Emergent Behavior on Flocks

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Introduction

General theme: reaching of consensus without a central direction.

Examples:

(1) emergence of a common belief in a price system when activity takes place in a given market.

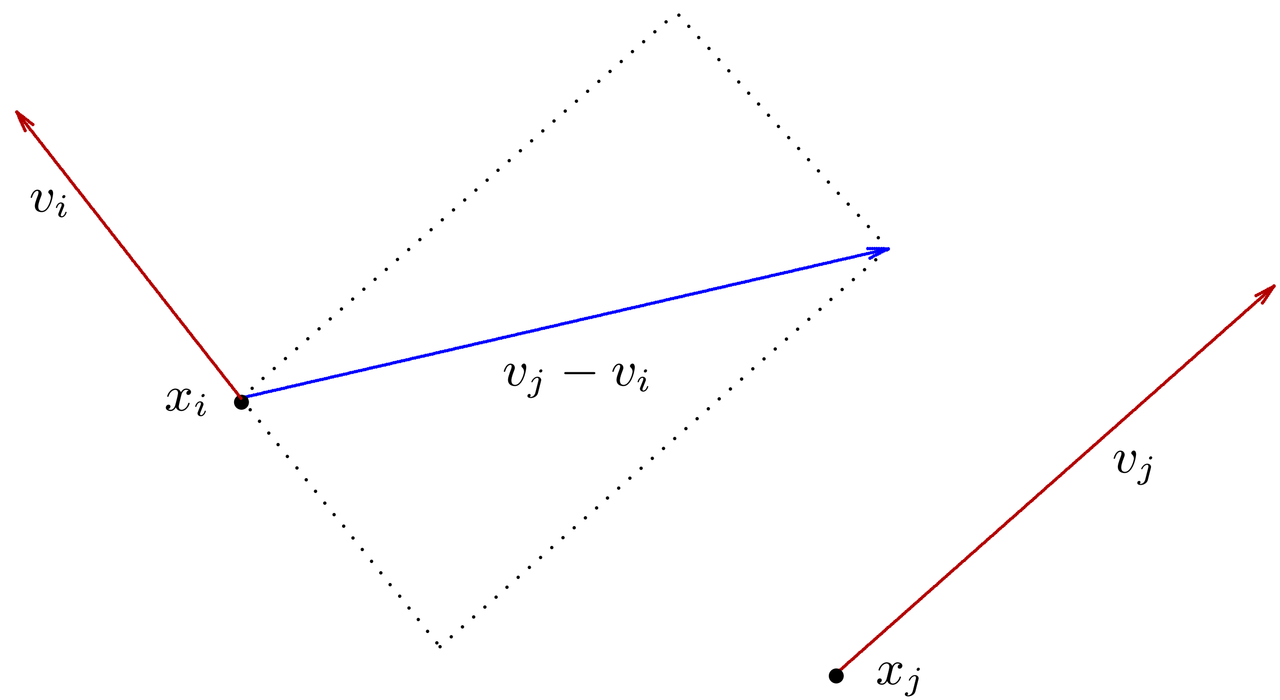
(2) emergence of common languages in primitive societies, or the dawn of vowel systems.

(3) flocking in a population, say of birds or fish, whose members are moving in $\mathbb{E} := \mathbb{R}^3$.

It has been observed that under some initial conditions, for example on their positions and velocities, the state of the flock converges to one in which all birds fly with the same velocity. Our goal is to give a possible model of this behavior.

Model: every bird adjusts its velocity by adding to it a weighted average of the differences of its velocity with those of the other birds. That is, at time $t \in \mathbb{N}$, and for bird i,

$$v_i(t+1) - v_i(t) = \sum_{j=1}^{k} a_{ij}(v_j(t) - v_i(t)). \tag{1}$$



Here the weights $\{a_{ij}\}$ quantify the way the birds influence each other.

We assume that this influence is a function of the distance between birds, namely

$$a_{ij} = \frac{K}{(\sigma^2 + \|x_i - x_j\|^2)^\beta}. \tag{2}$$

for some fixed $K, \sigma > 0$ and $\beta \geq 0$.

