INTERNATIONAL JOURNAL OF CIRCUIT THEORY AND APPLICATIONS, VOL. 21, 371–384 (1993)

AN APPROACH TO MODELLING DC–DC CONVERTER CIRCUITS USING GRAPH THEORETIC CONCEPTS

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SUMMARY

An approach to modelling DC–DC converter circuits based on graph theoretic concepts is discussed in this paper. The DC–DC converter circuits are treated as networks containing switches, with the magnitudes of their associated eigenvalues much smaller than the switching frequency. The procedure for modelling this class of networks essentially involves separating the original network into two N-port networks, one containing those branches responsible for all phenomena peculiar to the switching action and the other containing the remaining branches of the network. The two N-port networks thus formed lead to a systematic and convenient way of developing low-frequency circuit models for DC–DC converter circuits.

1. INTRODUCTION

The key concept of switched mode power conversion lies in the presence of switches which are turned on and off in an *orderly pattern* to produce a sequence of circuit topologies. In almost all existing converter circuits the topological sequence consists of a number of circuit configurations which are repeated in a *periodic* manner. Figure 1 shows the family of basic switching cells. In each of these switching cells two ideal switches can be found, and in the continuous mode of operation the switches are turned on and off in a *complementary* fashion, i.e. when one is *on*, the other is *off*, and vice versa. However, in the discontinuous mode of operation an additional switch state exists where both switches are *off* simultaneously. As a result of such toggling between topologies, the analysis of this kind of network becomes rather difficult. In the past two decades many authors have contributed—in one way or another—to the modelling of DC–DC converter circuits.[1–5] In much of the published work the technique of *averaging* has been employed to simplify the analysis and to generate an 'averaged' circuit model which is capable of describing the low-frequency behaviour of the switching circuits. In Reference 5 such a model is obtained via a kind of *circuit-oriented* averaging technique. The derivation outlined in Reference 5, although not explicitly mentioned as such, has emphasized the dependence of the form of

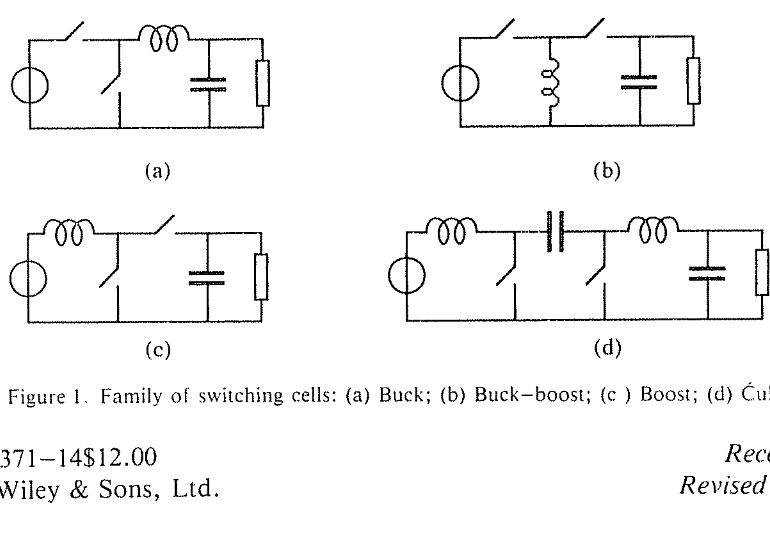


Figure 1. Family of switching cells: (a) Buck; (b) Buck–boost; (c) Boost; (d) Ćuk

0098–9886/93/040371–14$12.00

Received 27 April 1992
Revised 23 November 1992

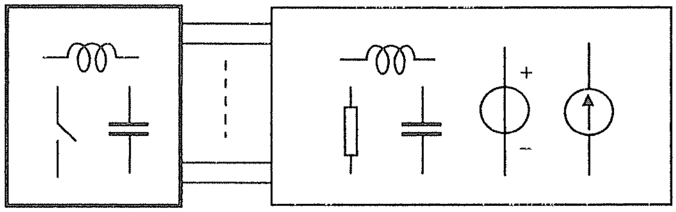


Figure 2. General converter circuit represented by two N-ports

the model upon the *topology of the circuit* to be modelled. Essentially, the derivation involves identifying the part of the circuit responsible for the behaviour peculiar to the switching action. The result is a representation of the original network by two N-port networks connected together as shown in Figure 2. The branches in the first N-port network correspond to those of switches and some capacitors and inductors whose behaviours are likely to be related to the switching action of the circuit. The second N-port will contain the remaining branches. Consequently, *averaging* can be conveniently applied to the first N-port, with the second N-port providing the terminal conditions. For the sake of consistency, the first N-port will be referred to as the MISSCO (MInimum Separable Switching COnfiguration) of the circuit as was originally introduced in Reference 5.

From the foregoing introductory remarks it would seem to make much sense to develop, based on *graph theoretic concepts* and some *basic facts about switching*, a systematic way of generating a low-frequency circuit model. Our purpose in this paper is to develop a simple graph theoretic approach based on the idea proposed in Reference 5, to modelling DC–DC converter circuits. We shall show as the main result of this study how a *graph* corresponding to any circuit containing switches can be systematically divided into two *disconnected subgraphs*. The voltages and currents in one of these subgraphs exhibit switching behaviour including possible discontinuity, while those in the other subgraph are well defined. Finally, for the particular case of DC–DC converter circuits the usual averaging technique is invoked to deal with the part exhibiting switching behaviour.

2. GENERAL DC–DC CONVERTER CIRCUITS

Before embarking on the derivation of the method for modelling DC–DC converter circuits, some important properties and assumptions must be recalled which represent a *de facto* definition for the class of circuits to be studied in this paper.

