Model Reduction of Linear and Nonlinear Control Systems

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Model Reduction for Control Systems

- **Full Order Model**

$$
\begin{aligned}
\dot{x} &= f(x, u) \\
y &= h(x)
\end{aligned}
$$

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u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p, \quad x \in \mathbb{R}^n, \quad n >> 1
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- **Reduced Order Model**

$$\begin{aligned} \dot{z} &= a(z,u) \\ y &= c(z) \end{aligned}$$

$$u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p, \quad z \in \mathbb{R}^k, \quad k << n$$

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- **We shall focus on the first goal.**

Model Reduction of Dynamical Systems

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- **The reduced order model should display the "essential" behaviour of the full order model.**

Model Reduction of Dynamical Systems

- **The model reduction problem for dynamical systems can be viewed as one for control systems by adding an input and output,**

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- **But because the input and output dimensions are now large, it may be difficult to reduce the model.**

Model Reduction of Dynamical Systems

- **Separation into slow and fast modes.**

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \bar{f}(x_1, \ldots, x_n)$$

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- **Other approaches: Petrov Galerkin, nonlinear Galerkin, singular perturbations, center manifolds, inertial manifolds..**

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The output may not be sensitive to changes in the slow modes.

Model Reduction of Linear Control Systems

- **We shall focus on state space methods which generalize to nonlinear control systems.**

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= Hx \\ x(0) &= 0 \\ x \in \mathbb{R}^n, \quad u &\in \mathbb{R}^m, \quad y \in \mathbb{R}^p \\ \lambda(F) &< 0 \end{aligned}$$

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- **Since the unforced system is Hurwitz, it defines an input-output map**

$$
\begin{aligned}
\mathcal{IO}_n : L^2(-\infty, \infty; \mathbb{R}^m) &\rightarrow L^2(-\infty, \infty; \mathbb{R}^p) \\
\mathcal{IO}_n : u(-\infty : \infty) &\mapsto y(-\infty : \infty) \\
y(t) &= \int_{-\infty}^{t} He^{F(t-s)}Gu(s)\, ds
\end{aligned}
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- **Any system can be reduced to one that is minimal.**

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- **The Hankel map takes past inputs to future outputs**

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- **It factors through the current state $x(0)$ and so it is of finite rank hence compact.**

$$u(-\infty : 0) \mapsto x(0) = \int_{-\infty}^{0} e^{-Fs} G u(s)\, ds$$

$$y(t) = H e^{Ft} x(0)$$

B.C. Moores's Balanced Realization Theory.

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= Hx \end{aligned}$$

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- **Hurwitz is needed to insure the existence of the input-output map.**

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- **Controllablity Function**

$$\pi_c(x^0) \quad = \quad \inf_{u(-\infty:0)} \frac{1}{2} \int_{-\infty}^{0} |u(t)|^2 \, dt$$

subject to the system dynamics and
$x(-\infty) = 0, \; x(0) = x^0.$

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- **Observability Function**

$$\pi_o(x^0) = \frac{1}{2} \int_{0}^{\infty} |y(t)|^2 \, dt$$

subject to the system dynamics and
$x(0) = x^0, \; u(t) = 0, \; t \geq 0.$

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- $\pi_c(x)$ and $\pi_o(x)$ are quadratic functions because the system is linear and the energies are quadratic,

$$\pi_c(x) = \frac{1}{2} x' P_c^{-1} x, \qquad \pi_o(x) = \frac{1}{2} x' P_o x$$

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- $\pi_o(x)$ small implies that changes in the direction x lead to small changes in the output energy and so this direction might be ignored in a reduced order model.

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- $P_c,\ P_o$ **transform differently under a linear change of states coordinates** $x = Tz$

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- **Its eigenvalues are the squares of the singular values of the Hankel map.**

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- **There is a linear change of state coordinates so that the controllability and observability gramians are diagonal and equal,**

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- **If the Hankel singular values are distinct then the balanced coordinates are unique up to changes of signs $x_i \mapsto -x_i$.**

Balanced Truncation

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  Let x_1 denote the first k components of x
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$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} H_1 & H_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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- **Balanced Truncation obtained by Galerkin projection.**

$$
\begin{aligned}
\dot{z} &= F_{11} z + G_1 u \\
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- **The Hankel singular values of the reduced model are $\sigma_1, \ldots, \sigma_k$.**
- **But the reduced model is not an optimal Hankel norm approximation of the full model because the singular vectors are different. Typically**

$$\|\mathcal{H}_n - \mathcal{H}_k\| > \sigma_{k+1}$$

Adamjan-Arov-Krein, Glover

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- Glover has shown that for balanced truncation

$$\|\mathcal{IO}_n - \mathcal{IO}_k\| \leq 2 \sum_{j=k+1}^{n} \sigma_j$$

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- **Moreover, a very stable state direction damps out quickly and so the observability function tends to be small in such a direction.**
- **Hence the very stable directions of the dynamics tend to be ignored in the reduction process.**

Balanced Reduction

Suppose we have a linear dyamical system in modal coordinates

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$0 > \lambda_1 \geq \ldots \geq \lambda_n$.

