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Chaos induced by coupled-expanding maps

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Outline

1. Introduction

A general discrete system

$$x_{n+1} = f(x_n), \quad n \geq 0,$$

where $f : D \subset X \to X$ is a map and (X, d) is a metric space.

Orbit: $x_0,\ x_1 = f(x_0),\ x_2 = f(x_1),\ \ldots$

What is chaos?

$\diamondsuit$ No general definition of chaos

Li and Yorke [1975], Devaney [1987], Wiggins [1990]

1. Li-Yorke chaos

The system has an uncountable scrambled set.

Scrambled set:

Let $S \subset D$, containing at least two distinct points. Then, S is called a scrambled set if $\forall\ x_0, y_0 \in S,\ x_0 \neq y_0$,

(i) $\liminf_{n\to\infty} d(x_n, y_n) = 0$;

(ii) $\limsup_{n\to\infty} d(x_n, y_n) > 0$.

2. Devaney chaos [1987]:

$f : V \subset D \to V$ satisfies

(i) dense periodic points in V;

(ii) topologically transitive in V;

(iii) sensitive dependence on initial conditions in V.

3. Wiggins chaos [1990]: (ii) + (iii)

Relationships: they are not equivalent in general.

1. Devaney chaos $\Longrightarrow$ Wiggins chaos.

2. Let V be a compact set of X, containing infinitely many points, and let $f : V \to V$ be continuous and surjective.

(i) Devaney chaos $\Longrightarrow$ Li-Yorke chaos.

(ii) Wiggins chaos + one periodic point in V $\Longrightarrow$ Li-Yorke chaos.

3. The converses are not true in general.

How to determine whether a given system is chaotic?

1. One-dimensional systems: period $k \neq 2^n$, positive entropy, turbulence

2. Higher-dimensional systems: snap-back repeller

3. Infinite-dimensional systems?

2. *Concept of coupled-expanding map*

Block LS and Coppel WA [1992] Lecture Notes in Mathematics, Vol.1513.

A C^0 map $f : I \to I$ is said to be turbulent if $\exists$ closed and bounded subintervals J and K, with at most one common point, s.t.

$$f(J) \supset J \cup K, \quad f(K) \supset J \cup K.$$

Further, it is said to be strictly turbulent if $J \cap K = \phi$.

* J and K are compact and connected.

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$\lozenge$ A turbulent map f is chaotic in the sense of Li-Yorke.

Example 1. The logistic map: $f(x) = \mu x(1-x)$, $\mu \geq 4$.

Example 2. The tent map

$$T(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1/2 \\ 2(x-1/2), & \text{if } 1/2 < x \leq 1. \end{cases}$$

Extended to maps in metric spaces, 2005.

"Turbulent map" $\Longrightarrow$ "Coupled-expanding map"

DEFINITION $f : D \subset X \to X$.

Assume that $\exists\, m\,(\geq 2)$ subsets $V_i \subset D$, $1 \leq i \leq m$, s.t.

$$V_i \cap V_j = \partial_D V_i \cap \partial_D V_j, \quad 1 \leq i \neq j \leq m$$

$$f(V_i) \supset \bigcup_{j=1}^{m} V_j, \ 1 \leq i \leq m.$$

Then f is said to be coupled-expanding in V_i, $1 \leq i \leq m$.

Strictly coupled-expanding: $d(V_i, V_j) > 0$ for all $1 \leq i \neq j \leq m$.

coupled-expanding — CE

strictly coupled-expanding — SCE

DEFINITION [Shi, Ju and Chen, 2006]

Let $f : D \subset X \to X$ and $A = ((A)_{ij})$ be an $m \times m$ transitive matrix ($m \geq 2$). Assume that $\exists\, m$ subsets $V_i \subset D$ s.t.

$$V_i \cap V_j = \partial_D V_i \cap \partial_D V_j, \quad 1 \leq i \neq j \leq m,$$

$$f(V_i) \supset \bigcup_{\substack{j \\ (A)_{ij}=1}} V_j,\ 1 \leq i \leq m$$

Then f is said to be CE for matrix A in V_i, $1 \leq i \leq m$.

SCE for matrix A: $d(V_i, V_j) > 0$ for all $1 \leq i \neq j \leq m$.

