Equivalent Convolutional Codes

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Introduction

- Conventional trellis of an \((n, k, \nu)\) binary convolutional code:
  (i) \(n\)-bit branches
  (ii) \(2^k\) branches departing from or entering a node
  (iii) \(2^\nu\) nodes at each depth

- Minimal trellis of an \((n, k)\) binary convolutional code:
  (i) one-bit branches
  (ii) One or two branches departing from or entering a node
  (iii) Number of nodes at each depth varies

We will show that, for \(\nu \neq \nu'\), an \((n, k, \nu)\) convolutional code may be equivalent to an \((n, k, \nu')\) convolutional code in the sense that their minimal trellises are shifted versions of each other.
Some Basics of Convolutional Codes (I)

- An \((n,k)\) convolutional code \(C\) is defined by a \(k \times n\) polynomial generator matrix \(G(D) = [g_{i,j}(D)]\), where \(g_{i,j}(D)\) is a polynomial over \(F\), \(1 \leq i \leq k, 0 \leq j < n\). The memory size of the encoder \(G(D)\) is the sum

\[
\nu_G = \sum_{i=1}^{k} \max_j \{\deg g_{i,j}(D)\}.
\]  

- \(G(D)\) is said to be a minimal encoder for \(C\) if it has the minimum memory size \(\nu\) over all encoders realizing the code.

- Let \(g_i(D) = (g_{i,0}(D), g_{i,1}(D), \ldots, g_{i,n-1}(D)) = \sum_{j=0}^{m_i} g_{i,j}^j D^j\), where \(g_{i,j}^j\) is an \(n\)-tuple over \(F\) and \(m_i = \max_j \{\deg g_{i,j}(D)\}\).

- Rewrite \(G(D)\) as \(\sum_{j=0}^{m} G_j D^j\), where \(G_j\) is a \(k \times n\) matrix over \(F\) and \(m = \max_i \{m_i\}\).
Some Basics of Convolutional Codes (II)

- The convolutional code $C$ with $G(D)$ can be viewed as a block code over $F$ that has a generator matrix $G_{scalar}$ over $F$ for $C$

$$G_{scalar} = \begin{pmatrix}
G_0 & G_1 & \cdots & G_m \\
G_0 & G_1 & \cdots & G_m \\
G_0 & G_1 & \cdots & G_m \\
\vdots & \vdots & \ddots & \vdots
\end{pmatrix}. \quad (2)$$

- A vertical slice of $G_{scalar}$: $[0^T, \cdots, 0^T, G_m^T, \cdots, G_1^T, G_0^T, 0^T, \cdots, 0^T]^T$ can be represented by a minimal trellis module.
Some Basics of Convolutional Codes (III)

- A code $\hat{C}$ is the reciprocal code of $C$ if there is a minimal encoder $\hat{G}(D)$ for $\hat{C}$ that is realized by the generators $\hat{g}_i(D) = \sum_{j=0}^{m_i} g_i^{m_i-j} D^j$.

- Thus $\hat{C}^\bot = \overline{C}^\bot$ and $\nu_G = \nu_{\hat{G}}$, where $C^\bot$ is the dual code of $C$. 

Minimal Trellis Module (I)

- Two matrices are interesting: \( G_0 \) and \( G_{\text{end}} = [(g_1^{m_1})^T \cdots (g_k^{m_k})^T]^T \).
- Two spaces are defined: \( V_0 = \text{span}(G_0) \) and \( V_{\text{end}} = \text{span}(G_{\text{end}}) \).
- Define \( i^- = \{0, \cdots, i - 1\} \), \( i^+ = \{i, \cdots, n - 1\} \) and \( 0^+ = n^- = I \) and \( 0^- = n^+ = \phi \) the empty set.
- The dimension of state space at depth \( i \) (mod \( n \)) for a minimal trellis module of an \((n, k, \nu)\) code \( C \) is

\[
s_i = k + \nu - \dim(V_0, i^+) - \dim(V_{\text{end}, i^-}).
\]

- The \( V_0, i^+ \) is a subspace of \( V_0 \) consisitng of all the vectors of \( V_0 \) for which the components with indices outside \( i^+ \) are zero.
- The \( V_{\text{end}, i^-} \) is a subspace of \( V_{\text{end}} \) consisitng of all the vectors of \( V_{\text{end}} \) for which the components with indices outside \( i^- \) are zero.
Minimal Trellis Module (II)

- The state complexity profile of a convolutional code $C$ is defined as \( \{s_0, s_1, \cdots, s_{n-1}\} \).

