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The University of Texas at Arlington

Adaptive Dynamic Programming (ADP) For Feedback Control Systems

Invited by
Hanxiong Li
Gary Feng
Ron Chen

Importance of Feedback Control

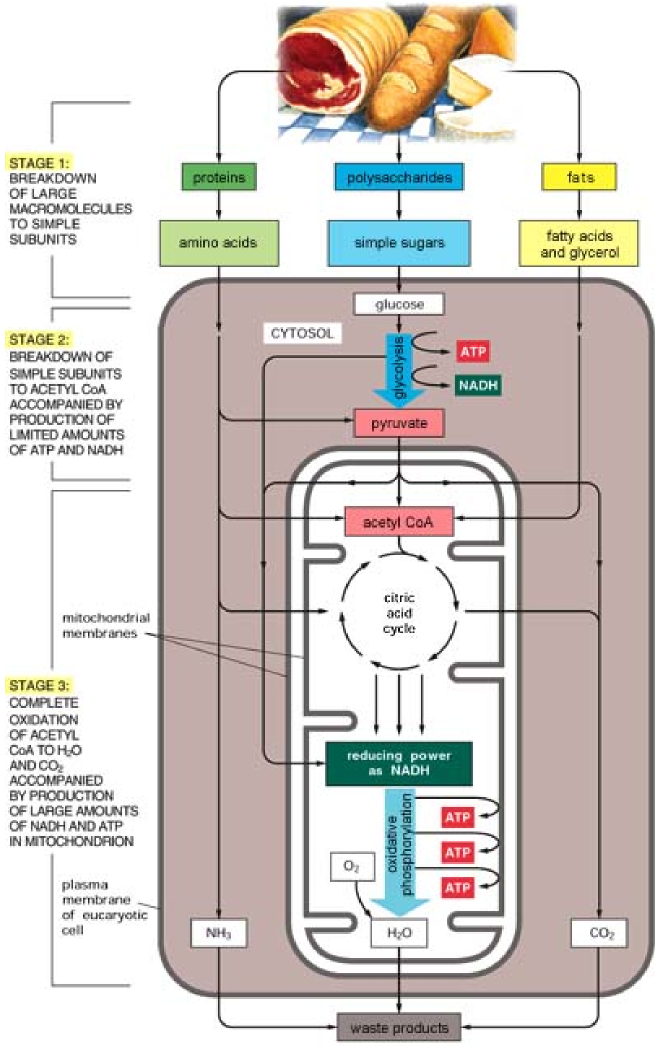
Darwin 1850- FB and natural selection
Vito Volterra 1890- FB and fish population balance
Adam Smith 1760- FB and international economy
James Watt 1780- FB and the steam engine
FB and cell homeostasis

The resources available to most species for their survival are meager and limited

Nature uses Optimal control

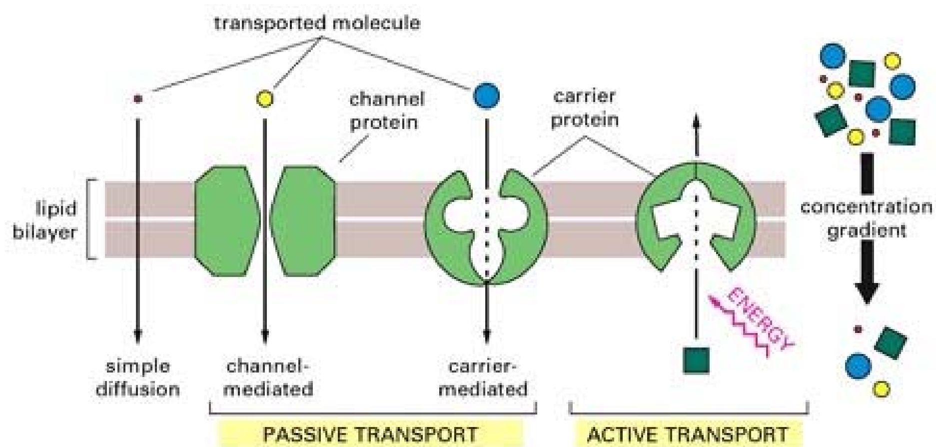
Optimality in Biological Systems

Cell Homeostasis



Cellular Metabolism

The individual cell is a complex feedback control system. It pumps ions across the cell membrane to maintain homeostatis, and has only limited energy to do so.



Permeability control of the cell membrane

http://www.accessexcellence.org/RC/VL/GG/index.html

R. Kalman 1960

Optimality in Control Systems Design

Rocket Orbit Injection

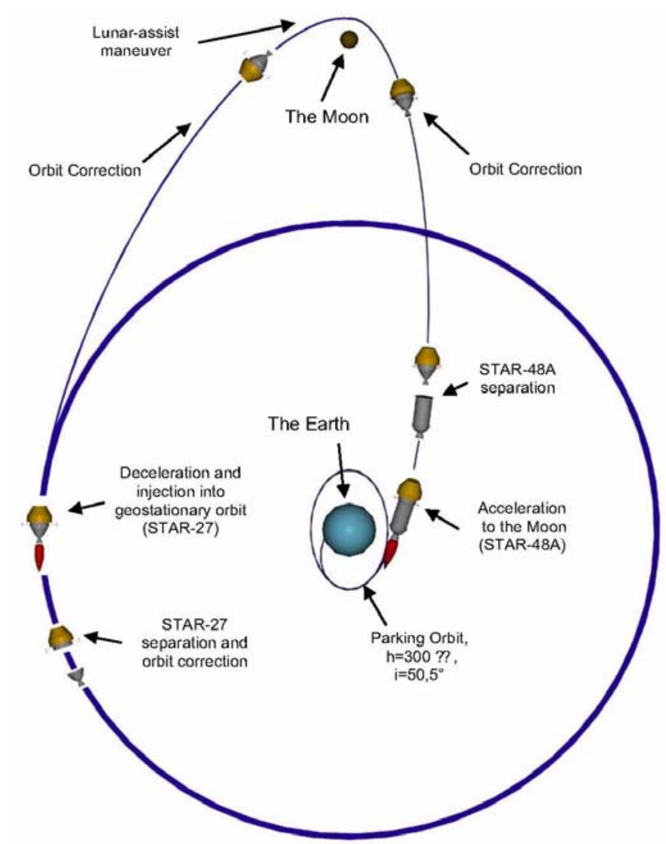


Fig. 1-1. Trajectory scheme

ISC Kosmotras Proprietary 9

http://microsat.sm.bmstu.ru/e-library/Launch/Dnepr_GEO.pdf

Dynamics

$$\dot{r} = w$$

$$\dot{w} = \frac{v^2}{r} - \frac{\mu}{r^2} + \frac{F}{m}\sin\phi$$

$$\dot{v} = \frac{-wv}{r} + \frac{F}{m}\cos\phi$$

$$\dot{m} = -Fm$$

Objectives

Get to orbit in minimum time

Use minimum fuel

Adaptive Control-
Online in real time
No dynamics knowledge needed

Adaptive control generally minimizes a squared (tracking error)

Inverse Optimal adaptive control

Minimizes a cost not of our choosing.