* A transitive matrix $A = ((A)_{ij})_{m\times m}$ ($m \geq 2$)

$(A)_{ij} = 0$ or 1 for all $i,\ j$;

$\sum_{j=1}^{m}(A)_{ij} \geq 1$ for all i; $\sum_{i=1}^{m}(A)_{ij} \geq 1$ for all j.

3. SCE maps and subshifts of finite type

3.1. One-sided symbolic dynamical systems

$S := \{1, 2, \ldots, m\}$, $m \geq 2$,

$$\sum\nolimits_m^+ := \{\alpha = (a_0, a_1, a_2, \ldots) : \ a_i \in S, \ i \geq 0\}$$

$$\rho(\alpha, \beta) := \sum_{i=0}^{\infty} \frac{d(a_i, b_i)}{2^i},$$

$$d(a_i, b_i) := 0 \text{ if } a_i = b_i,$$

$$d(a_i, b_i) := 1 \text{ if } a_i \neq b_i,$$

where $\alpha = (a_0, a_1, a_2, \ldots)$ and $\beta = (b_0, b_1, b_2, \ldots)$.

$\diamondsuit$ $(\sum_m^+, \rho)$ is a complete metric space and a Cantor set (compact, perfect, and totally disconnected).

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The shift map

$$\sigma : \sum\nolimits_m^+ \to \sum\nolimits_m^+, \quad \sigma(a_0, a_1, a_2, \ldots) = (a_1, a_2, a_3, \ldots).$$

$(\sum_m^+, \sigma)$ is called the one-sided symbolic dynamical system on m symbols.

It is chaotic in the sense of both Devaney and Li-Yorke.

3.2 Subshift of finite type

Let A be an $m \times m$ transitive matrix.

$$\sum_m^+(A) := \{\alpha = (a_0, a_1, \ldots) \in \sum_m^+ : (A)_{a_i a_{i+1}} = 1, \; i \geq 0\}$$

is a compact invariant set under σ.

The subshift of finite type

$$\sigma_A := \sigma|_{\sum_m^+(A)} : \sum_m^+(A) \to \sum_m^+(A).$$

Q: Under what conditions the subshifts are chaotic in the sense of Li-Yorke or Devaney?

THEOREM [Shi, Ju and Chen, 2006]

Assume that A is irreducible. Then the following statements are equivalent:

(i) σ_A is chaotic in the sense of Devaney on the infinite set $\sum_m^+(A)$;

(ii) σ_A is chaotic in the sense of Li-Yorke;

(iii) $\sum_m^+(A)$ is infinite;

(iv) $\sum_m^+(A)$ is a Cantor set;

(v) $\sum_{j=1}^m (A)_{i_0 j} \geq 2$ for some i_0;

(vi) $\sum_{i=1}^m (A)_{i j_0} \geq 2$ for some j_0.

* Irreducible transitive matrix: $\forall\,(i,j),\ 1 \leq i,j \leq m$, $\exists\ k \geq 1$ s.t. $(A^k)_{ij} > 0$.

3.3. Relationships

THEOREM [Shi, Ju and Chen, 2006] $f : D \subset X \to X$ is C^0.

f is topologically conjugate to σ_A

if and only if

$\exists\, m$ disjoint compact subsets $V_i \subset D$, $1 \le i \le m$, s.t.

(i) f is SCE for A in $V_i,\ 1 \le i \le m$;

(ii) $\forall\, \alpha = (a_0, a_1, \ldots) \in \sum_m^+(A)$,

$$\bigcap_{n=0}^{\infty} f^{-n}(V_{a_n})$$

is a singleton.

THEOREM [Shi, Ju and Chen, 2006] $f : D \subset X \to X$.

Assume that $\exists\ m(\geq\ 2)$ nonempty bounded and closed subsets $V_i \subset D$ with $d(V_i, V_j) > 0$ s.t. f is C^0 in $\bigcup_{i=1}^m V_i$ and satisfies

(i) f is SCE for some A in V_i, $1 \leq i \leq m$;

(ii) $\forall\, \alpha = (a_0, a_1, \ldots) \in \sum_m^+(A)$,

$$d(\bigcap_{i=0}^{n} f^{-i}(V_{a_i})) \to 0 \text{ as } n \to \infty.$$

Then f in some invariant set $V \subset \bigcup_{i=1}^m V_i$ is topologically conjugate to σ_A.

