\]
Or \( H(z) = \frac{2}{3} \cdot \frac{1}{1 + \frac{1}{2} z^{-1}} + \frac{1}{3} \cdot \frac{1}{1 - \frac{1}{4} z^{-1}} \)

The system is as the parallel of two systems:

\[
v_1[n] + \frac{1}{2} v_1[n-1] = \frac{2}{3} x[n] \quad \text{and} \quad v_2[n] - \frac{1}{4} v_2[n-1] = \frac{1}{3} x[n]
\]

and

\[
y[n] = v_1[n] + v_2[n]
\]