Indirect optimal adaptive control

identifies A and B and then solves the Riccati equation

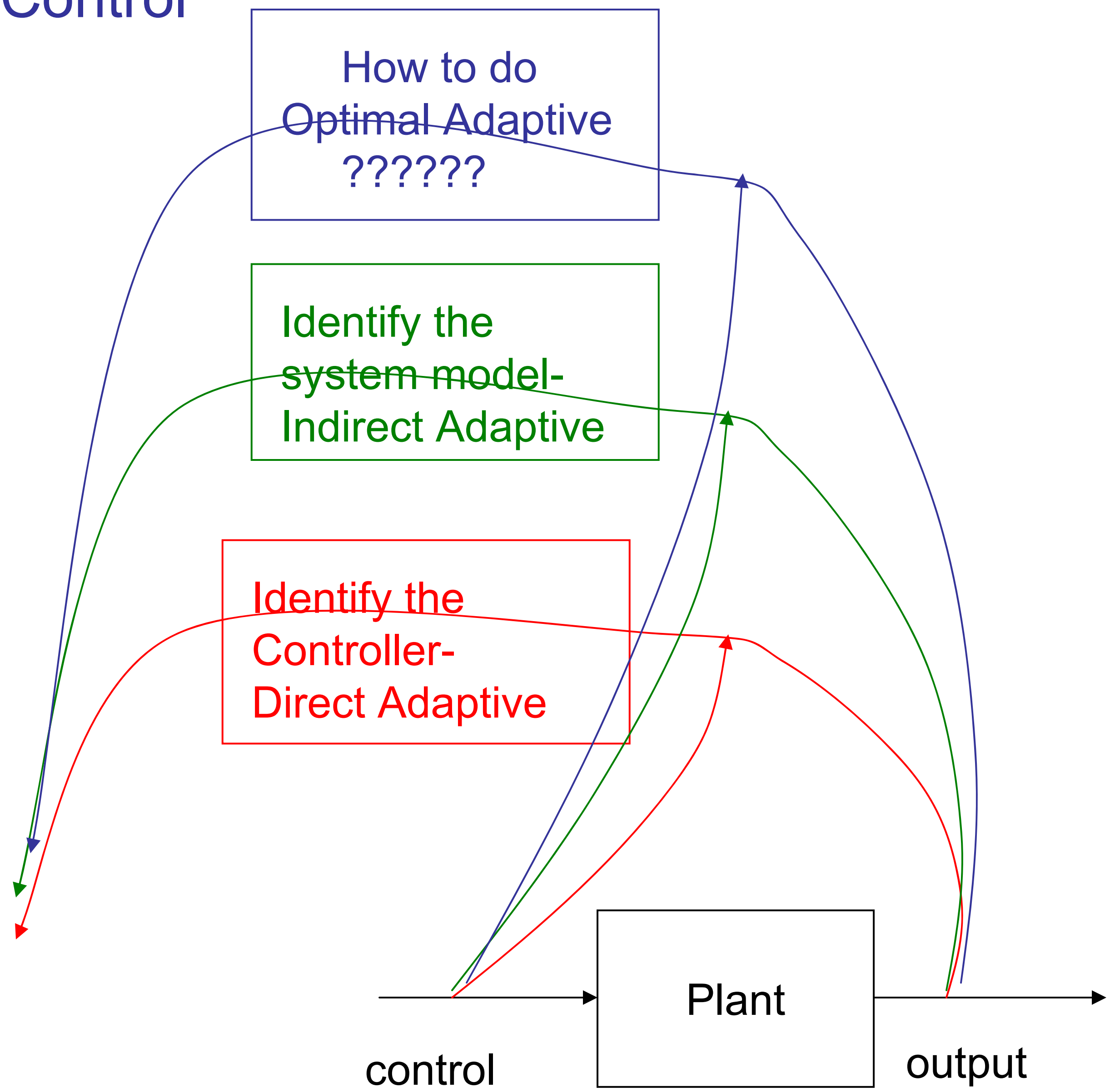
Adaptive Control is generally not Optimal

Optimal Control is off-line,
and needs to know the system dynamics to solve design eqs.
e.g. Riccati equation needs A, B

We want ONLINE DIRECT ADAPTIVE OPTIMAL Control
For any performance cost of our own choosing

Solve Riccati eq. online without knowing full dynamics

Adaptive Control



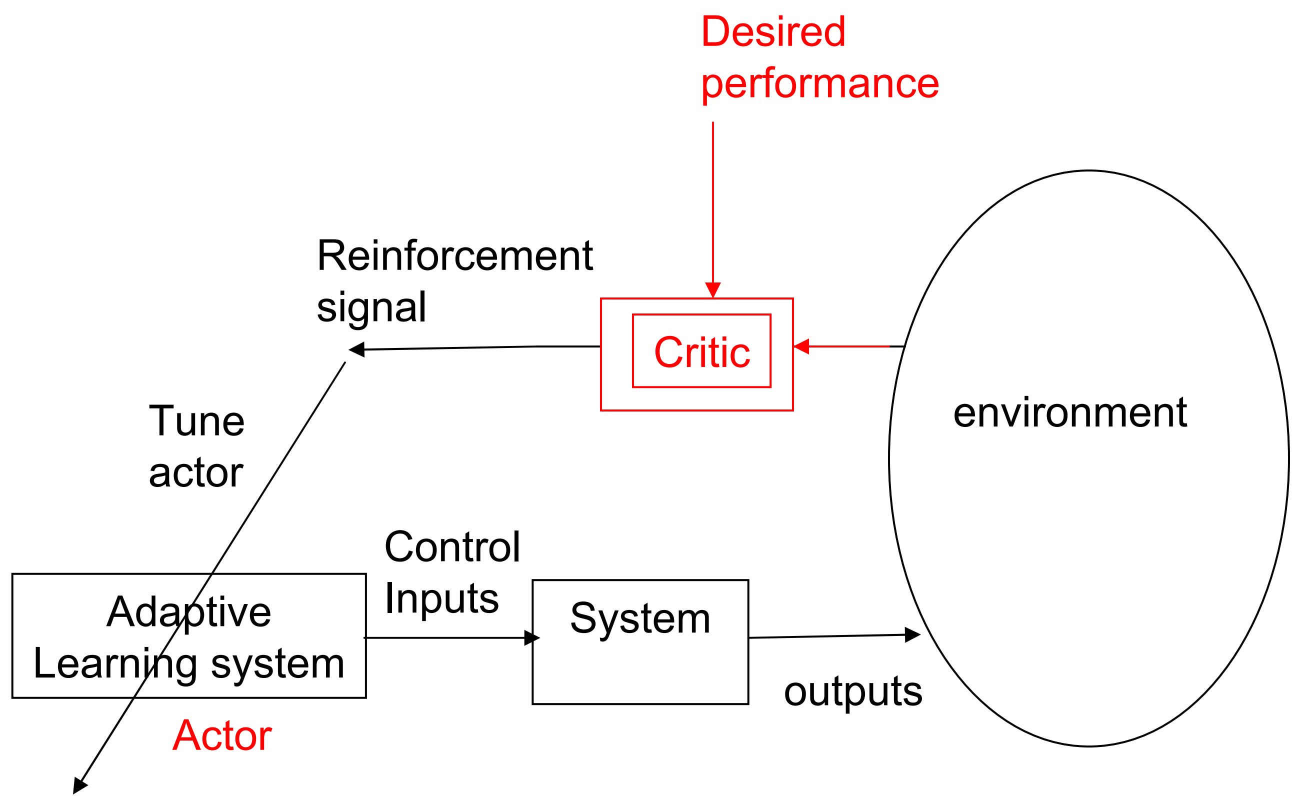
Reinforcement Learning turns out to be the key to this!