4. Chaos induced by coupled-expanding maps

4.1. SCE maps in compact sets

THEOREM [Shi and Chen, 2004]

Let $V_j,\ 1 \le j \le m$, be disjoint compact sets of X, and $f : \bigcup_{j=1}^m V_j \to X$ be C^0. If

(i) f is SCE in $V_j,\ 1 \le j \le m$;

(ii) $\exists\ \lambda > 1$ s.t.

$$d(f(x), f(y)) \ge \lambda\, d(x, y),\ \forall\ x, y \in V_j,\ 1 \le j \le m;$$

then $\exists$ a Cantor set $\Lambda \subset \bigcup_{j=1}^m V_j$ s.t. $f : \Lambda \to \Lambda$ is topologically conjugate to $\sigma : \sum_m^+ \to \sum_m^+$.

$\Longrightarrow$ Chaotic on Λ in the sense of Devaney as well as Li-Yorke.

THEOREM [Shi and Yu, 2005]

If f satisfies (i) and

(ii′) $\exists$ a j_0, and $\lambda > 1$ s.t.

$$d(f(x), f(y)) \geq \lambda\, d(x, y) \;\; \forall\, x, y \in V_{j_0};$$

and f is injective in the other sets V_j, $j \neq j_0$;

then f is chaotic in the sense of both Wiggins and Li-Yorke on a perfect and compact invariant set.

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THEOREM [Shi, Ju and Chen, 2006]

Assume that $\exists$ an $m \times m$ irreducible transitive matrix A with $\sum_{j=1}^{m}(A)_{i_0 j} \geq 2$ for some i_0; m disjoint compact subsets V_i of D, $1 \leq i \leq m$, and f is C^0 in $\bigcup_{i=1}^{m} V_i$.

If f satisfies that

(i′) f is SCE for A in V_i, $1 \leq i \leq m$;

(ii) $\exists\ \lambda > 1$ s.t.

$$d(f(x), f(y)) \geq \lambda\, d(x, y) \ \ \forall\, x, y \in V_i,\ 1 \leq i \leq m.$$

Then, f in a Cantor set V is topologically conjugate to σ_A.

$\Longrightarrow$ f is chaotic on V in the sense of Devaney as well as Li-Yorke.

4.2. SCE maps in bounded and closed sets

In the following, (X, d) is a complete metric space.

THEOREM [Shi and Chen, 2004] Let $V_j \subset X$, $1 \leq j \leq m$, be bounded and closed subsets of X with $d(V_i, V_j) > 0$, and $f : \bigcup_{i=1}^m V_i \to X$ be C^0. If

(i) f is SCE in V_j, $1 \leq j \leq m$;

(ii) $\exists\, \mu \geq \lambda > 1$ s.t.

$$\lambda\, d(x,y) \leq d(f(x), f(y)) \leq \mu\, d(x,y)\ \forall\, x, y \in V_j,\ \ 1 \leq j \leq m;$$

then $\exists$ a Cantor set $\Lambda \subset \bigcup_{i=1}^m V_i$ s.t. $f : \Lambda \to \Lambda$ is topologically conjugate to $\sigma : \sum_m^+ \to \sum_m^+$.
$\Longrightarrow$ f is chaotic on Λ in the sense of Devaney as well as Li-Yorke.

* Similarly, these two conditions can be weakened as we have done before.

5. Some applications

1. Anti-control of chaos (or chaotification)

The original system

$$x_{n+1} = f(x_n), \quad n \geq 0.$$

Objective: Design a control input sequence $\{u_n\}$ s.t.

$$x_{n+1} = f(x_n) + u_n, \quad n \geq 0$$

is chaotic.

$$u_n = \mu\, g(x_n) \text{ or } u_n = g(\mu\, x_n).$$

* Shi Y and Chen G [2005] Int J Bifur Chaos, 15, 547–556. (R^n)