- **Theorem 1** The state complexity profiles of $C$ and its reciprocal dual code $\hat{C}^\perp$ are identical.

Minimal trellis modules of a (8, 7) convolutional code $C$ and its reciprocal dual $\hat{C}^\perp$ with state complexity profile \{4, 4, 4, 4, 3, 4, 4, 4\}
Equivalent Codes — Block Codes (I)

• In (3), we have the block code for $\nu = 0$ and $V_0 = V_{end}$.

• By applying permutation to $n$ bits of each code word of an $(n, k)$ block code $V$, we have an equivalent code $V'$ of $V$.

• The state complexity $s(V)$ of $V$ and all its equivalent codes is defined as $\min_{V'} \{s_{\max}(V')\}$, where $V'$ is an equivalent code of $V$ and $s_{\max}(V') = \max_i \{s_i(V')\}$ is the largest value among the state complexity profile of $V'$. 
Equivalent Codes — Block Codes (II)

- Let $\underline{x} = \{x_0, x_1, x_2, \cdots \}$ be a nonzero sequence of symbols.

- Its left index, denoted $L(\underline{x})$, is the smallest index $i$ such that $x_i \neq 0$. Similarly, the right index of $x$ if exists, denoted $R(\underline{x})$ is the largest index $i$ such that $x_i \neq 0$.

- Whenever $R(\underline{x})$ exists, the span of $\underline{x}$, denoted $\text{Span}(\underline{x})$, is the discrete interval $[L(\underline{x}), R(\underline{x})] = [L(\underline{x}), L(\underline{x}) + 1, \cdots, R(\underline{x})]$. Otherwise $\text{Span}(\underline{x}) = [L(\underline{x}), \infty]$.

- A nonzero sequence $x$ is said to be active at depth $i$ if both $i - 1$ and $i$ are in $\text{Span}(\underline{x})$. 
Equivalent Codes — Block Codes (III)

- A generator matrix of a linear block code is said to be a minimal span generator matrix (MSGM) if for any two distinct rows \( \underline{x}_p \) and \( \underline{x}_q \) of it, we have \( L(\underline{x}_p) \neq L(\underline{x}_q) \) which is termed the \( L \)-property and \( R(\underline{x}_p) \neq R(\underline{x}_q) \) which is termed the \( R \)-property. Equivalently, we may say an MSGM is with \( LR \)-property.

- (Schuurman) Suppose an \((n, k)\) block code \( V \) has its largest value among the state complexity profile termed \( s_{max}(V) \). Then there exists an equivalent code \( V' \) with a generator matrix \( G' \) in which the rows are \( \underline{x}_1, \ldots, \underline{x}_k \). We may have
  - \( G' \) is an MSGM.
  - \( w(\underline{x}_1) = R(\underline{x}_1) + 1 \), where \( w(\underline{x}_1) \) denotes the Hamming weight of \( \underline{x}_1 \).
  - \( R(\underline{x}_j) > R(\underline{x}_1) \), for \( j > 1 \); and
  - \( s_{max}(V') \leq s_{max}(V) \).
Equivalent Codes — Convolutional Codes (I)

1. \( G(D) = \sum_{j=0}^{m} G_j D^j \) is said to be a minimal encoder for \( C \) if it has the minimum memory size \( \nu \) over all encoders realizing the code.

2. For the two matrices \( G_0 \) and \( G_{\text{end}} \), suppose that \( G_0 \) and \( G_{\text{end}} \) are with \( L \)-property and \( R \)-property respectively.

3. According to (3), \( s_i(C') \) is the number of rows in \( G_{\text{scalar}} \) which are active at depth \( pn + i \) for \( p \geq m \).