We can write the set of equalities (1) in a more concise form. Let

$$A_x = (a_{ij})$$

be the *adjacency matrix*, D_x be the $k \times k$ diagonal matrix whose ith diagonal entry is $d_i = \sum_{j \leq k} a_{ij}$ and $L_x = D_x - A_x$.

Then

$$
\begin{aligned}
v_i(t+1) - v_i(t) &= -\sum_{j=1}^{n} a_{ij}(v_i(t) - v_j(t)) \\
&= -\left(\sum_{j=1}^{n} a_{ij}\right) v_i(t) + \sum_{j=1}^{n} a_{ij} v_j(t) \\
&= -[D_x v(t)]_i + [A_x v(t)]_i \\
&= -[L_x v(t)]_i
\end{aligned}
$$

and (adding a natural equation for the change of positions) we obtain the system

$$
\begin{aligned}
x(t+1) &= x(t) + v(t) \\
v(t+1) &= (\mathrm{Id} - L_x)\, v(t).
\end{aligned}
\tag{3}
$$

We also consider evolution for continuous time. The corresponding model is given by the system of differential equations

$$\begin{aligned} x' &= v \\ v' &= -L_x v. \end{aligned} \tag{4}$$

Our two main results give conditions to ensure that the birds' velocities converge to a common one and the distance between birds remain bounded, for both continuous and discrete time.

Convergence in continuous time

For $x, v \in \mathbb{E}^k$ we denote

$$\Gamma(x) = \frac{1}{2} \sum_{i \neq j} \|x_i - x_j\|^2$$

and

$$\Lambda(v) = \frac{1}{2} \sum_{i \neq j} \|v_i - v_j\|^2.$$

In the following we fix a solution (x, v) of (4). To simplify notation we write

$$\Lambda(t) := \Lambda(v(t)), \qquad \Gamma(t) := \Gamma(v(t)).$$

Theorem 1 *Assume that*

$$a_{ij} = \frac{K}{(\sigma^2 + \|x_i - x_j\|^2)^\beta}.$$

Assume also that one of the three following hypothesis hold:

(i) $\beta < 1/2$,

(ii) $\beta = 1/2$ *and* $\Lambda_0 < \frac{K^2}{18k^2}$,

(iii) $\beta > 1/2$ *and*

$$\left[\left(\frac{1}{2\beta}\right)^{\frac{1}{2\beta-1}} - \left(\frac{1}{2\beta}\right)^{\frac{2\beta}{2\beta-1}} \right] \left(\frac{K^2}{18k^2\Lambda_0}\right)^{\frac{1}{2\beta-1}} > 2\Gamma_0 + \sigma^2.$$

Then there exists a constant B_0 *such that* $\Gamma(t) \le B_0$ *for all* $t \in \mathbb{R}_+$. *In addition, when* $t \to \infty$, $\Lambda(t) \to 0$ *and the vectors* $x_i - x_j$ *tend to a limit vector* $\widehat{x_{ij}}$, *for all* $i, j \le k$.

Convergence in discrete time

A motivation to consider discrete time is that we want to derive (possibly a small variation of) our model from a mechanism based on exchanges of signals. The techniques to do so, learning theory, are better adapted to discrete time.

Theorem 2 *Assume that* $K < \frac{\sigma^{2\beta}}{(k-1)\sqrt{k}}$ *and*

$$a_{ij} = \frac{K}{(\sigma^2 + \|x_i - x_j\|^2)^\beta}.$$

Assume also that one of the three following hypothesis hold:

(i) $\beta < 1/2$,

(ii) $\beta = 1/2$ *and* $\|v(0)\| \leq \frac{K}{3\sqrt{2}k^2}$,

(iii) $\beta > 1/2$ *and*

$$\left(\frac{1}{\boldsymbol{a}}\right)^{\frac{2}{\alpha-1}} \left[\left(\frac{1}{\alpha}\right)^{\frac{2}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha+1}{\alpha-1}}\right] > 2k^2 \left(V_0^2 + 2V_0((\alpha\boldsymbol{a})^{-\frac{2}{\alpha-1}} - \sigma^2)(\sqrt{2}k)^{-1}\right) + \boldsymbol{b}.$$

Here $\alpha = 2\beta$, $V_0 := \|v(0)\|$, $\boldsymbol{a} = \frac{3\sqrt{2}k^2}{K}V_0$, $\boldsymbol{b} = \sqrt{2}k\|x(0)\| + \sigma$.