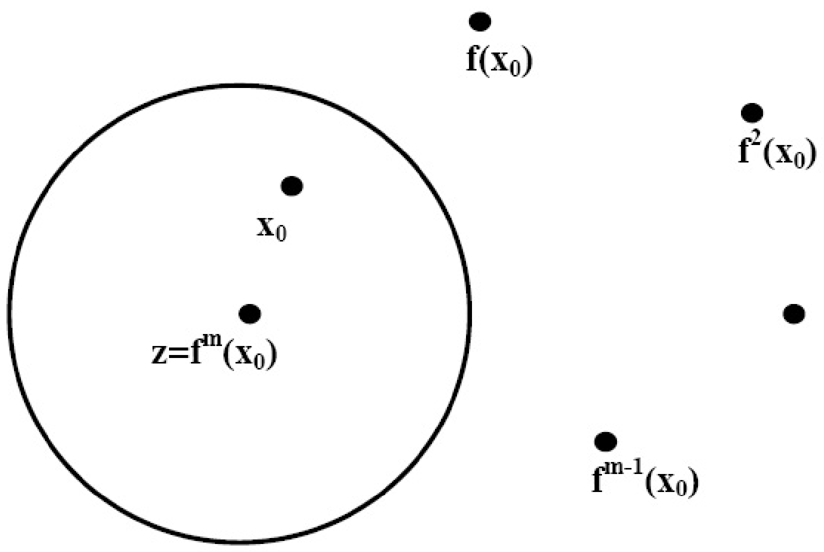
* Lu J [2005] Chinese Physics 14, 1082-1087; 1342-1346. (R^n)

* Shi Y, Yu P and Chen G [2006] Int J Bifur Chaos, 16, 2615-2636. (Banach spaces)

2. Snap-back repeller theory

Marotto FR [1978] J. Math. Anal. Appl. 63, 199-223.

$f : \mathrm{R}^n \to \mathrm{R}^n$ is C^1, $f(z) = z$.



A snap-back repeller implies Li-Yorke chaos.

Recent developments:

(1) The concept is extended to maps in metric spaces in 2004:

Regular and singular; nondegenerate and degenerate.

* In the Marotto paper, a snap-back repeller is regular and nondegenerate.

(2) C^1 maps in R^n:

- Improvement of the Marotto theorem:

A snap-back repeller in the Marotto paper implies Devaney chaos as well as Li-Yorke chaos.

- The assumptions of the Marotto theorem were weakened:

A regular snap-back repeller implies Li-Yorke chaos.

(3) Snap-back repellers in Banach spaces and in general complete metric spaces.

3. Partial difference equations [Y Shi, 2006]

4. Time-varying discrete systems [Y Shi and G Chen, 2005]

5. PDEs, FDEs?

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6. Examples with simulations

Example 1.

$$f(x,y) = \begin{cases} 4(x,y), & \text{if } (x,y) \in B_2(0) \\ \big(\sin^3(x-3) + \sin x \, \cos^2(y-3), \, \sin(y-3)\big), & \text{if } (x,y) \notin B_2(0). \end{cases}$$

The origin is a regular and nondegenerate snap-back repeller.

$\Longrightarrow$ f is chaotic in the sense of both Devaney and Li-Yorke.

