1. Using the window design technique with rectangular window, design a length-21 casual linear-phase FIR filter that approximates an ideal low-pass filter with cutoff frequency of 500Hz. The sampling frequency is 8000Hz.

2. Compute the energy for $h_d[n]$:

$$h_d[n] = \frac{\omega_c}{\pi} \text{sinc}\left(\frac{\omega_c n}{\pi}\right), \quad \omega_c = 0.1\pi$$

which corresponds to an ideal low-pass filter:

$$H_d(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \text{otherwise} \end{cases}$$
3. Given a discrete-time sinusoid:

\[ x[n] = 2 \cos(0.7\pi n + 1), \quad n = 0, 1, \ldots, 20 \]

Suggest how to use DFT to find its frequency.
Solution

1. With the use of Example 11.6, the required impulse response is:

\[ h[n] = \begin{cases} h_d[n], & 0 \leq n \leq 20 \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad \alpha = 10 \]

where

\[ h_d[n] = \frac{\sin(\omega_c (n - \alpha))}{\pi(n - \alpha)} \]

To determine \( \omega_c \), we need the information: cutoff frequency of 500Hz and sampling frequency of 8000Hz. Following Example 11.7, we have \( \omega_c = 2\pi \cdot 500 / 8000 = 0.125\pi \)
Alternatively, recall (6.3):

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n T} \]

which is a general DTFT formula for any sampling period \( T \). Comparing it with the DTFT formula:

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \]

We see that \( \omega \) in the DTFT formula corresponds to \( \omega T \) in (6.13). As a result, \( \omega_c \) is determined as:

\[ \omega_c = 2\pi \cdot \frac{500}{8000} = 0.125\pi \]
The required impulse response is:

\[ h[n] = \begin{cases} 
0.125, \\
\sin(0.125\pi(n-10)) / \pi(n-10), \\
20, & n = 10 \\
11, & n = 10 \\
9, & n = 10 \\
10, & n = 10 \\
\end{cases} \]

\[ n = 0, 1, \ldots, 9, 11, \ldots, 20 \]
2. Using Parseval’s relation for DTFT

\[
\text{Signal Energy} = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega
\]

The signal energy is easily computed in the frequency domain:

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \frac{1}{2\pi} \int_{-0.1\pi}^{0.1\pi} 1 d\omega = \frac{0.2\pi}{2\pi} = 0.1
\]
3. First we consider the continuous-time case. Suppose we have a sinusoidal signal \( x(t) = \cos(\omega_o t) \). If we take the Fourier transform, two impulse functions will be observed in the frequency positions of \( \omega_o \) and \( -\omega_o \).

It is because
\[
x(t) = \cos(\omega_o t) = 0.5e^{j\omega_o t} + 0.5e^{-j\omega_o t}
\]

And from (2.8), we have
\[
\Rightarrow \quad e^{j\omega_0 t} \xrightarrow{\text{Fourier transform}} 2\pi \delta(\omega - \omega_0) \quad \xrightarrow{\text{Inverse Fourier transform}}
\]

This means that we can find the frequency from one of peak of \( X(\omega) \). This concept of determining the frequency will be utilized in discrete-time signals as follows.
We compute the DFT of \( x[n] \) using MATLAB:

\[
\begin{align*}
N &= 21; \quad \text{% number of samples is 21} \\
A &= 2; \quad \text{% tone amplitude is 2} \\
w &= 0.7\pi; \quad \text{% frequency is 0.7}\pi \\
p &= 1; \quad \text{% phase is 1} \\
n &= 0: N-1; \quad \text{% define a vector of size N} \\
x &= A \cos(wn+p); \quad \text{% generate tone} \\
X &= \text{fft}(x); \\
\text{subplot}(2,1,1); \\
\text{stem}(n, \text{abs}(X)); \quad \text{% plot magnitude response} \\
\text{title('Magnitude Response');} \\
\text{subplot}(2,1,2); \\
\text{stem}(n, \text{angle}(X)); \quad \text{% plot phase response} \\
\text{title('Phase Response');}
\end{align*}
\]
\( X = \\
1.0806 \hspace{1cm} 1.0674+0.2939i \hspace{1cm} 1.0243+0.6130i \\
0.9382+0.9931i \hspace{1cm} 0.7756+1.5027i \hspace{1cm} 0.4409+2.3159i \\
-0.4524+4.1068i \hspace{1cm} -6.7461+15.1792i \hspace{1cm} 6.5451-7.2043i \\
3.8608-2.1316i \hspace{1cm} 3.3521-0.5718i \hspace{1cm} 3.3521+0.5718i \\
3.8608+2.1316i \hspace{1cm} 6.5451+7.2043i \hspace{1cm} -6.7461-15.1792i \\
-0.4524-4.1068i \hspace{1cm} 0.4409-2.3159i \hspace{1cm} 0.7756-1.5027i \\
0.9382-0.9931i \hspace{1cm} 1.0243-0.6130i \hspace{1cm} 1.0674-0.2939i \\