As before we add a dummy input and output,

$$\dot{x} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} x + u, \qquad y = x$$

Balanced Reduction

Then

$$P_c = P_o = \begin{bmatrix} -\frac{1}{2\lambda_1} & & 0 \\ & \ddots & \\ 0 & & -\frac{1}{2\lambda_n} \end{bmatrix}, \qquad \sigma_i = -\frac{1}{2\lambda_i}$$

Balanced Reduction

Then

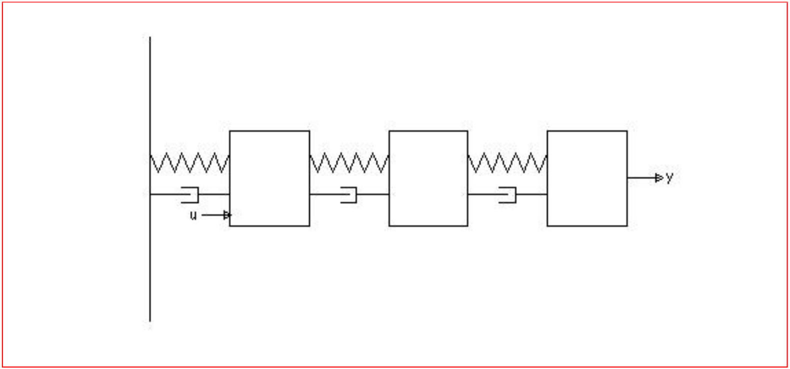
$$P_c = P_o = \begin{bmatrix} -\frac{1}{2\lambda_1} & & 0 \\ & \ddots & \\ 0 & & -\frac{1}{2\lambda_n} \end{bmatrix}, \qquad \sigma_i = -\frac{1}{2\lambda_i}$$

So balanced truncation is the usual truncation onto the slow modes.

$$-\frac{1}{2\lambda_1} \geq -\frac{1}{2\lambda_2} \geq \ldots \geq -\frac{1}{2\lambda_n} > 0$$

Example

Chain of three masses connected by springs and dashpots attached to a wall at one end. The input is a force applied to the mass next to the wall and the output is the displacement of the mass at the other end. Assume that each mass is μ , each spring constant is c and each dampening constant is b.



Example

The system is linear,

$$F = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{2c}{m} & \frac{c}{\mu} & 0 & -\frac{2b}{\mu} & \frac{b}{\mu} & 0 \\ \frac{c}{\mu} & -\frac{2c}{\mu} & \frac{c}{\mu} & \frac{b}{\mu} & -\frac{2b}{\mu} & \frac{b}{\mu} \\ 0 & \frac{c}{\mu} & -\frac{c}{\mu} & 0 & \frac{b}{\mu} & -\frac{b}{\mu} \end{bmatrix} \qquad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\mu} \\ 0 \\ 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Example

If $\mu = 1,\ c = 3,\ b = 0.5$ **then after balancing**

$$P_c = P_o = \begin{bmatrix} 1.6895 & & & & & \\ & 1.4901 & & & & \\ & & 0.1404 & & & \\ & & & 0.1079 & & \\ & & & & 0.0077 & \\ & & & & & 0.0076 \end{bmatrix}$$

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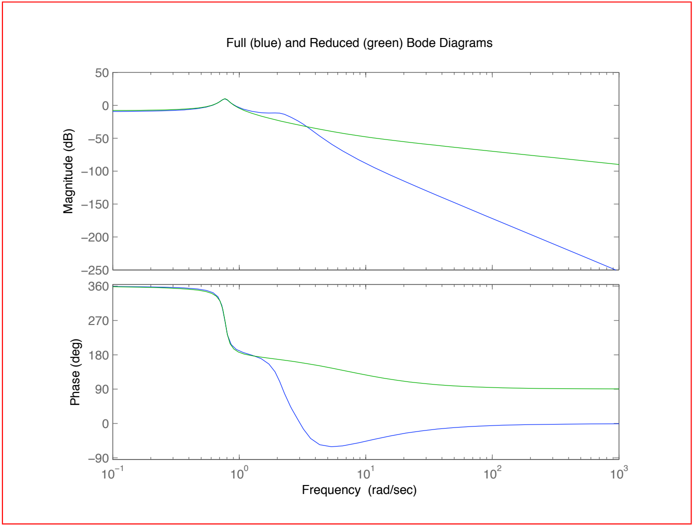
$$\sigma_2 = 1.4901 >> 0.1404 = \sigma_3$$

This suggests taking a reduced order model of dimension $k = 2$.