In general, DC–DC converter circuits may be assumed to be composed of ideal switches, independent voltage and current sources, linear capacitors and inductors, and strictly passive resistors as shown in Figure 3. Also of importance is the fact that the dynamics of DC–DC converter circuits depends not only

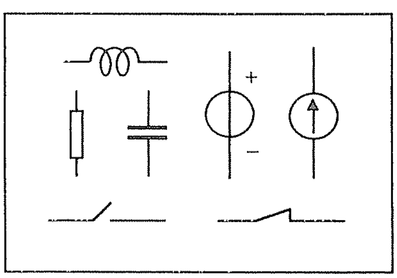


Figure 3. General DC–DC converter circuit

on the eigenvalues associated with the participating circuit configurations but also on the switching frequency. In the case of PWM and quasi-resonant DC-DC converters the associated eigenvalues are much smaller than the switching frequency. As will be explained in Section 6, this fact is crucial to the derivation of low-frequency models for PWM and quasi-resonant DC–DC converters.

It should be noted that the above definition of converter circuits covers a very general class of circuits and is not restricted to the commonly known family of DC–DC converters. The result obtained in this paper will therefore remain valid for a general class of switching networks.

3. SWITCHING ACTION AND THE CONTINUITY THEOREM

As regards the action of switches, two basic types can be identified, namely the 'make' and the 'break'. The action of a *make switch* can be described as one that causes two nodes to be in contact, while the action of a *break switch* is simply the dual of that of a make switch.[6] Thus the operation of DC–DC converters can be described in terms of the making and breaking actions of the switches.

In dealing with circuits containing switches, it is often important to study the effects of switching on the behaviour of other circuit parameters, in particular the capacitor voltages and inductor currents since their discontinuity may cause abnormal circuit conditions in which some currents and voltages may tend to infinity. The question of whether or not the current of an inductor (the voltage of a capacitor) may be discontinuous is partly answered by the *continuity theorem*,[6] which states that for circuits in their free oscillating state (i.e. excluding independent sources)

(i) if an inductor does not form any cutset exclusively with other inductors and break switches, then its current is continuous.
(ii) if a capacitor does not form any loop exclusively with other capacitors and make switches, then its voltage is continuous.

On the basis of the continuity theorem we can establish a set of rules governing the choice of branches to be included in the MISSCO, as will be explained in Section 4. At this point it should be noted that the current of an inductor forming a cutset exclusively with inductors and break switches is potentially discontinuous and that the voltage of a capacitor forming a loop with capacitors and make switches is also potentially discontinuous. Figure 4 illustrates this idea.

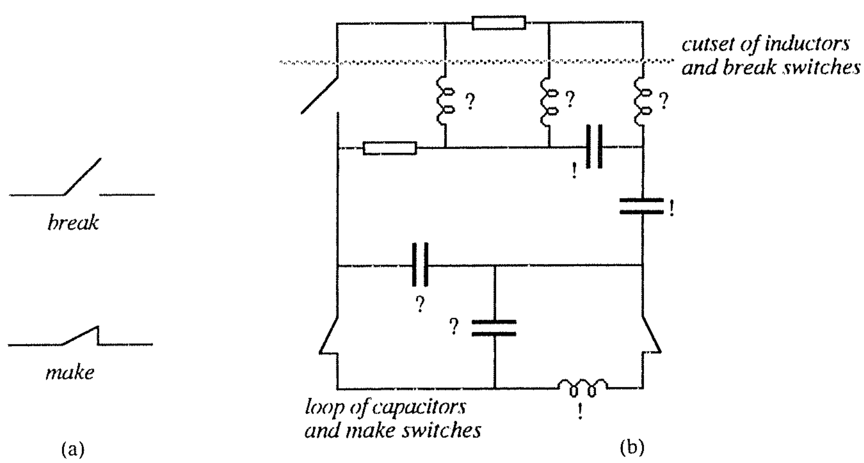


Figure 4. Switching action and the continuity theorem; (a) break and make switches; (b) application of Continuity Theorem: ? means continuity is uncertain; ! means continuity is certain

4. THE MINIMUM SEPARABLE SWITCHING CONFIGURATION (MISSCO)

Before we proceed to the modelling of DC–DC converter circuits, we need to identify the MISSCO which is responsible for all phenomena peculiar to the switching action. It should be obvious that the voltage or current may be discontinuous in a switch regardless of it being a break or a make. Furthermore, as discussed in the foregoing section, currents of some inductors and voltages of some capacitors may be discontinuous. Thus the following procedure may be taken to select the branches to be included in the MISSCO.

1. All switches are included in the MISSCO.
2. All inductors in inductor–break switch cutsets or inductor–break switch–current source cutsets are included in the MISSCO.
3. All capacitors in capacitor–make switch loops or capacitor–make switch–voltage source loops are included in the MISSCO.

In the Buck converter, for example, the switches form a cutset with the inductor but no loops with capacitors. When operating in the continuous mode, the inductor does not form any cutset with break switches at any time. The inductor is thus excluded from the MISSCO. When operating in the discontinuous mode, however, there is an instant at which the inductor forms a cutset with a break switch. (One switch is already open at that instant.) The inductor should therefore be included in the MISSCO for the case of the Buck converter operating in discontinuous mode. In a similar fashion the MISSCOs for all other switching cells can be obtained. For example, the MISSCOs of the Buck converter, the Ćuk converter and the quasi-resonant Buck converter are shown in Figure 5.

It should be noted that in converter circuits the MISSCO is always terminated with well-defined voltages or currents and as such its equivalent circuit can be readily established from the set of equations describing the external voltages and currents of the MISSCO.