4. The codes realized by \( G(D) \) and \( G'(D) = \sum_{j=0}^{m} G'_j D^j \) respectively are equivalent if all the \( G'_j \) are obtained by applying the same column permutation on the corresponding \( G_j \).
Equivalent Codes — Convolutional Codes (II)

- The decoding complexity according to the traditional trellis is identical for all the equivalent codes.

- The state complexity profiles are changed from one equivalent code to another. Thus the decoding complexity according to the minimum trellis modules is different for equivalent codes.

- An equivalent code is optimal if its minimum trellis module has the smallest amount of nodes over all equivalent codes.
Equivalent Codes — Convolutional Codes (III)

- For an \((n, 1)\) convolutional code \(C\) with memory size \(\nu\) and minimal encoder \(g(D) = \sum_{i=0}^{\nu} g_i D^i\), it is clear that the state complexity profile is completely determined by \(g_0\) and \(g_\nu\).

- Let \(I = \{0, 1, \cdots, n - 1\}\) be the index set. Let \(A = \text{supp}(g^\nu)\) and \(B = \text{supp}(g^0)\). Also denote \(S_A = \{0, 1, \cdots, R(g^\nu)\}\) and \(S_B = \{L(g^0), \cdots, n - 1\}\), then \(A \subseteq S_A\) and \(B \subseteq S_B\).

- For an \((n, 1)\) convolutional code \(C\), the permutation

\[
\pi = \begin{cases} 
(A - B) \land (A \cap B) \land (I - A) & \text{if } A \cap B \neq \phi; \\
A \land (I - (A \cup B)) \land B & \text{if } A \cap B = \phi
\end{cases}
\]  

(4)

will result in an optimally equivalent code of \(C\), where \(X \land Y\) means the concatenation of two ordered sets \(X\) and \(Y\).
Optimally Equivalent \((n, 1)\) and \((n, n - 1)\) Codes (I)

- **Example:** Consider an \((8,7,4)\) convolutional code \(C\) with minimal encoder

\[
G(D) = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 2 & 0 & 2 & 0 & 1 & 1 \\
0 & 1 & 2 & 0 & 0 & 0 & 0 & 1 \\
2 & 0 & 4 & 4 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]  

(5)

- The minimal encoder \(H(D)\) for \(C^\perp\) is \((26,32,3,17,5,11,20,34)\) which is represented in octal form.

- The state complexity profile of either \(C\) or \(\widehat{C^\perp}\) is \(\{4,5,5,5,5,4,4\}\).
Optimally Equivalent \((n, 1)\) and \((n, n - 1)\) Codes (II)

- The associated \(V_{\text{end}}^{\widehat{C}^\perp}\) and \(V_0^{\widehat{C}^\perp}\) are \(\text{span}\{(00111100)\}\) and \(\text{span}\{(11000011)\}\) respectively.

- For \(\widehat{C}^\perp\), we have \(A = \text{supp}(g^\nu) = \{2, 3, 4, 5\}\) and \(B = \text{supp}(g^0) = \{0, 1, 6, 7\}\). Hence an optimal permutation for the reciprocal dual code \(\widehat{C}^\perp\) is \(\pi = (2, 3, 4, 5, 0, 1, 6, 7)\).

- The state complexity profile for the equivalent code of either \(C\) or \(\widehat{C}^\perp\) is \(\{4, 4, 4, 4, 3, 4, 4, 4\}\) that is better than that of the original code.

- The minimal trellis modules of \(C\) and \(\widehat{C}^\perp\) shown in the next page are those shown in page 8.
Optimally Equivalent \((n, 1)\) and \((n, n - 1)\) Codes (III)

Figure 1: The minimal trellis module for optimally equivalent code of \(C\), for which its minimal encoder is given in (5).
Good \((n, n - 1)\) Convolutional Codes (I)

- The decoding complexity of an \((n, n - 1)\) code \(C\) with memory size \(\nu\) can be evaluated by
  \[
  \sum_{i=0}^{n-1} s_i = n\nu - \chi, \tag{6}
  \]
  where \(\chi \in \{-n + 1, -n + 2, \ldots, 0, \ldots, n - 1\}\).