Example



Full (blue) and Reduced (green) Bode Diagrams
Magnitude (dB)
50
0
−50
−100
−150
−200
−250
Phase (deg)
360
270
180
90
0
−90
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

Error Estimate

Glover has shown that for balanced truncation

$$\|\mathcal{IO}_n - \mathcal{IO}_k\| \leq 2 \sum_{j=k+1}^{n} \sigma_j$$

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For the spring mass example with $k = 2$ this yields

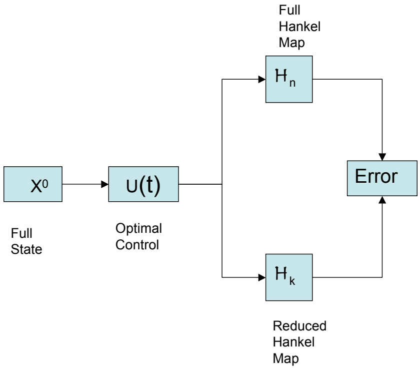
$$\sigma_3 = 0.1404 \leq \|\mathcal{H}_n - \mathcal{H}_k\| \leq \|\mathcal{IO}_n - \mathcal{IO}_k\| \leq 0.5672$$

Error Estimate

If we restrict the Hankel maps to optimal inputs of the full system then

$$\|\mathcal{H}_n - \mathcal{H}_k\| \leq 0.1432$$

Notice how close this is to $\sigma_3 = 0.1404$.

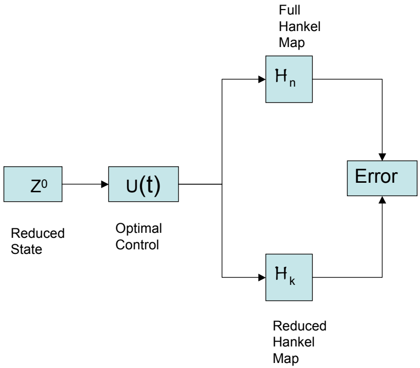


Error Estimate

If we restrict the Hankel maps to optimal inputs of the reduced system then

$$\|\mathcal{H}_n - \mathcal{H}_k\| \leq 0.0867$$

Notice how much smaller this is than $\sigma_3 = 0.1404$.



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- **Here is a new way of viewing and generalizing linear balanced truncation.**

Balanced Truncation

Full Order Model

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We restrict to those reduced order models that can be obtained by Petrov Galerkin projection. For this we need an injection Ψ and a surjection Φ

$$\begin{aligned}\Psi : \mathbb{R}^k &\longrightarrow \mathbb{R}^n \\ \Psi : z &\longmapsto x = \Psi z \\ \Phi : \mathbb{R}^n &\longrightarrow \mathbb{R}^k \\ \Phi : x &\longmapsto z = \Phi x \\ \Phi\Psi z = z, & \qquad (\Psi\Phi)^2 x = \Psi\Phi x\end{aligned}$$

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Reduced Order Model

$$\begin{aligned} \dot{z} &= \Phi F \Psi z + \Phi G u \\ y &= H \Psi z \end{aligned}$$

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To do this it is convenient to put the system in input normal form, that is, make a linear change of state coordinates so that

$$\pi_c(x) = \frac{1}{2}\sum_i x_i^2, \qquad \pi_o(x) = \frac{1}{2}\sum_i \tau_i x_i^2$$

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If $\tau_k >> \tau_{k+1}$ then we should take the range of $\Psi(z)$ to be $x_{k+1} = \cdots = x_n = 0$, e.g.,

$$\Psi(z_1, \ldots, z_k) = \ x \ = (z_1, \ldots, z_k, 0, \ldots, 0)$$

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Define the co-observability function

$$\pi_{oo}(x, \bar{x}) = \frac{1}{2} \int_0^\infty |y(t) - \bar{y}(t)|^2 \, dt$$

where $y(t),\ \bar{y}(t)$ are the outputs from $x(0) = x,\ x(0) = \bar{x}$.

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Because the system is linear

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If the system is in input normal form then the optimal Φ is

$$\Phi(x_1, \ldots, x_n) = \ \ z \ \ = (x_1, \ldots, x_k)$$

Nonlinear Balancing

Scherpen generalized Moore to nonlinear systems

$$\begin{aligned}\dot{x} &= f(x,u)\\ y &= h(x)\end{aligned}$$

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subject to the system dynamics and $x(-\infty) = 0, \; x(0) = x^0$.