Then there exists a constant B_0 *such that* $\|\Gamma(t)\| \leq B_0$ *for all* $t \in \mathbb{N}$. *In addition,* $v_i(t) - v_j(t) \to 0$ *and the vectors* $x_i - x_j$ *tend to a limit vector* $\widehat{x_{ij}}$, *for all* $i, j \leq k$, *when* $t \to \infty$.

Some previous (and posterior) related work

T. Vicsek, A. Czirók, E. Ben-Jacob, and O. Shochet. Novel type of phase transition in a system of self-driven particles. *Phys. Rev. Letters*, 75:1226–1229, 1995.

A. Jadbabaie, J. Lin, and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. on Autom. Control*, 48:988–1001, 2003.

J. Buhl, D.J.T. Sumpter, I.D. Couzin, J.J. Hale, E. Despland, E.R. Miller, and S.J. Simpson. From disorder to order in marching locusts. *Science*, 312:1402–1406, 2006.

J. Shen. Cucker-Smale flocking under hierarchical leadership. Preprint, 2007.

F.C. and E. Mordecki. Flocking in noisy environments. In preparation.

The proof for the continuous case.

The matrix L_x in (3) and (4) is the Laplacian of A_x. It acts on $\mathbb{E}^k$ and satisfies the following:

(a) For all $v \in \mathbb{E}$, $L_x(v, \ldots, v) = 0$.

(b) It is symmetric positive semidefinite,

(c) If $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of L_x then

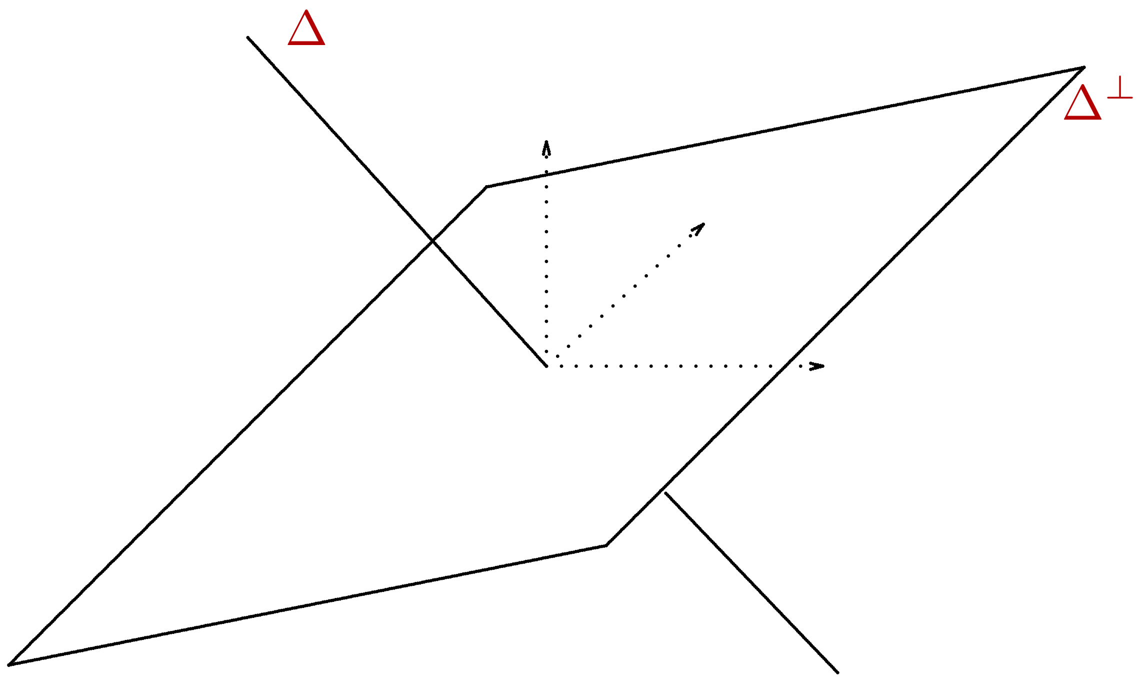
$$0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k = \|L_x\|.$$

The second eigenvalue λ_2 of L_x is called the *Fiedler number* of A_x. We denote it by ϕ_x.