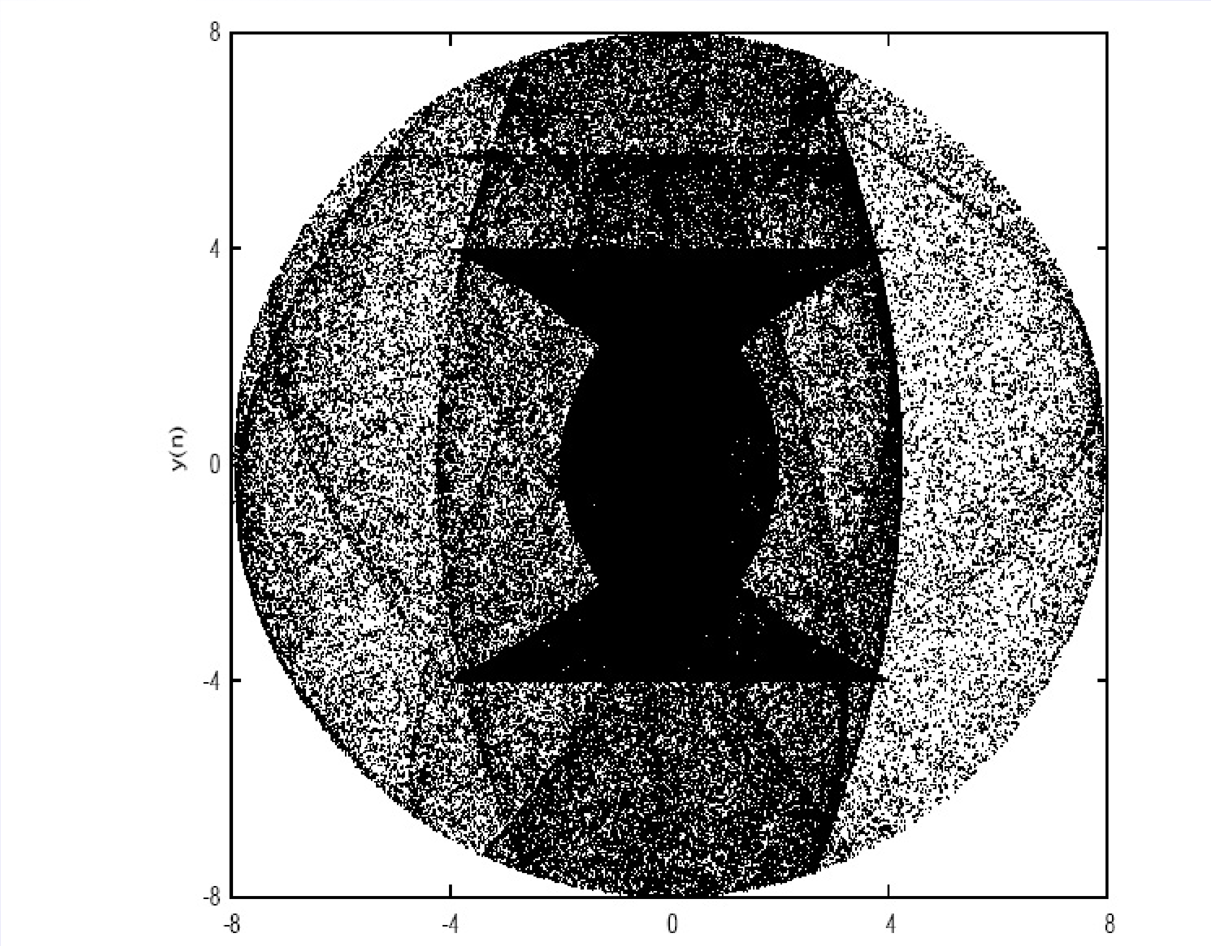

Figure 1a: Simulation result in the (x, y) space in the rectangular box $[-8, 8] \times [-8, 8]$.