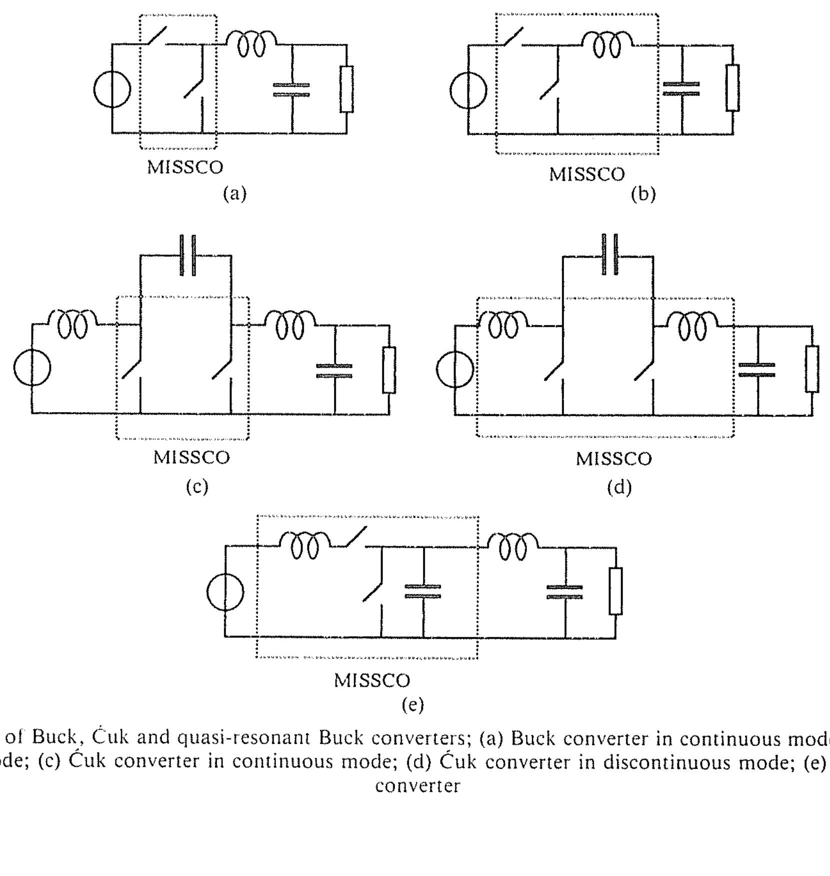


Figure 5. MISSCOs of Buck, Ćuk and quasi-resonant Buck converters; (a) Buck converter in continuous mode; (b) Buck converter in discontinuous mode; (c) Ćuk converter in continuous mode; (d) Ćuk converter in discontinuous mode; (e) quasi-resonant Buck converter

5. DYNAMICS OF MISSCO

As we have seen in Section 4, a MISSCO may contain reactive elements, depending upon the topology of the circuit and the way in which the circuit is operated. In the case where reactive elements are present, however, the MISSCO may not necessarily behave as a dynamic network. For example, it is well known that in the Buck, Boost or Buck–boost converter operating in discontinuous mode the inductor current does not serve as a state variable and in that case the MISSCO is a *zeroth-order* network. In this section we discuss the dynamics of the MISSCO in terms of the *number of independent (state) variables* present in the MISSCO, i.e. the *order* of the MISSCO.

Without loss of generality we assume that the MISSCO contains c cutsets formed of inductors and break switches, and p loops formed of capacitors and make switches. We define S and H as the sets of such cutsets and loops respectively:

$$S = \{s_1, s_2, \ldots, s_c\}, \qquad H = \{h_1, h_2, \ldots, h_p\}$$

Also define φ_i as the set of all inductors in s_i and ξ_j as the set of all capacitors in h_j:

$$\varphi_i = \{L\} \quad \text{for all } L \in s_i, \qquad \xi_j = \{C\} \quad \text{for all } C \in h_j$$

Suppose there are k_i inductors in cutset s_i and m_j capacitors in loop h_j, where $i = 1$ to c and $j = 1$ to p.

Two cases can be distinguished. In the first case each inductor forms *only one* cutset with break switches and possibly other inductors, and each capacitor forms *only one* loop with make switches and possibly other capacitors, i.e. $\varphi_i \cap \varphi_{i'} = \xi_i \cap \xi_{j'} = \varnothing$ (null set) for all $i \neq i'$ and $j \neq j'$, where i, $i' = 1$ to c^L and j, $j' = 1$ to p. In this case the total number of inductors, n_1, and the total number of capacitors, n_c, in the MISSCO are given by $n_L = k_1 + k_2 + \cdots + k_c$ and $n_c = m_1 + m_2 + \cdots + m_p$ respectively. In the second case there exists at least one inductor which forms more than one cutset with break switches (i.e. $\varphi_i \cap \varphi_{i'} \neq \varnothing$ for some $i \neq i'$, where i, $i' = 1$ to c) *or* at least one capacitor which forms more than one loop with make switches (i.e. $\xi_j \cap \xi_{j'} \neq \varnothing$ for some $j \neq j'$, where j, $j' = 1$ to p). Thus we have in this case $n_L < k_1 + k_2 + \cdots + k_c$ or $n_c < m_1 + m_2 + \cdots + m_p$.