Different methods of learning

Reinforcement learning
Ivan Pavlov 1890s

We want OPTIMAL performance
- ADP- Approximate Dynamic Programming

Actor-Critic Learning



Discrete-Time Optimal Control

system $\quad x_{k+1} = f(x_k) + g(x_k)u_k$

Difference eq. equivalent

cost $\quad V_h(x_k) = \sum_{i=k}^{\infty} \gamma^{i-k} r(x_i, u_i)$

Example $\quad r(x_k, u_k) = x_k^T Q x_k + u_k^T R u_k$

$$V_h(x_k) = r(x_k, u_k) + \gamma \sum_{i=k+1}^{\infty} \gamma^{i-(k+1)} r(x_i, u_i)$$

Value function recursion $\quad V_h(x_k) = r(x_k, h(x_k)) + \gamma V_h(x_{k+1}) \quad , \quad V_h(0) = 0$

Control policy $\quad u_k = h(x_k)$ = the prescribed control input function

Example $\quad u_k = -Kx_k \quad$ Linear state variable feedback

Bellman eq. – nonlinear Lyapunov equation

Discrete-Time Optimal Control

cost $$V_h(x_k) = \sum_{i=k}^{\infty} \gamma^{i-k} r(x_i, u_i)$$

Value function recursion
Bellman eq.

$$V_h(x_k) = r(x_k, h(x_k)) + \gamma V_h(x_{k+1})$$

$u_k = h(x_k)$ = the prescribed control policy

Hamiltonian $$H(x_k, \nabla V(x_k), h) = r(x_k, h(x_k)) + \gamma V_h(x_{k+1}) - V_h(x_k)$$

Optimal cost $$V^*(x_k) = \min_h (r(x_k, h(x_k)) + \gamma V_h(x_{k+1}))$$

Bellman's Principle $$V^*(x_k) = \min_{u_k} (r(x_k, u_k) + \gamma V^*(x_{k+1}))$$

Backwards in time solution

Optimal Control

$$h^*(x_k) = \arg\min_{u_k} (r(x_k, u_k) + \gamma V^*(x_{k+1})) \quad = \frac{\partial}{\partial u_k}(r(x_k, u_k) + \gamma V^*(x_{k+1}))$$

System dynamics does not appear

The Solution: Hamilton-Jacobi-Bellman Equation

System $$x_{k+1} = f(x_k) + g(x_k)u_k$$

$$V(x_k) = \sum_{i=k}^{\infty} x_i^T Q x_i + u_i^T R u_i$$

DT HJB equation

$$V^*(x_k) = \min_{u_k}\left[x_k^T Q x_k + u_k^T R u_k + V^*(x_{k+1}) \right]$$
$$= \min_{u_k}\left[x_k^T Q x_k + u_k^T R u_k + V^*\left(f(x_k) + g(x_k)u_k \right) \right]$$

Difficult to solve
Contains the dynamics

Minimize wrt u_k

$$2Ru_k + g(x_k)^T \frac{dV^*(x_{k+1})}{dx_{k+1}} = 0$$

$$u^*(x_k) = -\frac{1}{2} R^{-1} g(x_k)^T \frac{dV^*(x_{k+1})}{dx_{k+1}}$$

DT Optimal Control – Linear Systems Quadratic cost (LQR)

system

$$x_{k+1} = Ax_k + Bu_k$$

cost $$V(x_k) = \sum_{i=k}^{\infty} x_i^T Q x_i + u_i^T R u_i$$

Fact. The cost is quadratic $V(x_k) = x_k^T P x_k$ for some symmetric matrix P

HJB = DT Riccati equation

$$0 = A^T PA - P + Q - A^T PB(R + B^T PB)^{-1} B^T PA$$

Optimal Control $u_k = -Lx_k$

$$L = (R + B^T PB)^{-1} B^T PA$$

Optimal Cost

$$V^*(x_k) = x_k^T P x_k$$

Off-line solution
Dynamics must be known

Discrete-Time Optimal Adaptive Control
How to do it ONLINE

cost $V_h(x_k) = \sum_{i=k}^{\infty} \gamma^{i-k} r(x_i, u_i)$

Value function recursion
Bellman eq.

$$\boxed{V_h(x_k) = r(x_k, h(x_k)) + \gamma V_h(x_{k+1})}$$

$u_k = h(x_k)$ = the prescribed control policy

Hamiltonian $H(x_k, \nabla V(x_k), h) = r(x_k, h(x_k)) + \gamma V_h(x_{k+1}) - V_h(x_k)$

Optimal cost $V^*(x_k) = \min_h (r(x_k, h(x_k)) + \gamma V_h(x_{k+1}))$

Bellman's Opt. Principle $V^*(x_k) = \min_{u_k} (r(x_k, u_k) + \gamma V^*(x_{k+1}))$

Optimal Control

$$\boxed{h^*(x_k) = \arg\min_{u_k} (r(x_k, u_k) + \gamma V^*(x_{k+1}))}$$

Focus on these two eqs

Discrete-Time Optimal Control

Solutions by Comp. Intelligence Community

Value function recursion
Bellman eq.

$$V_h(x_k) = r(x_k, h(x_k)) + \gamma V_h(x_{k+1}), \qquad V_h(0) = 0$$

$u_k = h(x_k)$ = the prescribed control policy

The Lyapunov Equation

Theorem: Let $V_h(x_k)$ solve the Lyapunov equation. Then

$$V_h(x_k) = \sum_{i=k}^{\infty} \gamma^{i-k} r(x_i, h(x_i))$$

Gives value for any prescribed control policy

Policy Evaluation for any given current policy

Policy must be stabilizing

Optimal Control $\quad h^*(x_k) = \arg\min_{u_k}(r(x_k, u_k) + \gamma V^*(x_{k+1}))$

Bellman's result

What about? -

$$h'(x_k) = \arg\min_{u_k}(r(x_k, u_k) + \gamma V_h(x_{k+1})) \quad = \quad \frac{\partial}{\partial u_k}(r(x_k, u_k) + \gamma V_h(x_{k+1}))$$

for a given policy *h(.)* ?

Theorem. Bertsekas.

Let $V_h(x_k)$ be the value of any given policy $h(x_k)$.

Then

$$V_{h'}(x_k) \le V_h(x_k)$$

Policy Improvement

One step improvement property of Rollout Algorithms

DT Policy Iteration

e.g. Control policy = SVFB

$$h(x_k) = -Lx_k$$

Cost for any given control policy $h(x_k)$ satisfies the recursion

$$V_h(x_k) = r(x_k, h(x_k)) + \gamma V_h(x_{k+1})$$

Lyapunov eq.