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And the observability function,

$$\pi_o(x^0) = \frac{1}{2}\int_{0}^{\infty} |y(t)|^2 \, dt$$

subject to the system dynamics and $x(0) = x^0, \; u(t) = 0$.

Nonlinear Balancing

The controllability function $\pi_c(x)$ and the optimal control $u = \kappa(x)$ satisfy the HJB PDE

$$
\begin{aligned}
0 &= \frac{\partial \pi_c}{\partial x}(x) f(x, \kappa(x)) - \frac{1}{2}|\kappa(x)|^2 \\
0 &= \frac{\partial \pi_c}{\partial x}(x) \frac{\partial f}{\partial u}(x, \kappa(x)) - \kappa'(x)
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$$

The observability function $\pi_o(x)$ satisfies the Lyapunov PDE

$$
0 = \frac{\partial \pi_o}{\partial x}(x) f(x, 0) + \frac{1}{2} h'(x) h(x).
$$

Nonlinear Balancing

Suppose

- **The system is smooth with Taylor expansion**

$$\begin{aligned} \dot{x} &= f(x,u) &= Fx + Gu &+ f^{[2]}(x,u) &+ \dots \\ y &= h(x) &= Hx &+ h^{[2]}(x) &+ \dots \end{aligned}$$

where $^{[d]}$ denotes a vector field that is a homogeneous polynomial of degree d.

Nonlinear Balancing

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- **The linear part of the system is Hurwitz, controllable and observable**

Nonlinear Balancing

Then

- **there exist smooth, positive definite local solutions to the above PDEs around $x = 0$**

Nonlinear Balancing

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- $$\begin{aligned} \pi_c(x) &= \tfrac{1}{2}x'P_c^{-1}x &+\pi_c^{[3]}(x) &+ \dots \\ \pi_o(x) &= \tfrac{1}{2}x'P_o x &+\pi_o^{[3]}(x) &+ \dots \end{aligned}$$

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- **$P_c, \; P_o$ are the controllability and observability gramians of the linear part of the system.**

Nonlinear Balancing

Scherpen showed that there is a local change of coordinates that brings the system into the form

$$\pi_c(x) = \frac{1}{2}x'x, \qquad \pi_o(x) = \frac{1}{2}x' \begin{bmatrix} \tau_1(x) & & 0 \\ & \ddots & \\ 0 & & \tau_n(x) \end{bmatrix} x$$

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Unfortunately neither these coordinates nor the squared singular value functions are unique.

She obtained a reduced order model by Galerkin projection onto the states with large $\tau_i(x)$.

Input Normal Form of Degree One

The functions $\pi_c(x)$ and $\pi_o(x)$ have power series expansions

$$\pi_c(x) = \frac{1}{2}x'P_c^{-1}x + \pi_c^{[3]}(x) + \pi_c^{[4]}(x) + \ldots$$

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Following Moore and Scherpen we can make a linear change of coordinates so that

$$P_c = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}, \qquad P_o = \begin{bmatrix} \tau_1 & & 0 \\ & \ddots & \\ 0 & & \tau_n \end{bmatrix}$$

where $\tau_i = \sigma_i^2$ and $\tau_1 \geq \tau_2 \geq \dots \geq \tau_n > 0$. After this linear change of coordinates the system is said to be in input normal form of degree one.

Input Normal Form of Degree Two

For simplicity of exposition we shall assume that the τ_i are distinct, $\tau_1 > \tau_2 > \ldots > \tau_n > 0$.

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Now

$$\begin{aligned} \pi_c^{[3]}(x) &= \sum_{i \le j \le k} \gamma_c^{ijk} x_i x_j x_k \\ \pi_o^{[3]}(x) &= \sum_{i \le j \le k} \gamma_o^{ijk} x_i x_j x_k \end{aligned}$$

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\end{aligned}$$

Choose three indices $1 \le r \le s \le t \le n$, at least two indices are different, $r < t$. Consider the change of coordinates

$$\begin{aligned}
x_r &= \xi_r + \beta_r \xi_s \xi_t \\
x_t &= \xi_t + \beta_t \xi_r \xi_s \\
x_l &= \xi_l \quad \text{otherwise}
\end{aligned}$$

Input Normal Form of Degree Two

Then the quadratic parts of π_c, π_o are left unchanged but the cubic parts each pick up an extra term,

$$\pi_c^{[3]}(\xi) = \sum_{i \leq j \leq k} \gamma_c^{ijk} \xi_i \xi_j \xi_k + (\beta_r + \beta_t) \xi_r \xi_s \xi_t$$

$$\pi_o^{[3]}(\xi) = \sum_{i \leq j \leq k} \gamma_o^{ijk} \xi_i \xi_j \xi_k + (\tau_r \beta_r + \tau_t \beta_t) \xi_r \xi_s \xi_t$$

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Since $\tau_r > \tau_t$ we can solve the linear system

$$\begin{bmatrix} 1 & 1 \\ \tau_r & \tau_t \end{bmatrix} \begin{bmatrix} \beta_r \\ \beta_t \end{bmatrix} = - \begin{bmatrix} \gamma_c^{rst} \\ \gamma_o^{rst} \end{bmatrix}$$

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This change of coordinates cancels the monomials $\xi_r\xi_s\xi_t$ from $\pi_c^{[3]}(\xi),\ \pi_o^{[3]}(\xi)$.