Let Δ be the diagonal of $\mathbb{E}^k$, i.e.,

$$\Delta = \{(v, v, \dots, v) \mid v \in \mathbb{E}\}$$

and $\Delta^\perp$ be the orthogonal complement of Δ in $\mathbb{E}^k$. Every $x \in \mathbb{E}^k$ decomposes in a unique way as $x = x_\Delta + x_\perp$ with $x_\Delta \in \Delta$ and $x_\perp \in \Delta^\perp$.



Proposition 1 *The projections over $\Delta^\perp$ of the solutions of (4) are the solutions of the restriction of (4) to $\Delta^\perp$. That is,*

$$\begin{aligned} x'_\perp &= v_\perp \\ v'_\perp &= -L_{x_\perp} v_\perp . \end{aligned} \tag{5}$$

Remark 1 The condition "the velocities $v_i(t)$ tend to a common limit $\widehat{v} \in \mathbb{E}$" is equivalent to the condition "$v_\perp(t) \to 0$." Also, the condition "the vectors $x_i - x_j$ tend to a limit vector $\widehat{x_{ij}}$, for all $i, j \leq k$" is equivalent to "$x_\perp(t)$ tend to a limit vector $\widehat{x}$ in $\Delta^\perp$." This suggests that we are actually interested on the solutions —on the space $\Delta^\perp \times \Delta^\perp$— of the system (5) induced by (4). Since (4) and (5) have the same form we will keep referring to (4) but we will consider positions in

$$X := \mathbb{E}^k/\Delta \simeq \Delta^\perp$$

and velocities in

$$V := \mathbb{E}^k/\Delta \simeq \Delta^\perp.$$

Consider $Q : \mathbb{E}^k \times \mathbb{E}^k \to \mathbb{R}$ defined by

$$Q(u, v) = \frac{1}{2} \sum_{i,j=1}^{k} \langle u_i - u_j, v_i - v_j \rangle.$$

Then Q is bilinear, symmetric, and, when restricted to $\Delta^\perp \times \Delta^\perp$, positive definite. It therefore defines an inner product $\langle\ ,\ \rangle_Q$ on $\mathbb{E}^k/\Delta \simeq \Delta^\perp$. Now note that that $\Gamma(x) = \Gamma(x_\perp)$ and $\Lambda(v) = \Lambda(v_\perp)$ and that $\Lambda(v) = \|v\|_Q^2$ and $\Gamma(x) = \|x\|_Q^2$.

Proposition 2 *For all* $x \in X$, $\phi_x \geq \min_{i \neq j} a_{ij}$. *In particular, if* $a_{ij} = \frac{K}{(\sigma^2 + \|x_i - x_j\|^2)^\beta}$ *then*

$$\phi_x \geq \frac{K}{(\sigma^2 + \Gamma_x)^\beta}.$$

Denote $\Phi_t = \min_{\tau \in [0,t]} \phi_\tau$.

Proposition 3 *For all $t \geq 0$, $\Lambda(t) \leq \Lambda_0 e^{-2t\Phi_t}$.*

PROOF. Let $\tau \in [0,t]$. Then

$$\begin{aligned} \Lambda'(\tau) &= \frac{d}{d\tau}\langle v(\tau), v(\tau)\rangle_Q = 2\langle v'(\tau), v(\tau)\rangle_Q \\ &= -2\langle L_\tau v(\tau), v(\tau)\rangle_Q \leq -2\phi_{x(\tau)}\Lambda(\tau). \end{aligned}$$