Recursive form
Consistency equation

Recursive solution

Pick stabilizing initial control

Policy Evaluation

$$V_{j+1}(x_k) = r(x_k, h_j(x_k)) + \gamma V_{j+1}(x_{k+1})$$

f(.) and g(.) do not appear

Policy Improvement

$$h_{j+1}(x_{k+1}) = \arg\min_{u_k}(r(x_k, u_k) + \gamma V_{j+1}(x_{k+1}))$$

Howard (1960) proved convergence for MDP

Temporal difference error

$$\boxed{e_k = -V_{j+1}(x_k) + r(x_k, h_j(x_k)) + \gamma V_{j+1}(x_{k+1})}$$

Forward in time method of computing optimal control

DT Policy Iteration – Linear Systems Quadratic Cost- LQR

$$x_{k+1} = Ax_k + Bu_k$$

For any stabilizing policy, the cost is

$$V(x_k) = \sum_{i=k}^{\infty} x_i^T Q x_i + u^T(x_i) R u(x_i)$$

LQR value is quadratic $V(x) = x^T P x$

DT Policy iterations

Solves Lyapunov eq. without knowing A and B

$$V_{j+1}(x_k) = x_k^T Q x_k + u_j^T(x_k) R u_j(x_k) + V_{j+1}(x_{k+1})$$

$$u_{j+1}(x_k) = -\frac{1}{2} R^{-1} g(x_k)^T \frac{dV_{j+1}(x_{k+1})}{dx_{k+1}}$$

Equivalent to an Underlying Problem- DT LQR:

$$(A - BL_j)^T P_{j+1} (A - BL_j) - P_{j+1} = -Q - L_j^T R L_j$$

DT Lyapunov eq.

$$L_{j+1} = (R + B^T P_{j+1} B)^{-1} B^T P_{j+1} A$$

$$u_{j+1}(x_k) = -L_{j+1} x_k$$

Hewer proved convergence in 1971

ADP Solves Riccati equation WITHOUT knowing internal System Dynamics f(x)

DT Policy Iteration

DT Policy iterations

$$V_{j+1}(x_k) = x_k^T Q x_k + u_j^T(x_k) R u_j(x_k) + V_{j+1}(x_{k+1})$$

$$u_{j+1}(x_{k+1}) = -\frac{1}{2} R^{-1} g(x_k)^T \frac{dV_{j+1}(x_{k+1})}{dx_{k+1}}$$

LQR case

$$(A - BL_j)^T P_{j+1}(A - BL_j) - P_{j+1} = -Q - L_j^T R L_j$$

$$L_{j+1} = (R + B^T P_{j+1} B)^{-1} B^T P_{j+1} A$$

$$u_{j+1}(x_k) = -L_{j+1} x_k$$

How to implement online?

DT Policy Iteration – How to implement online?
Linear Systems Quadratic Cost- LQR

$$x_{k+1} = Ax_k + Bu_k \qquad V(x_k) = \sum_{i=k}^{\infty} x_i^T Q x_i + u(x_i) R u(x_i)$$

LQR cost is quadratic $V(x) = x^T P x$ for some matrix P

DT Policy iterations

Solves Lyapunov eq. without knowing A and B

$$V_{j+1}(x_k) = x_k^T Q x_k + u_j^T(x_k) R u_j(x_k) + V_{j+1}(x_{k+1})$$

$$x_k^T P_{j+1} x_k - x_{k+1}^T P_{j+1} x_{k+1} = x_k^T Q x_k + u_j^T R u_j$$

$$\begin{bmatrix} x_k^1 & x_k^2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} - \begin{bmatrix} x_{k+1}^1 & x_{k+1}^2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{bmatrix}$$

$$= \begin{bmatrix} p_{11} & p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} (x_k^1)^2 \\ 2x_k^1 x_k^2 \\ (x_k^2)^2 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} (x_{k+1}^1)^2 \\ 2x_{k+1}^1 x_{k+1}^2 \\ (x_{k+1}^2)^2 \end{bmatrix}$$

Quadratic basis set

$$W_{j+1}^T \left[\varphi(x_k) - \varphi(x_{k+1}) \right] = x_k^T Q x_k + u_j^T(x_k) R u_j(x_k)$$

Then update control using

$$h_j(x_k) = L_j x_k = (R + B^T P_j B)^{-1} B^T P_j A x_k$$

Need to know A AND B for control update

Implementation- DT Policy Iteration
Nonlinear Case

Value Function Approximation (VFA)

$$V(x) = W^T \varphi(x)$$

weights

basis functions

LQR case- V(x) is quadratic

$$V(x) = x^T P x = W^T \varphi(x)$$

$$\varphi(x) = \left[x_1^2, \ldots, x_1 x_n, x_2^2, \ldots, x_2 x_n, \ldots, x_n^2\right]'.$$

Quadratic basis functions

$$W^T = [\, p_{11} \quad p_{12} \quad \cdots]$$

Nonlinear system case- use Neural Network

Implementation- DT Policy Iteration

Value function update for given control

$$V_{j+1}(x_k) = r(x_k, h_j(x_k)) + \gamma V_{j+1}(x_{k+1})$$

Assume measurements of x_k and x_{k+1} are available to compute u_{k+1}

VFA $$V_j(x_k) = W_j^T \varphi(x_k)$$

Then

regression matrix

$$W_{j+1}^T \left[\varphi(x_k) - \gamma\varphi(x_{k+1})\right] = r(x_k, h_j(x_k))$$

Since x_{k+1} is measured,
do not need knowledge of f(x)
or g(x) for value fn. update

Indirect Adaptive control with identification of the optimal value

Solve for weights using RLS
or, many trajectories with different initial conditions over a compact set

Then update control using

$$u_{j+1}(x_{k+1}) = -\frac{1}{2} R^{-1} g(x_k)^T \frac{dV_{j+1}(x_{k+1})}{dx_{k+1}} = -\frac{1}{2} R^{-1} g(x_k)^T \varphi^T(x_{k+1}) W_{j+1}^T$$