- In case the complexity is measured by the number of branches, the decoding complexity is \((n - \chi) \cdot 2^{\nu+1} + (\chi - 1) \cdot 2^\nu\) for \(\chi \geq 0\) and is \((n - \chi - 1) \cdot 2^{\nu+1}\) for \(\chi < 0\).

- Note that a larger \(\chi\) value will result in lower decoding complexity.
**Good \((n, n - 1)\) Convolutional Codes (II)**

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\nu)</th>
<th>(G(D))</th>
<th>(d_{free})</th>
<th>(\chi)</th>
<th>Spectra (\frac{t_1, t_2, \ldots}{t_1, t_2, \ldots})</th>
</tr>
</thead>
</table>
| 3  | 2  | \[
\begin{pmatrix}
0 & 2 & 3 \\
3 & 3 & 1 \\
\end{pmatrix}
\] | 3 | -1 | 1, 4, 14, 40, 115, 331, 953, \ldots \\
| | | | | | 1, 10, 54, 226, 853, 3038, 10423, \ldots |
| 3  | 2  | \[
\begin{pmatrix}
3 & 2 & 0 \\
2 & 1 & 3 \\
\end{pmatrix}
\] | 3 | -2 | 1, 4, 14, 40, 116, 339, 991, \ldots \\
| | | | | | 1, 10, 57, 240, 911, 3275, 11366, \ldots |
| 3  | 3  | \[
\begin{pmatrix}
2 & 3 & 0 \\
6 & 2 & 3 \\
\end{pmatrix}
\] | 3 | 2 | 1, 4, 14, 40, 115, 331, 953, \ldots \\
| | | | | | 1, 10, 54, 226, 853, 3038, 10423, \ldots |
| 3  | 3  | \[
\begin{pmatrix}
0 & 3 & 2 \\
6 & 2 & 1 \\
\end{pmatrix}
\] | 3 | 1 | 1, 4, 14, 40, 116, 339, 991, \ldots \\
| | | | | | 1, 10, 57, 240, 911, 3275, 11366, \ldots |

Table 1: Good \((3, 2)\) convolutional codes

- There is an interesting phenomenon that the spectra of a good \((n, n - 1, \nu, \chi)\) code, for \(\chi < 0\), are close to those of a good \((n, n - 1, \nu + 1, \chi + n)\) code.
Another Type of Equivalent Codes (I)

- Consider three (5, 3) codes, all of which have the same free distance \( d_{\text{free}} = 4 \). The associated minimal encoders are shown respectively as follows:

\[
C_I : \quad G_I(D) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 & 2 \\ 0 & 3 & 3 & 0 & 1 \end{pmatrix}; \quad (7)
\]

\[
C_{II} : \quad G_{II}(D) = \begin{pmatrix} 2 & 2 & 2 & 0 & 1 \\ 2 & 1 & 3 & 2 & 0 \\ 3 & 3 & 0 & 1 & 0 \end{pmatrix}; \quad (8)
\]

\[
C_{III} : \quad G_{III}(D) = \begin{pmatrix} 2 & 2 & 0 & 1 & 1 \\ 1 & 3 & 2 & 0 & 1 \\ 6 & 0 & 2 & 0 & 3 \end{pmatrix}. \quad (9)
\]
Another Type of Equivalent Codes (II)

- The memory sizes of the encoders of $C_I$, $C_{II}$ and $C_{III}$ are 2, 3, and 4 respectively.

- The memory sizes of equivalent codes of conventional type remain unchanged.

- With traditional impression, although these three codes have similar weight spectra, their decoding complexities are significantly different.

- The state complexity profiles of $C_I$, $C_{II}$ and $C_{III}$ are $\{2, 3, 4, 4, 3\}$, $\{3, 4, 4, 3, 2\}$ and $\{4, 4, 3, 2, 3\}$, respectively. There is a cyclic relation among these state complexity profiles.
Another Type of Equivalent Codes (III)

- The minimal trellis can be easily obtained from the encoder of $C_I$, i.e.,

\[
G_{I, scalar} = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
& 1 & 1 & 1 & 1 & 0 \\
& 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
& 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
& & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}.
\]  

(10)
Minimal Trellis Modules of Equivalent Codes

- A portion of the minimal trellis of $C_I$, which is the cascaded of two minimal trellis modules with the junction of two modules being marked by a vertical dashed line.