Case 1. $\varphi_i \cap \varphi_{i'} = \varnothing$ *for all* $i \neq i'$, *where* i, $i' = 1$ *to* c, *and* $\xi_j \cap \xi_{j'} = \varnothing$ *for all* $j \neq j'$, *where* j, $j' = 1$ *to* p

It should be noted that for each inductor–switch cutset (capacitor–switch loop) in the MISSCO there is always a finite interval in every cycle during which all switches are open (closed). Let I_{L1,s_i}, I_{L2,s_i}, ..., I_{Lk_i,s_i} be the currents in the inductors of cutset s_i. The following equation relating I_{L1,s_i}, I_{L2,s_i}, ..., I_{Lk_i,s_i} must hold for the finite interval of time in every cycle during which all switches in s_i are open:

$$I_{L1,s_i} + I_{L2,s_i} + \cdots + I_{Lk_i,s_i} = 0 \quad \text{for } i = 1 \text{ to } c \tag{1}$$

Thus there are only $k_i - 1$ *independent inductor currents* in this cutset. By applying a dual argument to loop h_j, it can be concluded that there are only $m_j - 1$ *independent capacitor voltages* in h_j. Hence the *total number of independent variables,* N, in the MISSCO is given by

$$N = \sum_{i=1}^{c} (k_i - 1) + \sum_{j=1}^{p} (m_j - 1) = n_L + n_C - c - p \tag{2}$$

Case 2. $\varphi_i \cap \varphi_{i'} \neq \varnothing$ *for some* $i \neq i'$, *where* i, $i' = 1$ *to* c, *or* $\xi_j \cap \xi_{j'} \neq \varnothing$ *for some* $j \neq j'$, *where* j, $j' = 1$ *to* p

In this case some inductors (capacitors) may appear in more than one cutset (loop) in the MISSCO. Thus it is possible that $n_L < c$ or $n_C < p$. If $n_L < c$, then there will be no independent inductor current in the MISSCO. Similarly, if $n_C < p$, then there will be no independent capacitor voltage in the MISSCO. We therefore have

$$N = n_L + n_c - \min\{n_L, c\} - \min\{n_c, p\} \tag{3}$$

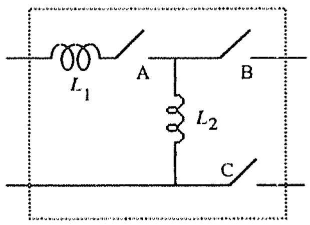


Figure 6. An example of inductors 'shared' among cutsets

So far we have assumed the validity of a classical scenario in which the presence of a cutset (loop) disqualifies one inductor current (capacitor voltage) from being an independent variable, as followed from a Kirchhoff's law equation such as (1). However, since many switches may be present in the MISSCO, it is possible that exactly the same set of inductors (capacitors) appears in more than one cutset (loop), i.e.

$$\varphi_i = \varphi_{i'} \quad \text{for some } i,\ i' = 1 \text{ to } c \tag{4}$$

$$\xi_j = \xi_{j'} \quad \text{for some } j,\ j' = 1 \text{ to } p \tag{5}$$

If either (4) or (5) is true, then equation (3) may have overcalculated the number of independent variables. Here such φ_i and ξ_j will be referred to as an *overlapping inductor set* and an *overlapping capacitor set* respectively. In this case we may assume with no loss of generality that there are α overlapping inductor sets $\varphi_1, \varphi_2, \ldots, \varphi_\alpha$, each of which forms more than one cutset with break switches, and that there are β overlapping capacitor sets $\xi_1, \xi_2, \ldots, \xi_\beta$, each of which forms more than one loop with make switches. Suppose in general that φ_i forms x_i cutsets with break switches and ξ_j forms y_j loops with make switches, where $i = 1$ to α and $j = 1$ to β. Then N is given by

$$N = n_L + n_C - \min\left\{c + \alpha - \sum_{i=1}^{\alpha} x_i,\ n_L\right\} - \min\left\{p + \beta - \sum_{j=1}^{\beta} y_j,\ n_c\right\} \tag{6}$$

Remarks

It should now be clear that the MISSCO of a converter circuit can be a zeroth-order N-port even though it contains reactive elements. For example, the MISSCO of the Buck converter in discontinuous mode, as shown in Figure 5(b), has one inductor which forms a cutset with break switches. Thus, putting $n_c = p = 0$, $n_L = 1$ and $c = 1$ in equation (2) of Case 1 gives $N = 0$. Indeed, the inductor current is identically zero during the interval when both switches are open and is therefore not expected to behave as a state variable. For the case of the Ćuk converter operating in discontinuous mode (Figure 5(d)), however, we have $n_c = p = 0$, $n_L = 2$ and $c = 1$. Again using (2) of Case 1 gives $N = 1$. The MISSCO of the Ćuk converter operating in discontinuous mode is therefore a first-order N-port network. Furthermore, an example of Case 2 is shown in Figure 6, where a switched inductor circuit has two inductors and switches S_A, S_B and S_C are switched periodically in such a way that cutsets s_1, s_2 and s_3 can be identified as $\{S_A, L_1\}$, $\{S_B, L_1, L_2\}$ and $\{S_C, L_1, L_2\}$ respectively. Clearly, there is one overlapping inductor set, namely $\varphi_1 = \{L_1, L_2\}$. Thus $\alpha = 1$, $x_1 = 2$, $n_L = 2$, $c = 3$ and $n_c = p = 0$. The number of state variables in this network can be calculated from (6) as $N = n_L - c + (x_1 - 1) = 2 - 3 + (2 - 1) = 0$. This network therefore exhibits no dynamical behaviour.

6. MODELLING THE MISSCO

In Sections 4 and 5 we have established the MISSCO of the converter circuit and discussed its order. The branches not contained in the MISSCO may be regarded as another N-port network which is connected directly to the MISSCO. In this N-port the associated eigenvalues are assumed to be much smaller than the switching frequency and all inductor currents and capacitor voltages are continuous. Thus any inductor in this N-port *appears to the MISSCO as* a constant current source of appropriate magnitude and any capacitor in this N-port *appears as* a constant voltage source of appropriate magnitude. An equivalent circuit of the MISSCO can thus be found by terminating it with appropriate sources and applying the technique of *averaging*.