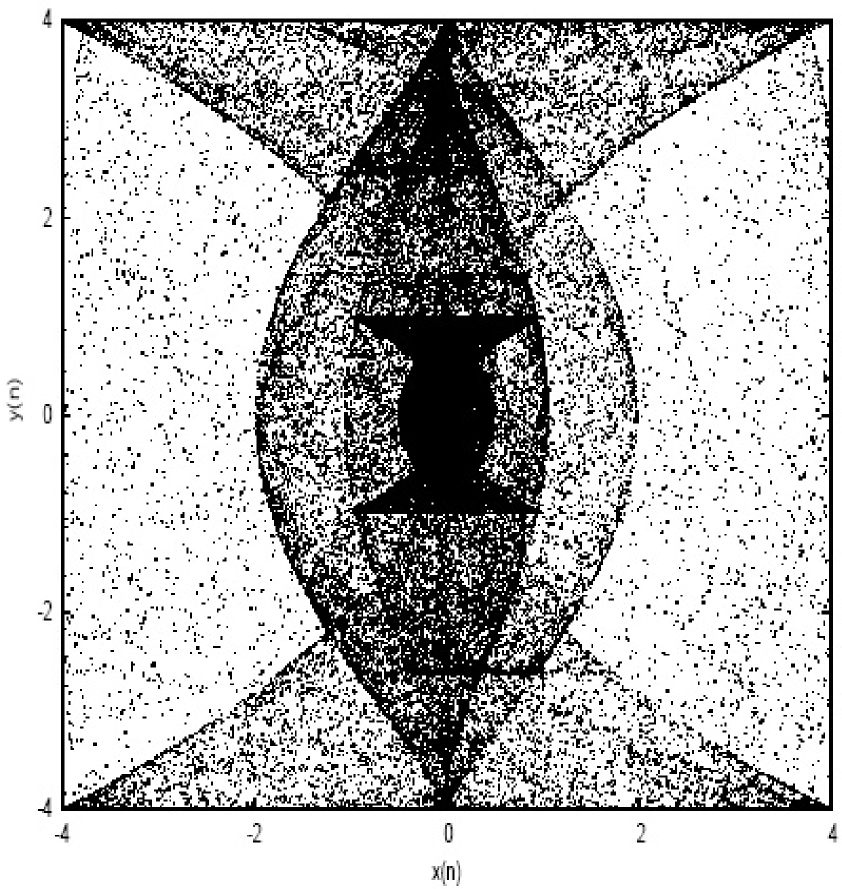

Figure 1b: Simulation result in the (x, y) space in the rectangular box $[-4, 4] \times [-4, 4]$.