Need to know $g(x_k)$ for control update

1. Select control policy

Solves Lyapunov eq. without knowing dynamics

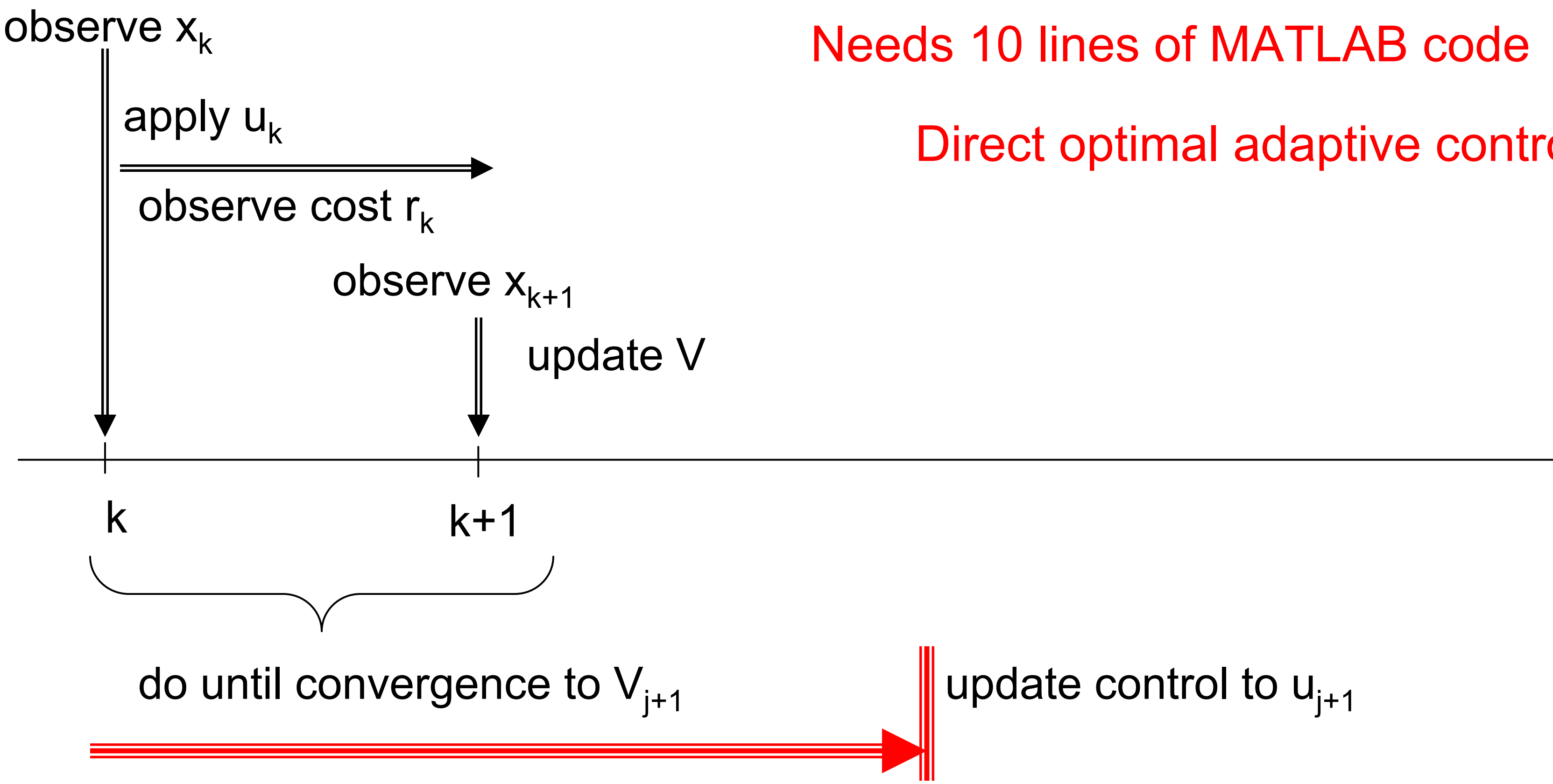
2. Find associated cost $V_{j+1}(x_k) = r(x_k, h_j(x_k)) + \gamma V_{j+1}(x_{k+1})$

$$W_{j+1}^T\left[\varphi(x_k) - \gamma\varphi(x_{k+1})\right] = r(x_k, h_j(x_k))$$