The following examples serve to illustrate the procedure for obtaining an equivalent circuit model of the MISSCO.

Example 1: Buck converter

Figure 7 shows an equivalent termination of the MISSCO of the Buck converter. In the continuous model, one switch opens while the other closes in such a way that the input current I_1 is pulsating between zero and I_0, and the voltage V_2 is pulsating between zero and E. The averaged values of I_1 and V_2 are given by

$$I_1 = DI_0 \tag{7}$$

$$V_2 = DE \tag{8}$$

where D is the duty cycle, normally defined as the fraction of the switching period during which switch A is on and switch B is off. The MISSCO of the Buck converter in continuous mode can thus be represented by a dependent current source and a dependent voltage source as shown in Figure 8(a).

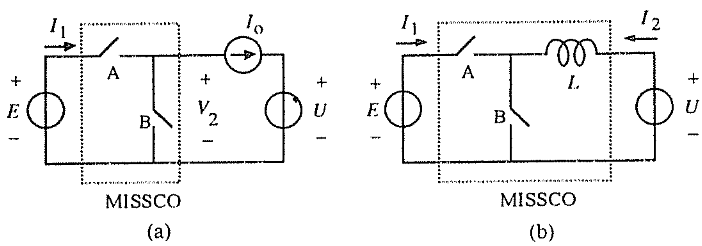


Figure 7. Equivalent termination of MISSCO of Buck converter; (a) continuous mode; (b) discontinuous mode

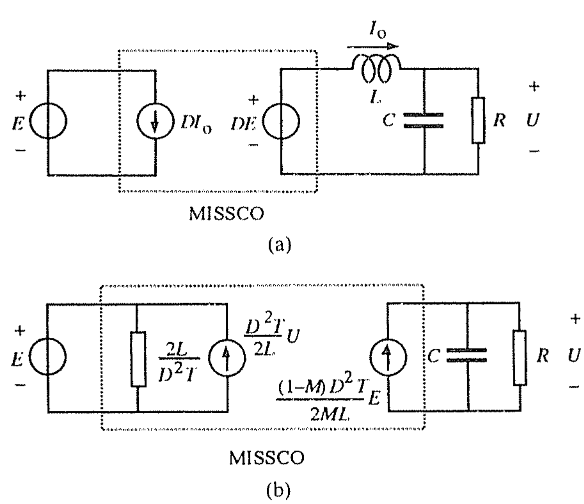


Figure 8. Equivalent circuit of MISSCO of Buck converter; (a) continuous mode; (b) discontinuous mode

For the Buck converter operating in discontinuous mode, three switched states can be identified; (i) switch A is on and switch B is off; (ii) switch A is off and switch B is on; (iii) both switches are off. In this case the application of averaging leads to

$$I_1 = E\frac{D^2T}{2L} + U\frac{-D^2T}{2L} \tag{9}$$

$$I_2 = E\frac{-D^2T}{2L}\left(1+\frac{2H}{D}\right) + U\frac{D^2T}{2L}\left(1+\frac{H}{D}\right)^2 \tag{10}$$

where H is the fraction of the switching period during which switch A is off and switch B is on and H is related to D by

$$\frac{H}{D} = \frac{E-U}{U} \tag{11}$$

Putting (11) in (10) yields

$$I_2 = -E\frac{1-M}{M}\frac{D^2T}{2L} \tag{12}$$

where M is the DC voltage transfer ratio U/E. An equivalent circuit model for the MISSCO of the Buck converter operating in discontinuous mode can be obtained from equations (9) and (12) as shown in Figure 8(b).

Example 2. Ćuk converter

In a similar fashion an equivalent termination of the MISSCO of the Ćuk converter is obtained by replacing the capacitors and inductors external to the MISSCO by constant voltage sources and current sources respectively as illustrated in Figure 9.

For the continuous mode case the switches are turned on and off in a complementary fashion and the averaged values of V_1, V_2 and I_3 are given by

$$V_1 = (1-D)V_c \tag{13}$$

$$V_2 = DV_c \tag{14}$$

$$I_3 = DI_0 - (1-D)I_{in} \tag{15}$$

Figure 10(a) shows an equivalent representation for the MISSCO of the Ćuk converter operating in continuous mode.

For the discontinuous mode case the topological sequence involves first turning switch A on and switch B off, then turning switch A off and switch B on, and finally turning both off. In deriving the averaged

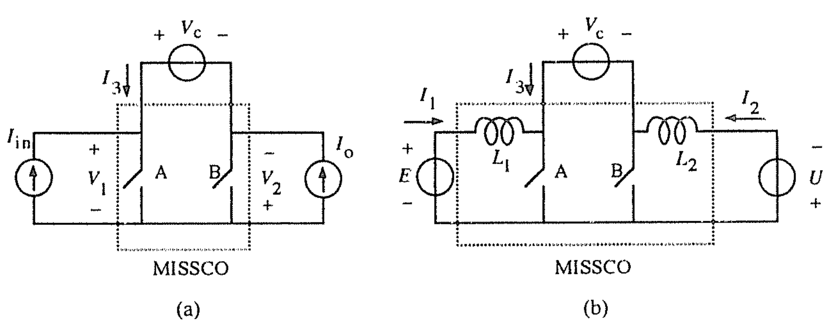
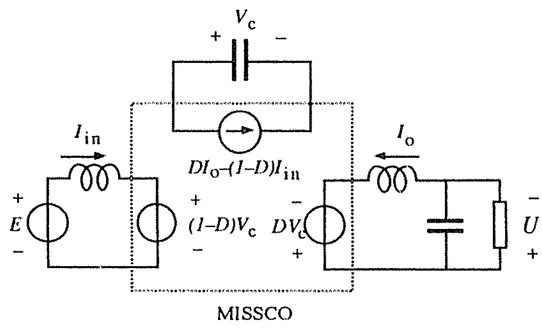