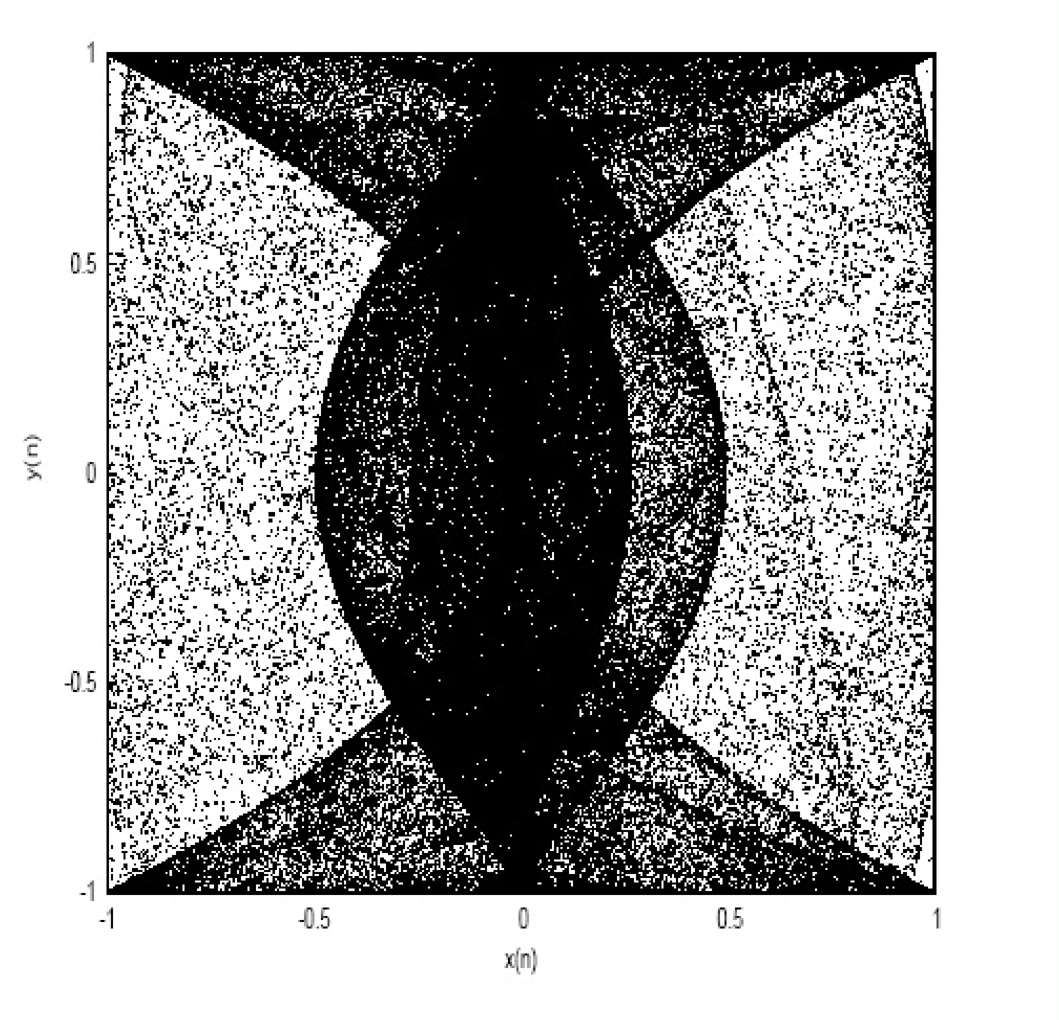

Figure 1c: Simulation result in the (x, y) space in the rectangular box $[-1, 1] \times [-1, 1]$.

Example 2.

$$f(x,y)=\begin{cases} 4(x,y), & \text{if } (x,y)\in B_2(0)\\ \left(\sin\left((x-3)^3+x(y-3)^2\right),\sin(y-3)\right), & \text{if } (x,y)\notin B_2(0). \end{cases}$$

The origin is a regular and degenerate snap-back repeller.

$\Longrightarrow$ f is chaotic in the sense of both Wiggins and Li-Yorke.