Here we have used that L_τ is symmetric positive definite on V. Using this inequality,

$$\ln(\Lambda(\tau))\Big|_0^t = \int_0^t \frac{\Lambda'(\tau)}{\Lambda(\tau)} d\tau \leq \int_0^t -2\phi_\tau d\tau \leq -2t\Phi_t$$

i.e.,

$$\ln(\Lambda(t)) - \ln(\Lambda_0) \leq -2t\Phi_t.$$

Proposition 4 *For $T > 0$, $\Gamma(T) \leq 2\left(\Gamma_0 + \frac{\Lambda_0}{\Phi_T^2}\right)$.*

PROOF. We have $|\Gamma'(t)| = |2\langle v(t), x(t)\rangle_Q| \leq 2\|v(t)\|_Q\|x(t)\|_Q$. But $\|x(t)\|_Q = \Gamma(t)^{1/2}$ and $\|v(t)\|_Q^2 = \Lambda(t) \leq \Lambda_0 e^{-2t\Phi_t}$, by Proposition 2. Therefore,

$$\Gamma'(t) \leq |\Gamma'(t)| \leq 2\left(\Lambda_0 e^{-2t\Phi_t}\right)^{1/2}\Gamma(t)^{1/2}$$

and, using that $t \mapsto \Phi_t$ is non-increasing,

$$\begin{aligned}
\int_0^T \frac{\Gamma'(t)}{\Gamma(t)^{1/2}} dt &\leq 2\int_0^T \left(\Lambda_0 e^{-2t\Phi_t}\right)^{1/2} dt \\
&\leq 2\int_0^T \Lambda_0^{1/2} e^{-t\Phi_T} dt \\
&= 2\Lambda_0^{1/2}\left(-\frac{1}{\Phi_T}\right) e^{-t\Phi(T)}\Big|_0^T \leq \frac{2\Lambda_0^{1/2}}{\Phi_T}
\end{aligned}$$

which implies

$$\Gamma(t)^{1/2}\Big|_0^T = \frac{1}{2}\int_0^T \frac{\Gamma'(t)}{\Gamma(t)^{1/2}}dt \leq \frac{\Lambda_0^{1/2}}{\Phi_T}$$

from where

$$\Gamma(T) \leq \left(\Gamma_0^{1/2} + \frac{\Lambda_0^{1/2}}{\Phi_T}\right)^2.$$

The result now follows from the inequality $(\alpha+\beta)^2 \leq 2(\alpha^2+\beta^2)$. ■

Lemma 1 *Let* $c_1, c_2 > 0$ *and* $s > q > 0$. *Then the equation*

$$F(z) = z^s - c_1 z^q - c_2 = 0$$

has a unique positive zero z_*. *In addition*

$$z_* \leq \max\left\{(2c_1)^{\frac{1}{s-q}}, (2c_2)^{\frac{1}{s}}\right\}$$

and $F(z) \leq 0$ *for* $0 \leq z \leq z^*$. ∎

Proof of Theorem 1. By Proposition 4, for all $x \in X$,

$$\phi_x \geq \frac{K}{(\sigma^2 + \Gamma_x)^\beta}.$$

Let $t^* \in [0, t]$ be the point maximizing Γ in $[0, t]$. Then

$$\Phi_t = \min_{\tau\in[0,t]} \phi_\tau \geq \min_{\tau\in[0,t]} \frac{K}{(\sigma^2 + \Gamma(\tau))^\beta} \geq \frac{K}{(\sigma^2 + \Gamma(t^*))^\beta}.$$

By Proposition 3

$$\Gamma(t) \leq 2\Gamma_0 + 2\Lambda_0 \frac{(\sigma^2 + \Gamma(t^*))^{2\beta}}{K^2}. \tag{6}$$

Since t^* maximizes Γ in $[0, t]$ it also does so in $[0, t^*]$. Thus, for $t = t^*$, (6) takes the form

$$(\sigma^2 + \Gamma(t^*)) - 2\Lambda_0 \frac{(\sigma^2 + \Gamma(t^*))^{2\beta}}{K^2} - (2\Gamma_0 + \sigma^2) \leq 0. \tag{7}$$

Let $z = \Gamma(t^*) + \sigma^2$,

$$\boldsymbol{a} = \frac{2\Lambda_0}{K^2}, \quad \text{and} \quad \boldsymbol{b} = 2\Gamma_0 + \sigma^2.$$