Figure 9. Equivalent termination of MISSCO of Ćuk converter; (a) continuous mode; (b) discontinuous mode



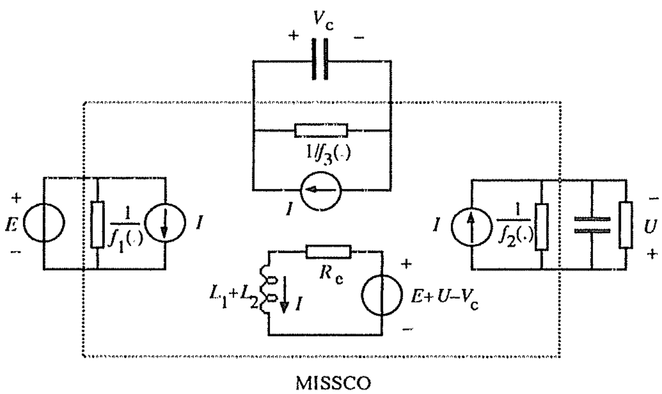


Figure 10. Equivalent circuit of MISSCO of Ćuk converter; (a) continuous mode; (b) discontinuous mode

values of I_1, I_2 and I_3, the usual integral average is taken over one complete cycle, i.e.

$$I_1 = \frac{1}{T}\int_0^T i_1(t)\,\mathrm{d}t, \qquad I_2 = \frac{1}{T}\int_0^T i_2(t)\,\mathrm{d}t, \qquad I_3 = \frac{1}{T}\int_0^T i_3(t)\,\mathrm{d}t$$

where $i_1(t)$, $i_2(t)$ and $i_3(t)$ can be expressed as piecewise linear functions of t. It can be shown that the averaged values of I_1, and I_3 are given by

$$I_1 = f_1(\cdot)E + I \tag{16}$$

$$I_2 = -f_2(\cdot)U - I \tag{17}$$

$$I_3 = f_3(\cdot)V_c - I \tag{18}$$

where

$$f_1(\cdot) = \frac{T}{2}\left(\frac{D(2-D)}{L_1} - \frac{(1-M_1)H(2-2D-H)}{L_1M_1} - \frac{(1-M_1-M_2)(1-D-H)^2}{M_1(L_1+L_2)}\right) \tag{19}$$

$$f_2(\cdot) = \frac{T}{2}\left(\frac{H(2-2D-H)}{L_2} - \frac{(1-M_2)D(2-D)}{L_2M_2} - \frac{(1-M_1-M_2)(1-D-H)^2}{M_2(L_1+L_2)}\right) \tag{20}$$

$$f_3(\cdot) = \frac{T}{2}\left(\frac{(1-M_2)D^2}{L_2} + \frac{(1-M_1)H(2-2D-H)-2M_1D(1-D)}{L_1} + \frac{(1-M_1-M_2)(1-D-H)^2}{L_1+L_2}\right) \tag{21}$$

$$H = D\,\frac{L_2M_1 + L_1(1-M_2)}{L_1M_2 + L_2(1-M_1)} \tag{22}$$

$$M_1 = \frac{E}{V_c}, \qquad M_2 = \frac{U}{V_c} \tag{23}$$

and I is the value of I_1, $-I_2$ and $-I_3$ at the start of the switching period, i.e. $I = I_1(kT) = -I_2(kT) = -I_3(kT)$. The energy absorbed by the MISSCO during one complete cycle is

$$(EI_1 - UI_2 + V_c I_3)T = \tfrac{1}{2}(L_1 + L_2)[(I + \Delta I)^2 - I^2] \approx (L_1 + L_2)I(\Delta I),$$

where $\Delta I = I((k+1)T) - I(kT)$. Writing $\mathrm{d}I/\mathrm{d}t = \Delta I/T$ gives the differential equation

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{1}{L_1 + L_2}\left(E + U - V_c + \frac{f_1(\cdot)E^2 + f_2(\cdot)U^2 + f_3(\cdot)V_c^2}{I}\right) = \frac{(E + U - V_c) - R_e I}{L_1 + L_2} \tag{24}$$

where R_e is a non-linear resistor given by

$$R_e = \frac{-[f_1(\cdot)E^2 + f_2(\cdot)U^2 + f_3(\cdot)V_c^2]}{I^2}$$

Equations (16)–(24) will conveniently lead to a circuit model for the MISSCO of the Ćuk converter operating in discontinuous modes as shown in Figure 10(b). It should be noted that the MISSCO in this case is a first-order network as predicted in Section 5.

Example 3. Zero-current-switched (ZCS) quasi-resonant Buck converter

In the ZCS quasi-resonant Buck converter there exists an inductor which forms a cutset with a break switch and a capacitor which forms a loop with a make switch. These reactive elements, together with the switches, are included in the MISSCO. An equivalent termination of this MISSCO is shown in Figure 11.

We shall demonstrate the modelling procedure for the ZCS quasi-resonant Buck converter operating in full-wave mode. The case of half-wave operation can be treated in a similar fashion. In the full-wave mode switch A is bidirectional whereas switch B is unidirectional, permitting only upward flow of current. The topological sequence corresponding to the full-wave operation can be described as follows.