The maximum value of \( X[7] \) corresponds to \( e^{j0.7\pi} \) while maximum value of \( X[14] \) corresponds to \( e^{-j0.7\pi} \). Recall Example 7.2, we know 21 corresponds to \( 2\pi \). As a result, we can get a coarse frequency estimate as:

\[
\hat{\omega}_0 = 7 \cdot \frac{2\pi}{21} = 0.6667\pi
\]
We also notice that if the signal has length 22:

\[ x[n] = 2 \cos(0.7\pi n + 1), \quad n = 0, 1, \ldots, 21 \]

\[
\begin{array}{cccc}
X &=& -0.9161 & -0.9376-0.0169i & -1.0075-0.0352i \\
    & & -1.1447-0.0566i & -1.3957-0.0843i & -1.8830-0.1261i \\
    & & -3.0267-0.2067i & -7.6747-0.4909i & 19.3903+1.0481i \\
    & & 5.0629+0.1984i & 3.4228+0.0705i & 3.0773 \\
    & & 3.4228-0.0705i & 5.0629-0.1984i & 19.3903-1.0481i \\
    & & -7.6747+0.4909i & -3.0267+0.2067i & -1.8830+0.1261i \\
    & & -1.3957+0.0843i & -1.1447+0.0566i & -1.0075+0.0352i \\
    & & -0.9376+0.0169i & & \\
\end{array}
\]

The peak appears at the 8th position of the DFT plot. The frequency estimate is then:

\[
\hat{\omega}_0 = 8 \cdot \frac{2\pi}{22} = 0.7273\pi
\]
From the two examples, we see:

\[
\text{Re}\{X[k]\} = \text{Re}\{X[N - k]\}
\]

\[
\text{Im}\{X[k]\} = -\text{Im}\{X[N - k]\}
\]

\[
|X[k]| = |X[N - k]|
\]

\[
\angle X[k] = -\angle X[N - k], \quad k = 0, 1, \ldots, N - 1
\]

⇒ For real signals, we only need to compute around half of the DFT coefficients.

To get a higher resolution frequency estimate, we should use DTFT. However, DTFT is continuous in frequency. Recall (7.11):

\[
\tilde{X}[k] = X(e^{j\omega}) \bigg|_{\omega = (2\pi / N)k}
\]
If we increase $N$, the DFS coefficients $\tilde{X}[k]$ (or DFT coefficients $X[k]$ between 0 and $N-1$) will contain more sample points of $X(e^{j\omega})$. This makes the DFT looks like the DTFT by using a sufficiently large value of $N$. It is achieved by appending zeros at the end of the finite-duration signals.

Consider to add 21 samples:

```matlab
N=42;
N1=21;
A=2;
w=0.7*pi;
p=1;
n=0:N1-1;
```

% number of samples is 42
% non-zero sample number is 21
% tone amplitude is 2
% frequency is 0.7*pi
% phase is 1
% define a vector of size N1
x = A*cos(w*n+p);   % generate tone
x = [x, zeros(1,N-N1)]; % append 21 zeros
X = fft(x);
subplot(2,1,1);
stem(0:N-1, abs(X)); % plot magnitude response
title('Magnitude Response');
axis([0 N-1 0 21]);
subplot(2,1,2);
stem(0:N-1, angle(X)); % plot phase response
title('Phase Response');
axis([0 N-1 -4 4]);
\[ X = \]

\[
\begin{array}{cccc}
1.0806 & -0.8603-0.0433i & 1.0674+0.2939i \\
-0.8841-0.1335i & 1.0243+0.6130i & -0.9382-0.2359i \\
0.9382+0.9931i & -1.0411-0.3647i & 0.7756+1.5027i \\
-1.2408-0.5511i & 0.4409+2.3159i & -1.6943-0.8903i \\
-0.4524+4.1068i & -3.3093-1.9168i & -6.7461+15.1792i \\
17.3059+10.0299i & 6.5451-7.2043i & 2.2018+1.0821i \\
3.8608-2.1316i & 1.2665+0.3671i & 3.3521-0.5718i \\
1.0806 & 3.3521+0.5718i & 1.2665-0.3671i \\
3.8608+2.1316i & 2.2018-1.0821i & 6.5451+7.2043i \\
17.3059-10.0299i & -6.7461-15.1792i & -3.3093+1.9168i \\
-0.4524-4.1068i & -1.6943+0.8903i & 0.4409-2.3159i \\
-1.2408+0.5511i & 0.7756-1.5027i & -1.0411+0.3647i \\
0.9382-0.9931i & -0.9382+0.2359i & 1.0243-0.6130i \\
-0.8841+0.1335i & 1.0674-0.2939i & -0.8603+0.0433i \\
\end{array}
\]
Consider to add 201 samples:
Consider to add 2001 samples:

```
Magnitude Response

Phase Response
```
It is observed that for this example, appending 201 or 2001 zeros is good enough for the DTFT plot from DFT.

Here the signal is a tone with frequency $\omega_0 = 0.7\pi$. The frequency can be obtained from the DTFT plot as follows.

The peak of the DTFT is located at the 709th position of the DFT plot. Note that it is not needed to find the second peak information as for real signals, the DFT is symmetric around the mid-point. The peak index is found using MATLAB:

\[
[Y,I]=\text{max}(\text{abs}(X))
\]

\[
Y = 20.6090
\]

\[
I = 710
\]

Note that $X$ starts at 1 in MATLAB so peak index is $I-1$. 
Recall Example 7.2, the interval \([0, N]\) in DFT corresponds to the interval \([0, 2\pi]\) in DTFT. Now \(N = 2022\).

As a result, 709 corresponds to the frequency value:

\[
\hat{\omega}_0 = 709 \cdot \frac{2\pi}{2022} = 0.7013\pi
\]