Then (7) can be rewritten as $F(z) \leq 0$ with

$$F(z) = z - \boldsymbol{a} z^{2\beta} - \boldsymbol{b}.$$

(i) Assume $\beta < 1/2$. By Lemma 3, $F(z) \leq 0$ implies that $z = (\sigma^2 + \Gamma(t^*)) \leq U_0$ with

$$U_0 = \max\left\{ \left(\frac{4\Lambda_0}{K^2}\right)^{\frac{1}{1-2\beta}}, 2(2\Gamma_0 + \sigma^2) \right\}.$$

That is $\Gamma(t^*) \leq B_0 := U_0 - \sigma^2$. Since B_0 is independent of t, we deduce that, for all $t \in \mathbb{R}_+$, $\Gamma(t) \leq B_0$. But this implies that $\phi_t \geq \frac{K}{(\sigma^2+B_0)^\beta}$ for all $t \in \mathbb{R}_+$ and therefore, the same bound holds for Φ_t. By Proposition 2

$$\Lambda(t) \leq \Lambda_0 e^{-2\frac{K}{(\sigma^2+B_0)^\beta} t} \tag{8}$$

which shows that $\Lambda(t) \to 0$ when $t \to \infty$.

(ii) Assume now $\beta = 1/2$. Then (7) takes the form

$$(\sigma^2 + \Gamma(t^*)) \left(1 - \frac{2\Lambda_0}{K^2}\right) - (2\Gamma_0 + \sigma^2) \leq 0.$$

Since $\Lambda_0 < \frac{K^2}{2}$ this implies that

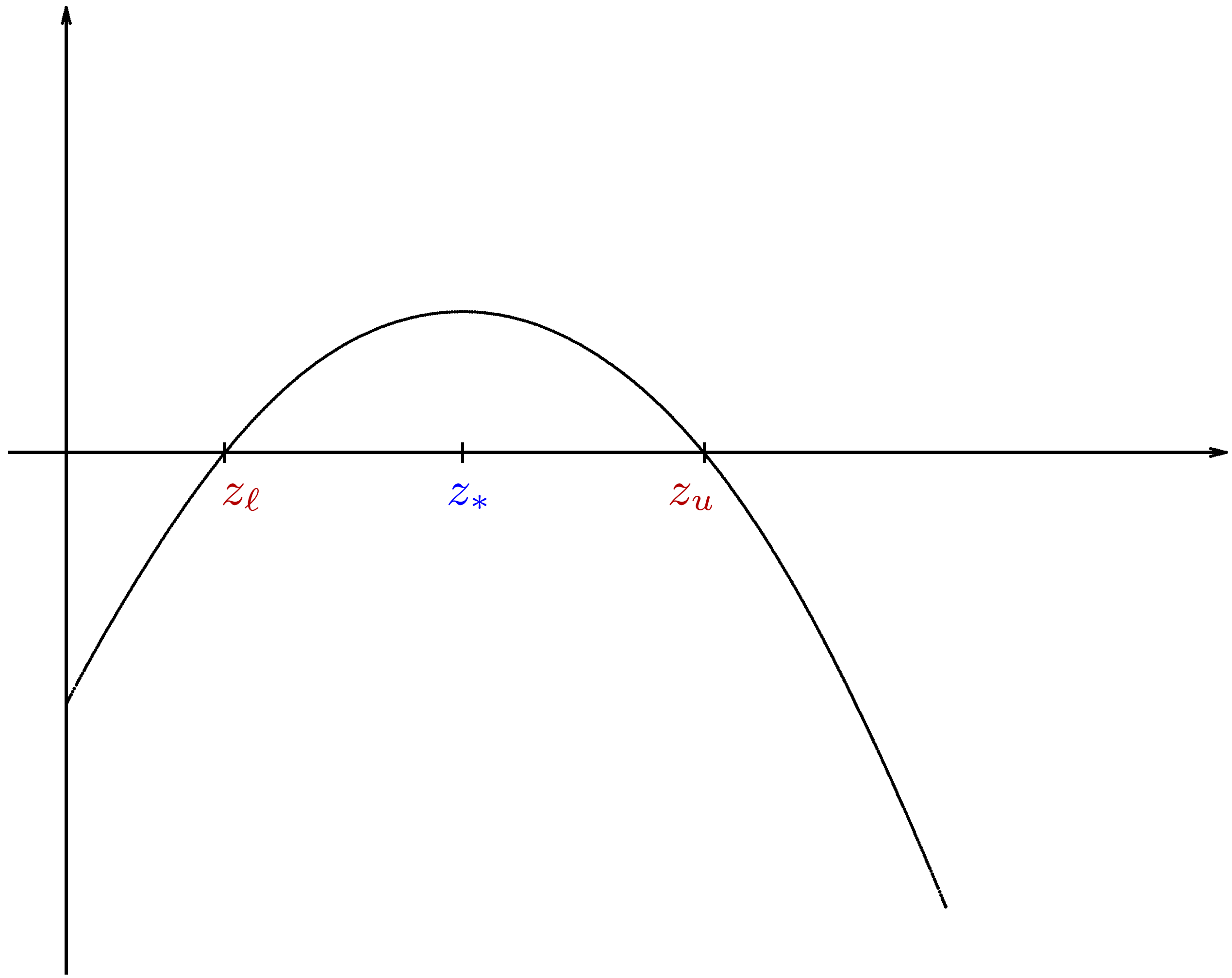
$$\Gamma(t^*) \leq B_0 := \frac{2\Gamma_0 + \sigma^2}{1 - \frac{2\Lambda_0}{K^2}} - \sigma^2 > 0.$$