1. *Switches A and B are closed.* I_1 grows linearly from zero to a value equal to I_0. The duration of this interval T_1 is given by

$$T_1 = \frac{L_r I_0}{E} \tag{25}$$

   and the averaged values of I_1 and V_2 in this interval are $I_0/2$ and zero respectively.
2. *Switch A is closed and switch B is open.* I_1 and V_2 vary sinusoidally according to the equations

$$i_1(t) = I_0 + \sqrt{\left(\frac{C_r}{L_r}\right)}E \sin\left(\frac{t}{\sqrt{(L_r C_r)}}\right) \tag{26}$$

$$v_2(t) = E - E\cos\left(\frac{t}{\sqrt{(L_r C_r)}}\right) \tag{27}$$

   The current I_1 swings positively and negatively (full-wave) and the interval ends when I_1 returns to

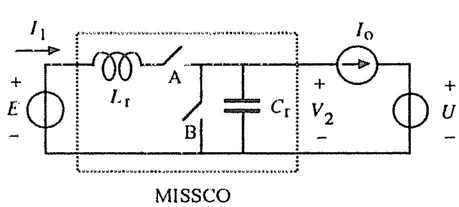


Figure 11. Equivalent termination of MISSCO of quasi-resonant Buck converter

zero from its first negative swing. The duration of this interval T_2 is

$$T_2 = \sqrt{(L_r C_r)}\left\{2\pi - \sin^{-1}\left[\sqrt{\left(\frac{I_0^2 L_r}{E^2 C_r}\right)}\right]\right\} \tag{28}$$

and the averaged values of I_1 and V_2 in this interval are

$$I_0 + \frac{EC_r}{T_2}\left[1 - \sqrt{\left(1 - \frac{I_0^2 L_r}{E^2 C_r}\right)}\right] \quad \text{and} \quad E + \frac{I_0 L_r}{T_2}$$

respectively.

3. *Switches A on B are open.* I_1 remains zero while V_2 falls linearly from V_a to zero, where V_a is the voltage of V_2 at the start of this interval and is given by

$$V_a = E\left[1 - \cos\left(\frac{T_2}{\sqrt{(L_r C_r)}}\right)\right] = E\left[1 - \sqrt{\left(1 - \frac{I_0^2 L_r}{E^2 C_r}\right)}\right] \tag{29}$$

The duration of this interval T_3 is

$$T_3 = \frac{V_a C_r}{I_0} \tag{30}$$

The averaged values of I_1 and V_2 in this interval are zero and $V_a/2$ respectively.

4. *Switch A is open and switch B is closed.* I_1 and V_2 both remain at zero. The duration of this interval is simply given by

$$T_4 = T - T_1 - T_2 - T_3 \tag{31}$$

From equations (25)–(31) and the averaged values of I_1 and V_2 in each subinterval, the averaged values of I_1 and V_2 over the whole switching period can be found as

$$I_1 = \frac{I_0}{T}\left\{\frac{T_1}{2} + T_2 + \frac{L_r C_r}{T_1}\left[1 - \sqrt{\left(1 - \frac{I_0^2 L_r}{E^2 C_r}\right)}\right]\right\} \tag{33}$$

$$V_2 = \frac{E}{T}\left\{T_1 + T_2 + \frac{T_3}{2}\left[1 - \sqrt{\left(1 - \frac{I_0^2 L_r}{E^2 C_r}\right)}\right]\right\} \tag{33}$$

Equations (32) and (33) can be put in the form

$$I_1 = I_0 h(\cdot) \tag{34}$$

$$V_2 = E g(\cdot) \tag{35}$$

where $h(\cdot)$ and $g(\cdot)$ are functions of T_1, T_2, T_3 and T, which are in turn functions of L_r, C_r, I_0 and E. An equivalent circuit for the MISSCO of the ZCS quasi-resonant Buck converter can now be obtained from equations (32) and (33) or equations (34) and (35) as shown in Figure 12.

It should be emphasized that models obtained from the foregoing method essentially describe the large-signal behaviour of converter circuits. They contain no switches and can be conveniently analysed by various computer simulation programs.[7]

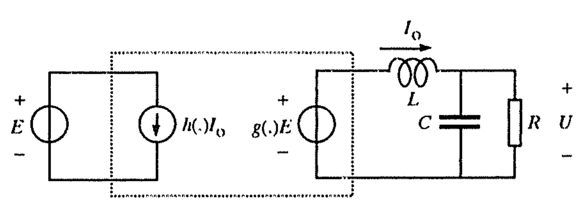


Figure 12. Equivalent circuit of MISSCO of ZCS quasi-resonant Buck converter

7. COMPUTER SIMULATIONS OF MISSCO MODELS

In order to be useful in design and analysis, a circuit model must be adequate in describing the salient characteristics of the physical circuit and yet be sufficiently simple to facilitate the use of various general-purpose simulators (e.g. SPICE) for simulating actual circuit behaviour. Study of the mathematics of averaging[8] has clearly proven the adequacy of averaged models in describing the low-frequency behaviour of DC–DC converter circuits. Since the method of modelling the MISSCO is based on averaging, it is guaranteed to be adequate, at least for low-frequency characterization. In addition, the resulting models are well suited for computer-aided analysis and design, as reflected by our experience with the use of SPICE in verifying the results of Section 6. Since detailed discussion on using SPICE is beyond the scope of this paper, we exemplify here the results from SPICE simulation for the Ćuk converter in discontinuous mode. Suffice to say, the circuit model shown in Figure 10(b) is readily described by a SPICE 'net-list' where all non-linear resistors are implemented by a number of controlled voltage and current sources. In our simulation the operating condition is chosen to resemble that of an experimental study of the Ćuk converter reported in Reference 9, as shown in Table I.