3. Improve control $u_{j+1}(x_{k+1}) = -\frac{1}{2}R^{-1}g(x_k)^T \frac{dV_j(x_{k+1})}{dx_{k+1}}$

Needs 10 lines of MATLAB code

Direct optimal adaptive control



Adaptive Critic

Use RLS until convergence

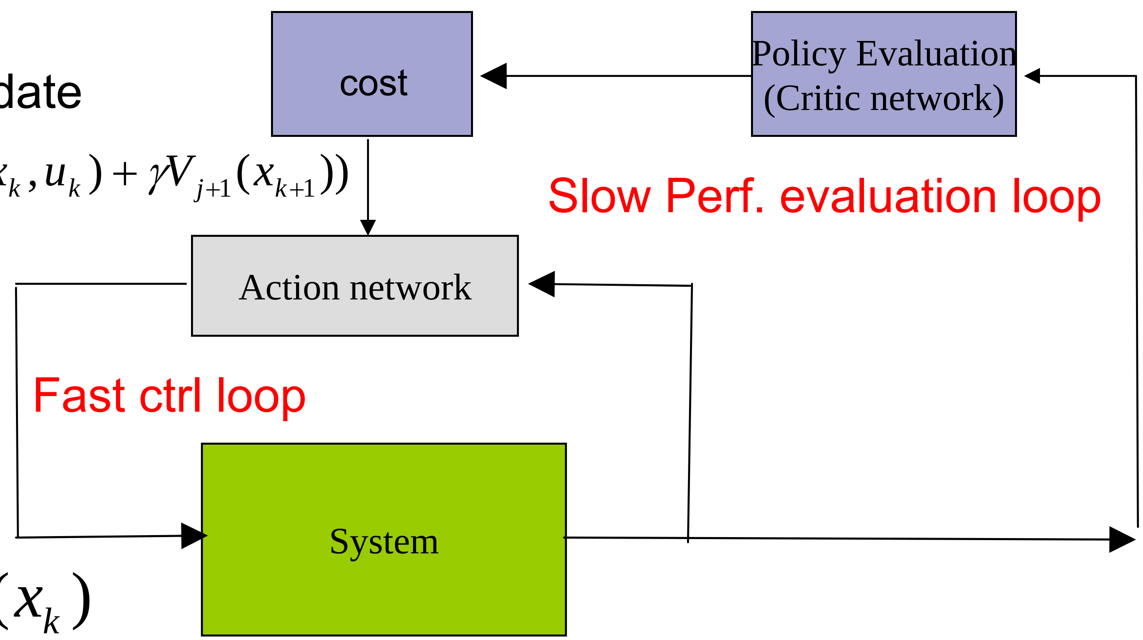
The Adaptive Critic Architecture

Value update

$$V_{j+1}(x_k) = r(x_k, h_j(x_k)) + \gamma V_{j+1}(x_{k+1})$$

Control policy update

$$h_{j+1}(x_k) = \arg\min_{u_k}(r(x_k, u_k) + \gamma V_{j+1}(x_{k+1}))$$



$h_j(x_k)$

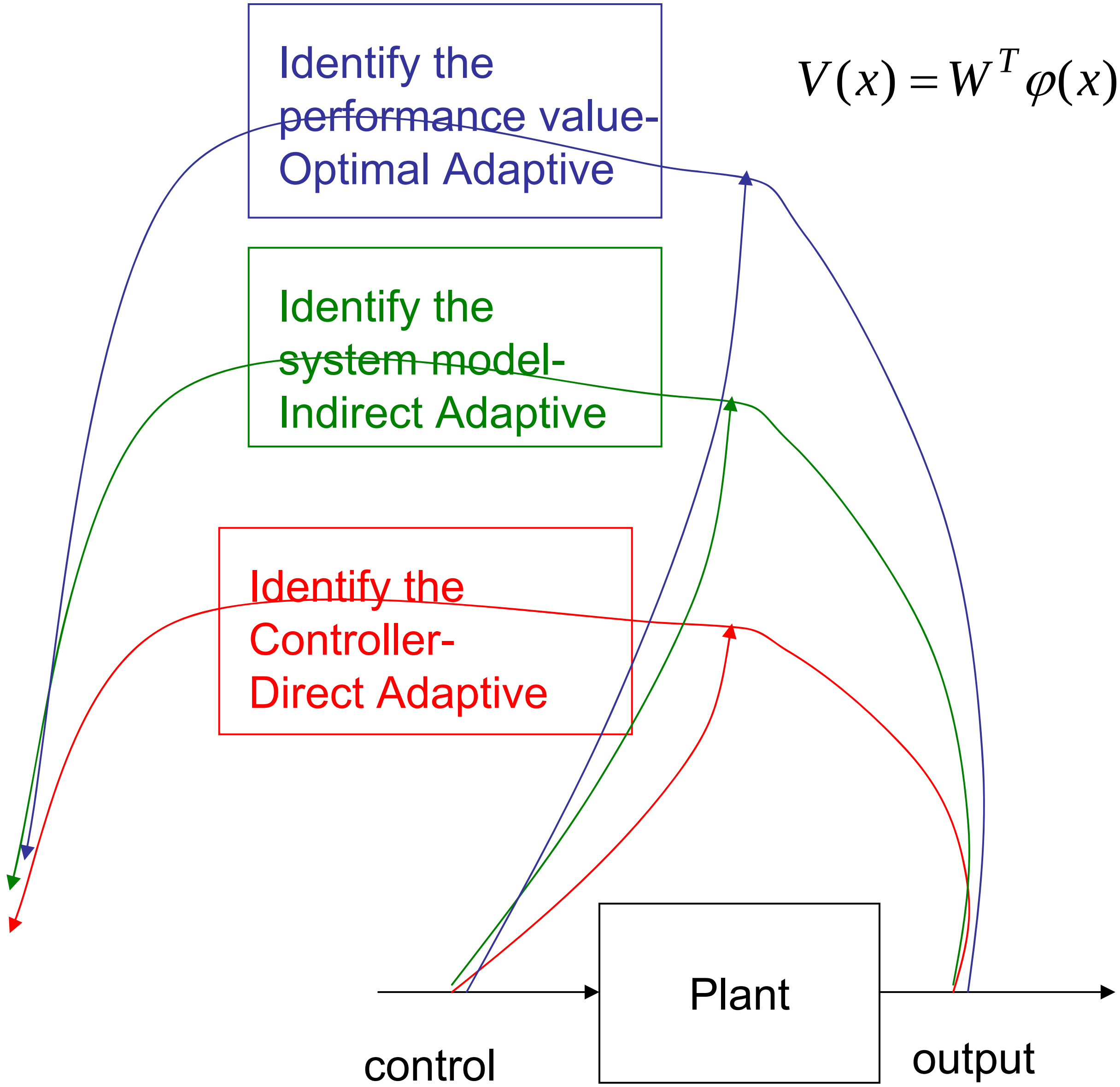
Control policy

Leads to ONLINE FORWARD-IN-TIME implementation of optimal control

Optimal Adaptive Control - A 2-time scale DT controller

Adaptive Control

$$V(x) = W^T \varphi(x)$$

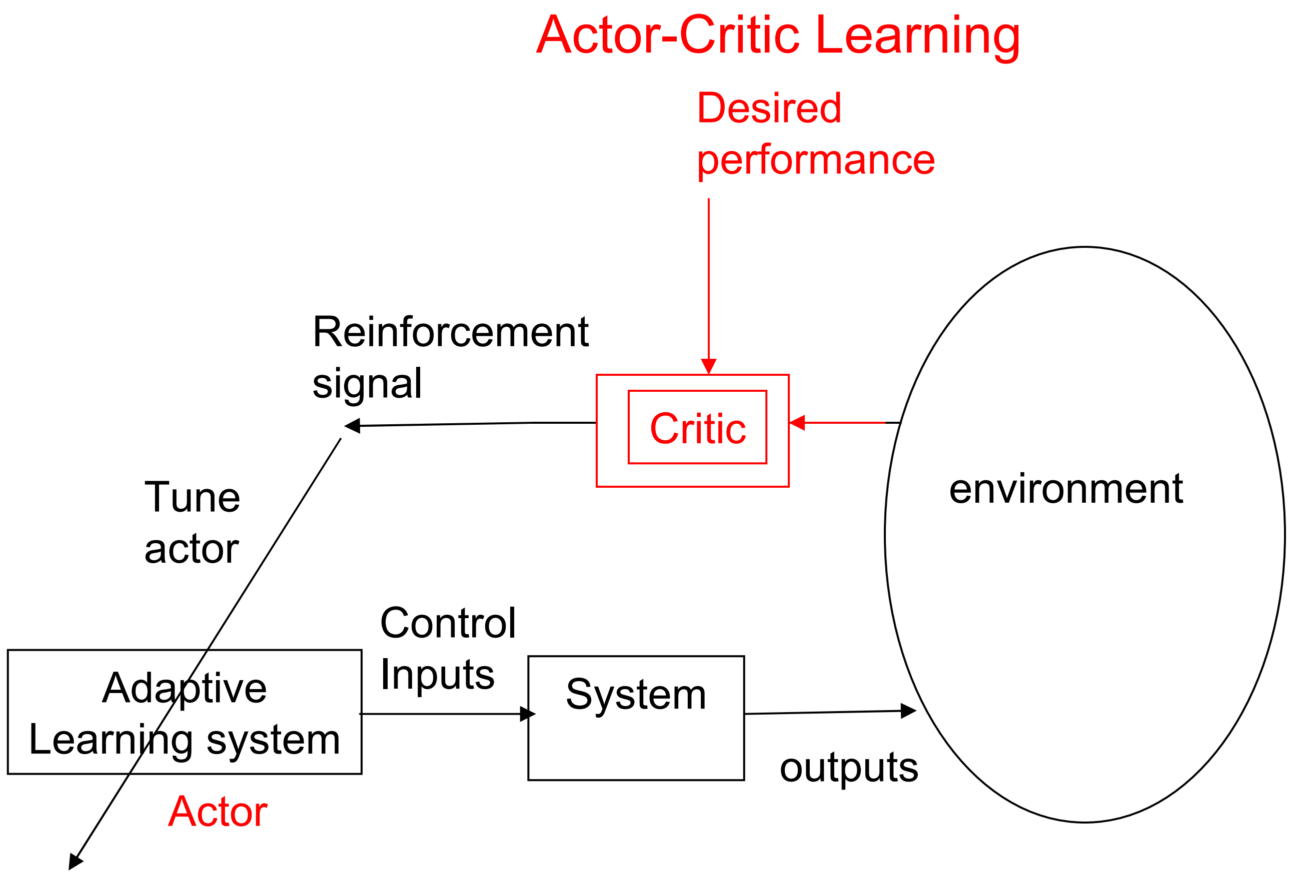


Reinforcement Learning!

Different methods of learning

Reinforcement learning
Ivan Pavlov 1890s

We want OPTIMAL performance
- ADP- Approximate Dynamic Programming



Greedy Value Fn. Update- Approximate Dynamic Programming
ADP Method 1 - Heuristic Dynamic Programming (HDP)

Paul Werbos

Policy Iteration

$$V_{j+1}(x_k) = r(x_k, h_j(x_k)) + \gamma V_{j+1}(x_{k+1})$$

$$h_{j+1}(x_k) = \arg\min_{u_k}(r(x_k, u_k) + \gamma V_{j+1}(x_{k+1}))$$

Lyapunov eq.