\[ C_I \quad C_{II} \quad C_{III} \]

\[ C_{IV} \quad C_V \]
Conversions of Equivalent Codes (I)

- Permuting rows numbered 1 (mode 3) and 2 (mod 3), we have $G_{I', \text{scalar}}$ for a code $C_{I'}$.

$$
G_{I', \text{scalar}} = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
\ldots
\end{pmatrix}.
$$

- Apparently, $C_I$ and $C_{I'}$ are the same code with distinct message mapping and have identical minimal trellises.
Conversions of Equivalent Codes (II)

- Delete the first row (numbered 0) and the first column (numbered 0) of $G_{I', scalar}$ hence we have $G_{II', scalar}$.

$$G_{II', scalar} = \begin{pmatrix}
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
\vdots & & & & & & & & & \\
\end{pmatrix}. $$
Conversions of Equivalent Codes (III)

- Interchanging rows of $G_{II', \text{scalar}}$ numbered 0 (mode 3) and 2 (mod 3), we have $G_{II, \text{scalar}}$, where

$$G_{II, \text{scalar}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
\vdots
\end{pmatrix}.$$  \hspace{1cm} (11)
Conversions of Equivalent Codes (IV)

- Delete the first two rows (numbered 0 and 1) and the two columns (numbered 0 and 1) of $G_{I'}^{'},scalar$. We have $G_{III'}^{'},scalar$ for $C_{III'}$, for which its minimal trellis module is obtained from cyclicly shifting the minimal trellis of $C_{I}$ (or $C_{I'}$) by two sections (or two depths).

$$G_{III'}^{'},scalar = \begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
\cdots
\end{pmatrix}.$$
Conversions of Equivalent Codes (V)

- Interchanging rows of $G_{III', scalar}$ numbered 0 (mod 3) and 1 (mod 3), we have $C_{III}$ with $G_{III', scalar}$. $C_{III}$ and $C_{III'}$ are equivalent and have the same minimal trellis.

$$G_{III, scalar} = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
. . .
\end{pmatrix}.$$  

(12)
Conversions of Equivalent Codes (VI)

- The minimal trellises of $C_I$, $C_{II}$ and $C_{III}$ are identical except for the shifting of one or two sections.

- We have an interesting result which indicates that convolutional codes with distinct number of memory sizes of encoders may be equivalent in the sense that they have minimal trellises each of which is a shifted version of one another.

- The cyclic-shift relation of the minimal trellis modules of these equivalent codes implies the cyclic shift relation of the associated state complexity profiles.
Weight Spectra (I)

- The code weight spectra \( t_i \) and the information weight spectra \( f_i \):

\[
\begin{align*}
C_I & \quad \rightarrow \quad \{ t_i : 1, 12, 32, 68, 172, 488, 1364, \cdots \}, \text{ and} \\
& \quad \quad \{ f_i : 1, 32, 144, 424, 1264, 4116, 13224, \cdots \} \\
C_{II} & \quad \rightarrow \quad \{ t_i : 1, 12, 32, 68, 173, 506, 1484, \cdots \}, \text{ and} \\
& \quad \quad \{ f_i : 1, 32, 144, 424, 1266, 4185, 13916, \cdots \} \\
C_{III} & \quad \rightarrow \quad \{ t_i : 1, 12, 32, 68, 173, 508, 1512, \cdots \}, \text{ and} \\
& \quad \quad \{ f_i : 1, 32, 144, 424, 1266, 4190, 14030, \cdots \}
\end{align*}
\]

- \( t_i \) is the number of code paths of weight \( i \) departing from the zero state of the code trellis at depth 0 and return to the zero state for the first time.

- \( f_i \) is the number of message bits on all the code paths of weight \( i \) departing from the zero state of the code trellis at depth 0 and return to the zero state for the first time.
Weight Spectra (II)

- Although these spectra are very close, there are still noticeable differences among these equivalent codes.

- Conventionally, the weight spectra of an \((n, k, \nu)\) convolutional code are calculated based on a code trellis of \(2^n\) states and the transition from one state to a following state is represented by a branch or multiple branches of \(n\) code bits.