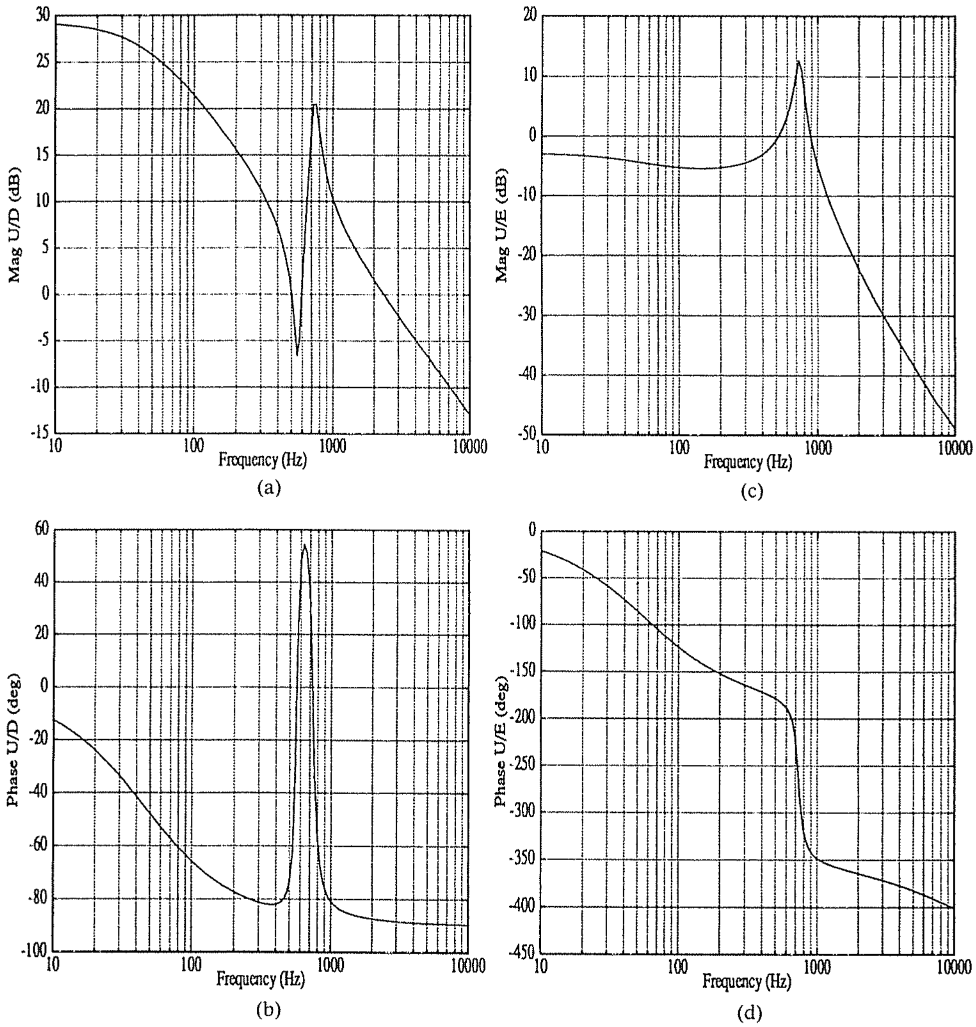


Figure 13. Small-signal frequency responses from SPICE simulation for Ćuk converter in discontinuous mode: (a) magnitude of U/D; (b) phase of U/D; (c) magnitude of U/E; (d) phase of U/E

Table I.

Parameter	Value
L_1, L_2	1 mH
D	0·37
T	50 μs
E	15 V
Load resistor	75 Ω
All capacitors	47 μF

The steady state conditions as reported by SPICE's calculation of the *biasing point* are given by $U = 10{\cdot}748$ V, $V_c = 25{\cdot}748$ V, $H = 0{\cdot}516$, $I_1 = 102{\cdot}7$ mA, $I_2 = 143{\cdot}3$ mA and $I_3 = 0{\cdot}0$ mA. More specifically, frequency responses of the small-signal duty-cycle-to-output-voltage and input-voltage-to-output-voltage transfer functions are obtained from two separate SPICE AC analyses as shown in Figure 13. It is found that the results are in perfect agreement with those determined experimentally in Reference 9.

8. CONCLUSIONS

In dealing with DC–DC converter circuits, the technique of averaging is often used. Generally speaking, the way in which averaging is applied to DC–DC converter circuits can be classified into two main categories. First, on a strictly mathematical basis, averaging produces an *averaged state equation* for a multistructural system. Secondly, averaging is applied to a physical circuit to generate an *averaged circuit model* which can be conveniently used in analysis and design. Our purpose in this paper has been to systematize the derivation of averaged circuit models. Starting from the simplest topological consideration and observing some basic properties of switches, a simple and systematic method for generating averaged circuit models has been discussed. Intuitively speaking, the proposed procedure for constructing the MISSCO of a circuit can be explained as follows. Suppose we want to simplify the modelling process by replacing all inductors and capacitors with constant current and voltage sources, so that applying averaging to the 'new' circuit would allow the switches to be modelled as controlled sources. However, this scheme will definitely fail if in any of the switched states there exists a current source forming a cutset with open switches or a voltage source forming a loop with closed switches, since such conditions would violate Kirchhoff's laws. Thus the construction of the MISSCO may be viewed as a systematic identification of elements (switches, capacitors and inductors) which, upon transforming into constant sources, would lead to violation of Kirchhoff's laws.

Since the modelling of the MISSCO is based on averaging, the accuracy and implications of the results thus obtained have been discussed in various publications.[1–4,8] In conclusion, we should reiterate that our intention is not to demonstrate the well-known averaging technique but to propose a graph theoretic approach to modelling DC–DC converter circuits which, in the way described in this paper, has not previously been applied in power electronics.

ACKNOWLEDGEMENTS

We wish to thank Professor K. M. Adams of the University of Melbourne for helpful suggestions. This work is supported in part by Hong Kong Polytechnic Research Sub-committee under Grant 341.166.A3.420.

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