Input Normal Form of Degree Two

But if $r = s = t$ then we can cancel the monomial x_r^3 from only one of $\pi_c^{[3]}(\xi),\ \pi_o^{[3]}(\xi)$ by a change of coordinates of the form

$$
\begin{aligned}
x_r &= \xi_r + \beta_r \xi_r^2 \\
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\end{aligned}
$$

We do so in $\pi_c^{[3]}(\xi)$ to obtain input normal form of degree two,

$$
\begin{aligned}
\pi_c^{[3]}(\xi) &= 0 \\
\pi_o^{[3]}(\xi) &= \sum_i \gamma_o^{iii} \xi_i \xi_i \xi_i
\end{aligned}
$$

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$$\begin{aligned} x_r &= \xi_r + \beta_r \xi_r^2 \\ x_l &= \xi_l \quad \text{otherwise} \end{aligned}$$

We do so in $\pi_c^{[3]}(\xi)$ to obtain input normal form of degree two,

$$\begin{aligned} \pi_c^{[3]}(\xi) &= 0 \\ \pi_o^{[3]}(\xi) &= \sum_i \gamma_o^{iii} \xi_i \xi_i \xi_i \end{aligned}$$

If the three indices are distinct $r < s < t$ then there are many ways to cancel $x_r x_s x_t$ from π_c, π_o.

Input Normal Form of Degree d

We can do similarly for higher degrees and in this way bring the system into input normal form of degree d ,

$$\pi_c(x) = \frac{1}{2}\sum_{i=1}^{n} x_i^2 + O(x)^{d+2}$$

$$\pi_o(x) = \frac{1}{2}\sum_{i=1}^{n} \tau_i^{[0:d-1]}(x_i)x_i^2 + O(x)^{d+2}$$

where the squared singular value polynomials $\tau_i^{[0:d-1]}(x_i)$ are of degrees 0 through $d-1$

$$\tau_i^{[0:d-1]}(x_i) = \tau_i + \tau_{i,1}x_i + \ldots + \tau_{i,d-1}x_i^{d-1}$$

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We have "simultaneously diagonalized" $\pi_c(x),\ \pi_o(x)$ through terms of degree $\leq d+1$. There are no cross terms, $x_i x_j \ldots$

Input Normal Form of Degree d

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If the system is odd

$$\begin{aligned} f(-x,-u) &= -f(x,u) \\ h(-x) &= -h(x) \end{aligned}$$

then $\pi_c(x),\ \pi_o(x)$ are even and the $\tau_i^{[0:d-1]}(x_i)$ are unique for $d \leq 12$.

Nonlinear Model Reduction

As with linear balancd truncation we restrict to reduced order models that can be found by nonlinear Galerkin projection.

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For this we need an embedding ψ and a submersion ϕ

$$\begin{aligned} \psi : \mathbb{R}^k &\rightarrow \mathbb{R}^n \\ \psi : z &\mapsto x = \psi(z) \\ \phi : \mathbb{R}^n &\rightarrow \mathbb{R}^k \\ \phi : x &\mapsto z = \phi(x) \\ \phi(\psi(z)) = z, & \qquad (\psi \circ \phi)^2(x) = (\psi \circ \phi)(x) \end{aligned}$$

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Reduced Order Model

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\begin{aligned}
\dot{z} &= \frac{\partial \phi}{\partial x}(\psi(z)) f(\psi(z), u) \\
y &= h(\psi(z))
\end{aligned}
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Nonlinear Model Reduction

We would like to choose the embedding $\psi : I\!R^k \to I\!R^n$ so the k dimensional submanifold that is its range "maximizes" the output energy $\pi_o(x)$ for given input energy $\pi_c(x)$.

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Assume the system is in input normal form of degree d , that the range of input energy of interest is

$$\pi_c(x) \approx \frac{1}{2}|x|^2 \leq \frac{1}{2}c^2$$

and that

$$\tau_i^{[0:d-1]}(x_i) >> \tau_j^{[0:d-1]}(x_j)$$

for $1 \leq i \leq k < j \leq n$ and $|x_i|, \ |x_j| \leq c$.