- Since \(C_I, C_{II}\) and \(C_{III}\) are equivalent in the sense that the minimal trellises of \(C_{II}\) and \(C_{III}\) can be obtained from that of \(C_I\) by shifting one or two bits (sections), the weight spectra calculated by using the trellis composed of \(n\)-bit branches may not be very appropriate.
Weight Spectra (III)

- A more accurate way of calculating the weight spectra is to use the minimal trellis.

- The number $t_i$ is the sum of code paths of weight $i$ departing from the zero states at depths $0, 1, \cdots, n - 1$ of the minimal trellis and return to the zero state for the first time. The number $f_i$ is similarly modified.

- The weight spectra derived by the conventional way will be lower bounded by those derived by the modified method since in the conventional method a path that returns to the zero state at depth $pn + i$, $0 < i \leq n - 1$ and leaves the zero state at depth $pn + j$, $0 < i \leq j \leq n - 1$ will not be considered as returning to the zero state at the time interval of $[pn, (p + 1)n - 1]$.

- With this modified method, the weight spectra of all the equivalent codes are identical.

- The code weight and information weight spectra are $1, 12, 32, 68, 172, 488, 1364, \cdots$ and $1, 32, 144, 424, 1264, 4116, 13224, \cdots$, respectively.
Systematic Construction (I)

- Suppose \( C \) has its \( G_{\text{scalar}} \), which is in MSGM (minimal span generator matrix) form, and assume it is in row echelon form.

- Let \( j_i \) be the location of the leading ”1” of the row numbered \( i \), where \( 0 \leq i \leq k - 1 \) and \( 0 \leq j_i \leq n - 1 \).

- For \( 0 \leq i \leq k - 1 \), delete the first \( i \) rows of \( G_{\text{scalar}} \) and the first \( j_i \) columns of \( G_{\text{scalar}} \). We then have \( G_{eq,i,\text{scalar}}, 0 \leq i \leq k - 1 \), which is the generator matrix of an equivalent code \( C_{eq,i} \).

- The minimal trellis module of \( C_{eq,i} \) is obtained from that of \( C \) by cyclicly shifting \( j_i \) bits (sections). In each equivalent code \( C_{eq,i}, 0 \leq i \leq k - 1 \), the associated minimal trellis has two branches emanating from each state at depth 0 (mod \( n \)).

- In fact, the \( n \) possible cyclic shifts of the minimal trellis module will imply \( n \) equivalent convolutional codes.
Systematic Construction (II)

- Suppose that $j_{i+1} - j_i \geq 2$ for any $i \in \{0, \cdots, k - 1\}$, for which $j_k = j_0 + n$.

- We already have $G_{eq,i, scalar}$. For each $\ell \in \{1, 2, \cdots, j_{i+1} - j_i - 1\}$, deleting the first row and the first $\ell$ columns of $G_{eq,i, scalar}$ yields a generator matrix of an equivalent code.

- In this way, we can obtain generator matrices of $C_{eq,k+z}, 0 \leq z \leq n - k - 1$, which is equivalent to $C$.

- For $C_{eq,k+z}, 0 \leq z \leq n - k - 1$, the associated minimal trellis has only one branch emanating from each state at depth 0 (mod $n$).

- Besides the three equivalent codes $C_I$, $C_{II}$ and $C_{III}$, there are two more equivalent codes.

$$
C_{IV} : \quad G_{IV}(D) = \begin{pmatrix}
2 & 0 & 1 & 1 & 1 \\
0 & 2 & 0 & 3 & 3 \\
6 & 4 & 0 & 2 & 1
\end{pmatrix}
\quad \text{and} \quad
C_{V} : \quad G_{V}(D) = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
2 & 0 & 3 & 3 & 0 \\
4 & 0 & 2 & 1 & 3
\end{pmatrix}.
$$

(13)
VI. Conclusions

• By examining the properties of minimal trellises of the convolutional codes, we find that convolutional codes with distinct memory sizes of minimal encoders may be equivalent.

• The associated weight spectra and decoding complexity of equivalent convolutional codes are identical.

• For the $(n, k, \nu)$ convolutional code, the complexity measured by the number of branches in the minimal trellis may vary.