For LQR
Underlying RE

$$(A - BL_j)^T P_{j+1}(A - BL_j) - P_{j+1} = -Q - L_j^T R L_j$$

Hewer 1971

$$L_j = -(R + B^T P_j B)^{-1} B^T P_j A$$

Initial stabilizing control is needed

Value Iteration

Two occurrences of cost allows def. of greedy update

$$V_{j+1}(x_k) = r(x_k, h_j(x_k)) + \gamma V_j(x_{k+1})$$

$$h_{j+1}(x_k) = \arg\min_{u_k}(r(x_k, u_k) + \gamma V_{j+1}(x_{k+1}))$$

Simple recursion

For LQR
Underlying RE

$$P_{j+1} = (A - BL_j)^T P_j (A - BL_j) + Q + L_j^T R L_j$$

$$L_j = -(R + B^T P_j B)^{-1} B^T P_j A$$

Lancaster & Rodman proved convergence

Initial stabilizing control is NOT needed

Motivation for Value Iteration

Draguna Vrabie-
For CT systems

PI Policy Evaluation Step

$$V_{j+1}(x_k) = r(x_k, h(x_k)) + \gamma V_{j+1}(x_{k+1})$$

Needs stabilizing gain

$$(A - BL)^T P_{j+1}(A - BL) - P_{j+1} = -Q - L^T RL$$

LE= Lyapunov equation

VI Policy Evaluation Step

$$V_{i+1}(x_k) = r(x_k, h(x_k)) + \gamma V_i(x_{k+1})$$

Does not need stabilizing gain

$$P_{i+1} = (A - BL)^T P_i \ (A - BL) + Q + L^T RL$$

MR= Matrix recursion

Theorem

Let gain L be fixed and (A-BL) stable.
Let $P_0 \geq 0$ in MR

Then $P_i \to P_{j+1}$

Idea of GPI

i.e. repeated application of the VI policy evaluation step
is the same as one application of the PI policy evaluation step
IF THE CONTROL POLICY IS NOT UPDATED

Implementation- DT HDP – Value Iteration

Value function update for given control

$$V_{j+1}(x_k) = r(x_k, h_j(x_k)) + \gamma V_j(x_{k+1})$$

Since x_{k+1} is measured, do not need knowledge of f(x) or g(x) for value fn. update

Assume measurements of x_k and x_{k+1} are available to compute u_{k+1}

VFA $\quad V_j(x_k) = W_j^T \varphi(x_k)$

Then

regression matrix

Old weights

$$W_{j+1}^T \left[\varphi(x_k)\right] = r(x_k, h_j(x_k)) + \gamma W_j^T \left[\varphi(x_{k+1})\right]$$

Solve for weights using RLS

or, many trajectories with different initial conditions over a compact set

Then update control using

$$h_j(x_k) = L_j x_k = -(R + B^T P_j B)^{-1} B^T P_j A x_k$$

Need to know $f(x_k)$ AND $g(x_k)$ for control update

1. Select control policy

Solves Lyapunov recursion without knowing dynamics

2. Find associated cost $V_{j+1}(x_k) = r(x_k, h_j(x_k)) + \gamma V_j(x_{k+1})$

$$W_{j+1}^T \left[\varphi(x_k) \right] = r(x_k, h_j(x_k)) + \gamma W_j^T \left[\varphi(x_{k+1}) \right]$$

3. Improve control $u_{j+1}(x_k) = -\frac{1}{2} R^{-1} g(x_k)^T \frac{dV_j(x_{k+1})}{dx_{k+1}}$

observe x_k

apply u_k

observe cost r_k

observe x_{k+1}

update V

k

k+1

do until convergence to V_{j+1}

update control to u_{j+1}

Needs 10 lines of MATLAB code

Direct optimal adaptive control

Initial stabilizing control is NOT needed

DT HDP vs. Receding Horizon Optimal Control

Forward-in-time HDP

$$P_{i+1} = A^T P_i A + Q - A^T P_i B (R + B^T P_i B)^{-1} B^T P_i A$$

$$P_0 = 0$$

Backward-in-time optimization – RHC

$$P_k = A^T P_{k+1} A + Q - A^T P_{k+1} B (R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A$$

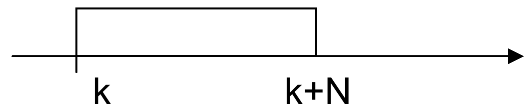
$P_N =$ Control Lyapunov Function overbounding P_∞

Adaptive Terminal Cost RHC

Hongwei Zhang
Dr. Jie Huang

Standard RHC

$$x_{k+1} = Ax_k + Bu_k$$



$$V(x_k) = \sum_{i=k}^{k+N-1} \left(x_i^T Q x_i + u_i^T R u_i \right) \quad + \quad x_{k+N}^T P_0 x_{k+N}$$

P_0 is the same for each stage

$$P_{i+1} = A^T P_i A + Q - A^T P_i B (R + B^T P_i B)^{-1} B^T P_i A , \quad P_0$$

$$u_{k+1}^{RH} = -(R + B^T P_{N-1} B)^{-1} B^T P_{N-1} A \, x_{k+1} = -L_N^{RH} \, x_{k+1}$$

Requires P_0 to be a CLF that overbounds the optimal inf. horizon cost, or large N

Our ATC RHC

Final cost from previous stage

$$V(x_k) = \sum_{i=k}^{k+N-1} \left(x_i^T Q x_i + u_i^T R u_i \right) \quad + \quad x_{k+N}^T P_{kN} x_{k+N}$$

$$P_{i+1} = A^T P_i A + Q - A^T P_i B (R + B^T P_i B)^{-1} B^T P_i A , \quad P_{kN}$$

HWZ Theorem- Let $N \geq 1$

under the usual suspect observability and controllability assumptions
ATC RHC guarantees ultimate uniform exponential stability
for ANY $P_0 > 0$.
Moreover, our solution converges to the optimal inf. horizon cost.

Adaptive Terminal Cost RHC

Let N=1. Then

$$V(x_k) = \sum_{i=k}^{k+N-1} \left(x_i^T Q x_i + u_i^T R u_i \right) \quad + \quad x_{k+N}^T P_{kN} x_{k+N}$$

$$u_{k+1}^{RH} = \arg\min_{\bar{u}} \{ \sum_{i=k}^{k+N-1} \left(x_i^T Q x_i + u_i^T R u_i \right) \quad + \quad x_{k+N}^T P_{kN} x_{k+N} \} = -L_N^{RH} x_{k+1}$$

becomes

$$V_{j+1}(x_k) = x_k^T Q x_k + u_k^T R u_k + V_j(x_{k+1})$$

$$u_{k+1}^{RH} = \arg\min_{u} \{ x_k^T Q x_k + u_k^T R u_k + V_j(x_{k+1}) \}$$

i.e. value iteration

So, for N=1, ATC RHC can be implemented using Value Iteration
without knowing the system A matrix