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Then a "reasonable" choice of $\psi(z) = x$ is

$$\psi(z_1, \ldots, z_k) = \quad x \quad = (z_1, \ldots, z_k, 0, \ldots, 0)$$

How should we choose the submersion ϕ ?

Nonlinear Model Reduction

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We choose $\phi : \mathbb{R}^n \to \mathbb{R}^k$ to minimize the L^2 norm of the difference of the outputs from x and $\psi(\phi(x))$.

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Define the co-observability function as before

$$\pi_{oo}(x, \bar{x}) = \frac{1}{2} \int_0^\infty |y(t) - \bar{y}(t)|^2 \, dt$$

where $y(t),\ \bar{y}(t)$ are the outputs from $x(0) = x,\ x(0) = \bar{x}$. when $u(t) = 0$.

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Then π_{oo} satisfies the Lyapunov PDE

$$0 \;=\; \frac{\partial \pi_{oo}}{\partial(x,\bar{x})}(x,\bar{x}) \begin{bmatrix} f(x,0) \\ f(\bar{x},0) \end{bmatrix} + \frac{1}{2}\,|h(x) - h(\bar{x})|^2$$

Nonlinear Model Reduction

As before π_{oo} has a power series expansion

$$\pi_{oo}(x,\bar{x}) \;=\; \frac{1}{2}\sum_i \tau_i (x_i-\bar{x}_i)^2 + \pi_{oo}^{[3]}(x,\bar{x}) + \dots$$

that can be computed term by term.

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so $\phi(x)$ satisfies

$$0 \;=\; \frac{\partial \pi_{oo}}{\partial \bar{x}}(x, \psi(\phi(x)))\frac{\partial \psi}{\partial z}(\phi(x))$$

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$$\frac{\partial \psi}{\partial z}(\phi(x)) \;=\; \begin{bmatrix} I \\ 0 \end{bmatrix}$$

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so $\phi_i(x) = z_i$ satisfies for $1 \leq i \leq k$

$$\phi_i(x) = x_i + \frac{1}{\tau_i}\left(\frac{\partial \pi_{oo}^{[3]}}{\partial \bar{x}_i}(x, (\phi(x), 0)) + \frac{\partial \pi_{oo}^{[4]}}{\partial \bar{x}_i}(x, (\phi(x), 0)) + \ldots\right)$$

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This can be solved term by term via repeated substitution.

$$\phi_i^{(1)}(x) = x_i$$

$$\phi_i^{(2)}(x) = x_i + \frac{1}{\tau_i}\frac{\partial \pi^{[3]}_{oo}}{\partial \bar{x}_i}(x, (\phi^{(1)}(x), 0))$$

$$\vdots$$

Nonlinear Error Estimate

Full Order Model $x \in \mathbb{R}^n$

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned}$$

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What is the error between their input-output maps?

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What is the error between their Hankel maps restricted to optimal inputs?

Nonlinear Error Estimate

Full Order Optimal Feedback

$$\begin{aligned} u &= \kappa(x) = \left(\frac{\partial \pi_c}{\partial x}(x) \frac{\partial f}{\partial u}(x, \kappa(x)) \right)' \\ &= G'x + \left(x' \frac{\partial f^{[2]}}{\partial u}(x, G'x) \right)' + \ldots \end{aligned}$$

This can be solved term by term via repeated substitution.

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Combined closed loop system

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} F + GG' & 0 \\ BG' & A \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \dots$$

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$F + GG'$ **is antistable.**

A **is stable.**

Nonlinear Error Estimate

So there is an unstable manifold $z = \theta(x)$ satisfying

$$a(\theta(x), \kappa(x)) = \frac{\partial \theta}{\partial x}(x) f(x, \kappa(x))$$

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This PDE can be solved term by term.

Nonlinear Error Estimate

Define the cross-observability function

$$\rho(x^0, z^0) = \frac{1}{2}\int_0^\infty |y(t) - w(t)|^2 dt$$

where

$$\begin{aligned} \dot{x} &= f(x, 0) & \dot{z} &= a(z, 0) \\ y &= h(x) & w &= c(z) \\ x(0) &= x^0 & z(0) &= z^0 \end{aligned}$$

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Then ρ satisfies the Lyapunov PDE

$$0 = \frac{\partial \rho}{\partial (x,z)}(x,z) \begin{bmatrix} f(x,0) \\ a(z,0) \end{bmatrix} + \frac{1}{2}|h(x) - c(z)|^2$$

which can be solved term by term.