We now proceed as in case (i).

(iii) Assume finally $\beta > 1/2$ and let $\alpha = 2\beta$ so that $F(z) = z - \boldsymbol{a}z^\alpha - \boldsymbol{b}$. The derivative $F'(z) = 1 - \alpha\boldsymbol{a}z^{\alpha-1}$ has a unique zero at $z_* = \left(\frac{1}{\alpha\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}}$ and

$$\begin{aligned}
F(z_*) &= \left(\frac{1}{\alpha\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}} - \boldsymbol{a}\left(\frac{1}{\alpha\boldsymbol{a}}\right)^{\frac{\alpha}{\alpha-1}} - \boldsymbol{b} \\
&= \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(\frac{1}{\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\left(\frac{1}{\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}} - \boldsymbol{b} \\
&= \left(\frac{1}{\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}}\left[\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right] - \boldsymbol{b} \\
&\geq 0
\end{aligned}$$

the last by our hypothesis. Since $F(0) = -\boldsymbol{b} < 0$ and $F(z) \longrightarrow -\infty$ when $z \longrightarrow \infty$ we deduce that the shape of F is as follows:



Recall, (7) shows that, for all $t \geq 0$, $F(\Gamma(t^*) + \sigma^2) \leq 0$. And the mapping $t \mapsto F(\Gamma(t^*) + \sigma^2)$ is continuous. Thus the image of $t \mapsto \Gamma(t^*) + \sigma^2$ is either included in $[0, z_\ell]$ or in $[z_u, \infty)$.

In addition, when $t = 0$ we have $t^* = 0$ and

$$\begin{aligned}
\Gamma_0 + \sigma^2 &\leq 2\Gamma_0 + \sigma^2 = \boldsymbol{b} \\
&< \left(\frac{1}{\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}} \left[\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right] \\
&< \left(\frac{1}{\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}} \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} \\
&= z_*.
\end{aligned}$$

This implies that $\Gamma_0 + \sigma^2 < z_\ell$ and then, for all $t \geq 0$,

$$\Gamma(t^*) + \sigma^2 \leq z_\ell \leq z_*.$$

We conclude that

$$\Gamma(t^*) \leq B_0 := \left(\frac{1}{\alpha \boldsymbol{a}}\right)^{\frac{1}{\alpha-1}} - \sigma^2 = \left(\frac{K^2}{2\alpha\Lambda_0}\right)^{\frac{1}{\alpha-1}} - \sigma^2.$$

We now proceed as in case (i). ■