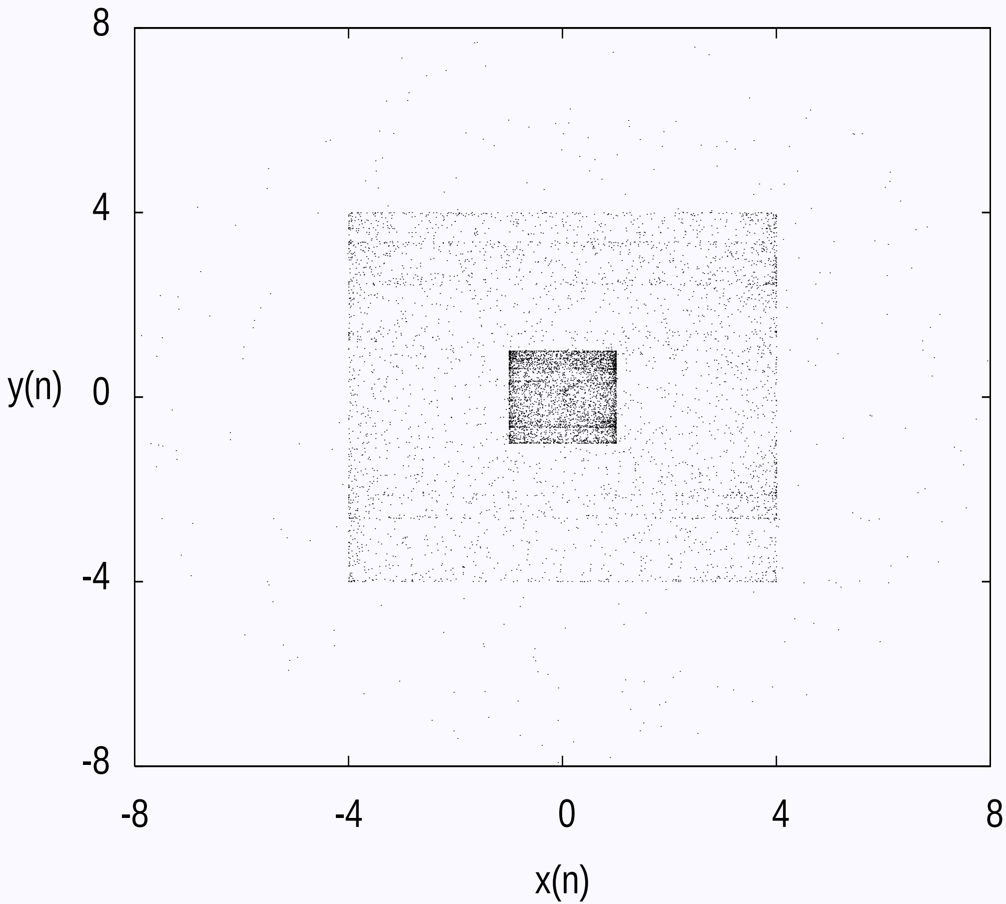

Figure 2a. Simulation result in the rectangular box $[-8, 8] \times [-8, 8]$

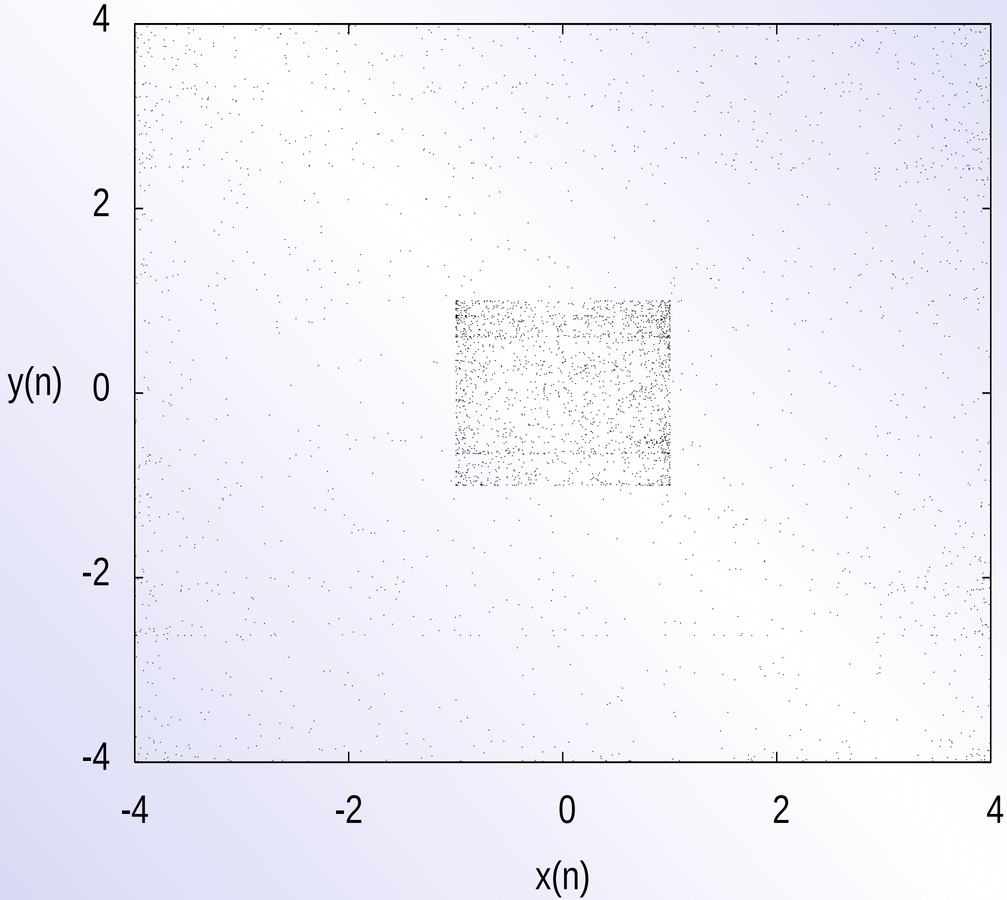

Figure 2b. Zoom area of the rectangular box $[-4, 4] \times [-4, 4]$

Thanks for your attention!

DEFINITION [Shi and Chen, 2004] $f: X \to X$, $f(z) = z$.

Expanding fixed point (EFP): $\exists\,\lambda > 1$ s.t.

$$d(f(x), f(y)) \geq \lambda\, d(x, y),\ \forall\, x, y \in \bar{B}_r(z).$$

Snap-back repeller (SBR): z is an EFP in $\bar{B}_r(z)$.

$\exists\, x_0 \in B_r(z)$, $x_0 \neq z$, s.t. $f^m(x_0) = z$ for some $m \geq 2$.

Nondegenerate SBR: $\exists\, \mu > 0$ s.t.

$$d(f^m(x), f^m(y)) \geq \mu\, d(x, y),\ \forall x, y \in \bar{B}_{r_0}(x_0) \subset \bar{B}_r(z).$$

Regular SBR: $f(B_r(z))$ is open, and $\exists\, \delta_0 > 0$ s.t. z is an interior point of $f^m(B_\delta(x_0))$ for any $\delta \leq \delta_0$.

* In the Marotto paper, a SBR is regular and nondegenerate.