Nonlinear Error Estimate

Let $u_x(-\infty : 0)$ be the optimal control that excites the full order system to $x(0) = x$. Then the nonlinear Hankel maps satisfy

$$|\mathcal{H}_n(u_x(-\infty : 0)) - \mathcal{H}_k(u_x(-\infty : 0))|^2 \leq \rho(x, \theta(x))$$

Nonlinear Error Estimate

Let $u_x(-\infty : 0)$ be the optimal control that excites the full order system to $x(0) = x$. Then the nonlinear Hankel maps satisfy

$$|\mathcal{H}_n(u_x(-\infty : 0)) - \mathcal{H}_k(u_x(-\infty : 0))|^2 \leq \rho(x, \theta(x))$$

A good estimate of the error between the nonlinear Hankel maps is

$$\sup \left(\frac{\rho(x, \theta(x))}{\pi_c(x)} \right)^{1/2}$$

Nonlinear Error Estimate

As with input normal form we can make a change of coordinates so that

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\begin{aligned}
\pi_c(x) &= \frac{1}{2}|x|^2 + O(x)^{d+2} \\
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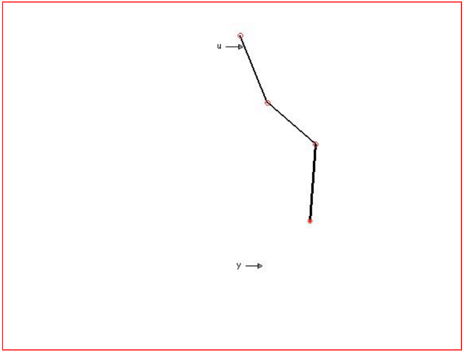
$\epsilon_i^{[0:d-1]}(x_i)$ are the error polynomials of degrees 0 through $d-1$.

The error polynomials are unique for $d \leq 6$.

Nonlinear Example

Three linked rods connected by planar rotary joints with springs and dampening hanging from the ceiling. The input is a torque applied to the top joint and the output is the horizontal displacement of the bottom. Each rod is uniform of length $l = 2$, mass $\mu = 1$, with spring constant $c = 3$, dampening constant $b = 0.5$ and gravity constant $g = 0.5$.



Nonlinear Example

We approximated the nonlinear system by its Taylor series through terms of degree $\mathbf{5}$.

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The Hankel singular values of the linear part of the system are

$$\left[\ 15.3437 \quad 14.9678 \quad 0.3102 \quad 0.2470 \quad 0.0156 \quad 0.0091\ \right]$$

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Apparently only two dimensions are linearly significant.

Nonlinear Example

Here are the squared singular value polynomials.

$$\boldsymbol{\tau_1^{[0:4]}(x_1) = 235.4298 - 3.4163x_1^2 - 0.3104x_1^4}$$

$$\boldsymbol{\tau_2^{[0:4]}(x_2) = 224.0356 - 3.2750x_2^2 - 0.2941x_2^4}$$

$$\boldsymbol{\tau_3^{[0:4]}(x_3) = 000.0962 + 0.0014x_3^2 - 0.0001x_3^4}$$

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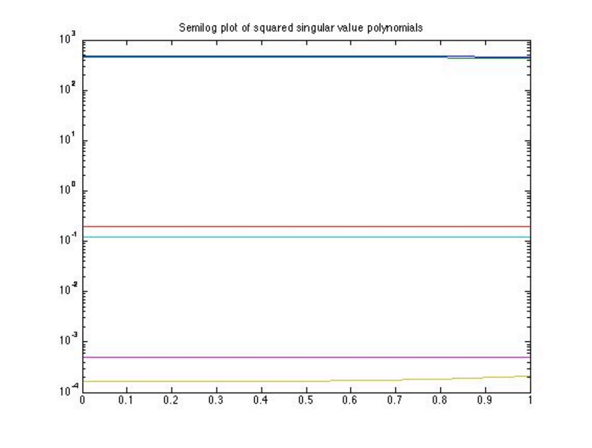
Apparently only two dimensions are nonlinearly significant.

Nonlinear Example

Semilog plot of the squared singular value polynomials $\tau_i^{[0:4]}$



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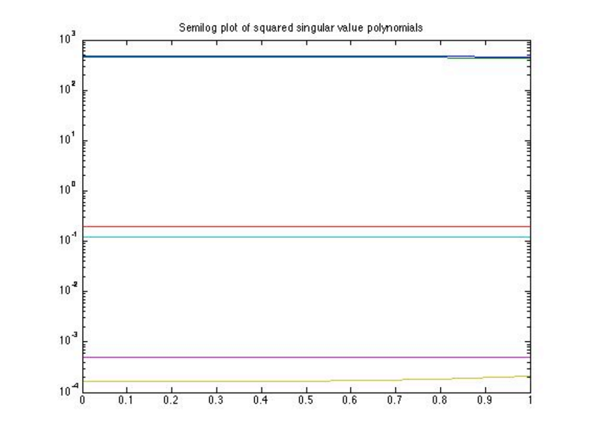

Notice the difference in scale and how flat they are.

Nonlinear Example

Let $u_x(-\infty : 0)$ be the optimal input that excites the full system to x.

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The error between the nonlinear Hankel maps satisfies

$$|\mathcal{H}_n(u_x(-\infty : 0)) - \mathcal{H}_k(u_x(-\infty : 0))|^2$$
$$\leq 0.0965|x|^2 - 0.0009|x|^4 + \dots$$

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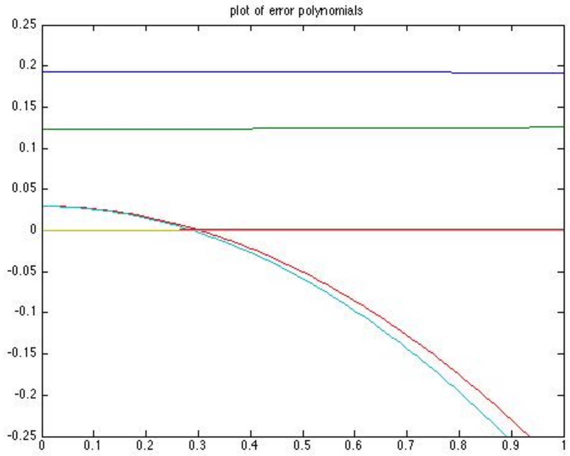
By way of comparison, the square of the third Hankel singular value is

$$0.0962$$

so this estimate is tight.

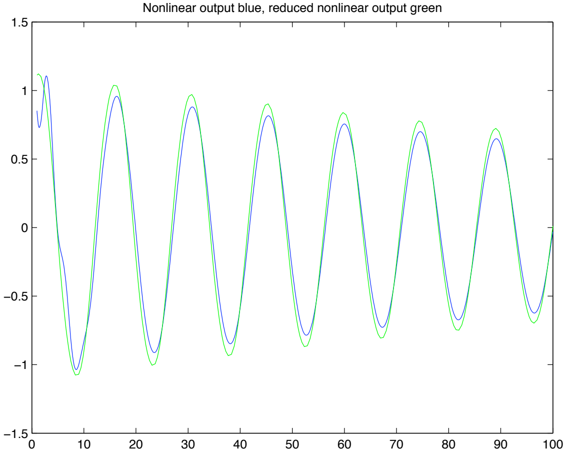
Nonlinear Example

Here are the error polynomials $\epsilon_i^{[0:2]}$



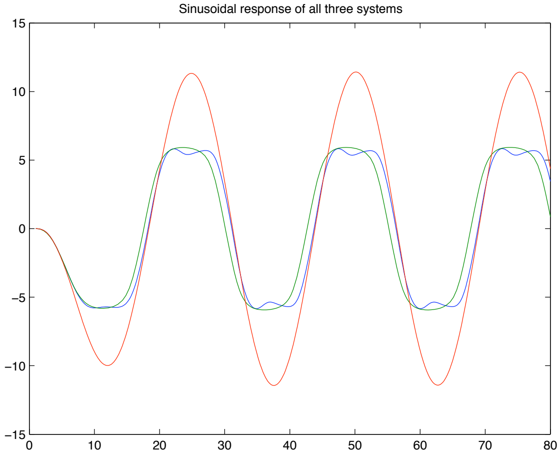
Nonlinear Example

Here are outputs of the Hankel maps of the full and reduced systems excited by an optimal control $u_x(-\infty : 0)$ for random x.



Nonlinear Example

Here are the responses of the full nonlinear model (blue), the reduced nonlinear model (green) and the linear part of the full model (red) to a sinusoidal input.



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- **We gave a new interpretation of linear balanced truncation that readily generalizes to nonlinear systems.**
- **The nonlinear method is compuationally feasible for moderate state dimensions and moderate degrees of approximation.**
- **For both linear and nonlinear model reduction, we gave new computable error bounds for the difference of the full and reduced Hankel maps restricted to optimal inputs.**
- **We believe that these bounds are typical of the errors for inputs that are used in practice.**

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- **We gave a new interpretation of linear balanced truncation that readily generalizes to nonlinear systems.**
- **The nonlinear method is compuationally feasible for moderate state dimensions and moderate degrees of approximation.**
- **For both linear and nonlinear model reduction, we gave new computable error bounds for the difference of the full and reduced Hankel maps restricted to optimal inputs.**
- **We believe that these bounds are typical of the errors for inputs that are used in practice.**
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- **Thank you for listening.**