Continuous-Time Optimal Control

Draguna Vrabie

System $\dot{x} = f(x,u)$

Off-line solution
Dynamics must be known

Cost $V(x(t)) = \int_t^\infty r(x,u)\, dt = \int_t^\infty (Q(x) + u^T Ru)\, dt$

Hamiltonian

$$H(x, \frac{\partial V}{\partial x}, u) = \dot{V} + r(x,u) = \left(\frac{\partial V}{\partial x}\right)^T \dot{x} + r(x,u) = \left(\frac{\partial V}{\partial x}\right)^T f(x,u) + r(x,u)$$

c.f. DT Hamiltonian $H(x_k, \nabla V(x_k), h) = r(x_k, h(x_k)) + \gamma V_h(x_{k+1}) - V_h(x_k)$

Optimal cost

Bellman $$0 = \min_{u(t)} \left(r(x,u) + \left(\frac{\partial V^*}{\partial x}\right)^T \dot{x} \right) = \min_{u(t)} \left(r(x,u) + \left(\frac{\partial V^*}{\partial x}\right)^T f(x,u) \right)$$

Optimal control $$h^*(x(t)) = -\tfrac{1}{2} R^{-1} g^T(x) \frac{\partial V^*}{\partial x}$$

HJB equation $$0 = \left(\frac{dV^*}{dx}\right)^T f + Q(x) - \tfrac{1}{4}\left(\frac{dV^*}{dx}\right)^T g R^{-1} g^T \frac{dV^*}{dx} \quad , \quad V(0) = 0$$

Discrete-Time Systems

$$H(x_k, \nabla V(x_k), h) = r(x_k, h(x_k)) + \gamma V_h(x_{k+1}) - V_h(x_k)$$

- Directly leads to temporal difference techniques
- System dynamics does not occur
- Two occurrences of value allow greedy value iteration methods

Continuous-Time Systems

$$H(x, \frac{\partial V}{\partial x}, u) = \dot{V} + r(x,u) = \left(\frac{\partial V}{\partial x}\right)^T \dot{x} + r(x,u) = \left(\frac{\partial V}{\partial x}\right)^T f(x,u) + r(x,u)$$

Leads to off-line solutions if system dynamics is known
Hard to do on-line learning

- How to define temporal difference?
- System dynamics DOES occur
- Only ONE occurrence of value gradient

How can one do Policy Iteration for Unknown Continuous-Time Systems?
What is Value Iteration for Continuous-Time systems?
How can one do ADP for CT Systems?

Asma Al-Tamimi

Discrete-time Nonlinear Heuristic Dynamic Programming:

System dynamics

$$x_{k+1} = f(x_k) + g(x_k)u(x_k)$$

$$V(x_k) = \sum_{i=k}^{\infty} x_i^T Q x_i + u_i^T R u_i$$

Value function recursion

$$V(x_k) = x_k^T Q x_k + u_k^T R u_k + \sum_{i=k+1}^{\infty} x_i^T Q x_i + u_i^T R u_i$$

$$= x_k^T Q x_k + u_k^T R u_k + V(x_{k+1})$$

HDP (Value Iteration)

$$u_i(x_{k+1}) = \arg\min_u (x_k^T Q x_k + u^T R u + V_i(x_{k+1}))$$

$$V_{i+1}(x_k) = x_k^T Q x_k + u^T R u + V_i(x_{k+1})$$

Asma Al-Tamimi

Proof of convergence of DT nonlinear HDP

Lemma 1 Let μ_i be any arbitrary sequence of control policies, and u_i is the policies as in (10). Let V_i be as in (11) and Λ_i as

$$\Lambda_{i+1}(x_k) = x_k Q x_k + \mu_i^T R \mu_i + \Lambda_i(x_{k+1}) . \qquad (12)$$

If $V_0 = \Lambda_0 = 0$, then $V_i \le \Lambda_i \quad \forall i$.

Lemma 2 Let the sequence $\{V_i\}$ be defined as in (11). If the system is controllable, then there is an upper bound Y such that $0 \le V_i \le Y \quad \forall i$.

Theorem 1 Define the sequence $\{V_i\}$ as in (11), with $V_0 = 0$. Then $\{V_i\}$ is a nondecreasing sequence in which $V_{i+1}(x_k) \ge V_i(x_k)$ $\forall i$, and converge to the value function of the DT HJB, *i.e.* $V_i \Rightarrow V^*$ as $i \Rightarrow \infty$.

Flavor of proofs

Proof: Let $V_0 = \Phi_0 = 0$ where V_i is updated as in (11) and, and Φ_i is updated as

$$\Phi_{i+1}(x_k) = (x_k Q x_k + u_{i+1}^T R u_{i+1} + \Phi_i(x_{k+1})) \qquad (11)$$

with the policies u_i as in (10). We will first prove by induction that $\Phi_i(x_k) \le V_{i+1}(x_k)$. Note that

$$V_1(x_k) - \Phi_0(x_k) = x_k^T Q x_k \ge 0$$
$$V_1(x_k) \ge \Phi_0(x_k)$$

Assume that $V_i(x_k) \ge \Phi_{i-1}(x_k) \quad \forall x_k$. Since

$$\Phi_i(x_k) = x_k Q x_k + u_i^T R u_i + \Phi_{i-1}(x_{k+1})$$
$$V_{i+1}(x_k) = x_k Q x_k + u_i^T R u_i + V_i(x_{k+1}) ,$$

then

$$V_{i+1}(x_k) - \Phi_i(x_k) = V_i(x_{k+1}) - \Phi_{i-1}(x_{k+1}) \ge 0 ,$$

and therefore

$$\Phi_i(x_k) \le V_{i+1}(x_k) . \qquad (12)$$

From Lemma 1 $V_i(x_k) \le \Phi_i(x_k)$ and therfore

$$V_i(x_k) \le \Phi_i(x_k) \le V_{i+1}(x_k)$$
$$V_i(x_k) \le V_{i+1}(x_k)$$

hence proving that $\{V_i\}$ is a nondecreasing sequence bounded from above as shown in Lemma 2. Hence $V_i \to V^*$ as $i \to \infty$. ■

Use Neural Network VFA for On-Line Implementation

NN for Value - Critic

$$\hat{V}_i(x_k, W_{Vi}) = W_{Vi}^T \phi(x_k)$$

NN for control action

$$\hat{u}_i(x_k, W_{ui}) = W_{ui}^T \sigma(x_k)$$

(can use 2-layer NN)

HDP

$$V_{i+1}(x_k) = x_k^T Q x_k + u^T R u + V_i(x_{k+1})$$

$$u_i(x_k) = \arg\min_u (x_k^T Q x_k + u^T R u + V_i(x_{k+1}))$$

Define target cost function

$$d(\phi(x_k), W_{Vi}^T) = x_k^T Q x_k + \hat{u}_i^T(x_k) R \hat{u}_i(x_k) + \hat{V}_i(x_{k+1})$$
$$= x_k^T Q x_k + \hat{u}_i^T(x_k) R \hat{u}_i(x_k) + W_{Vi}^T \phi(x_{k+1})$$

Explicit equation for cost – use LS for Critic NN update

$$W_{Vi+1} = \arg\min_{W_{Vi+1}} \{ \int_\Omega | W_{Vi+1}^T \phi(x_k) - d(\phi(x_k), W_{Vi}^T) |^2 \, dx_k \} \Longrightarrow W_{Vi+1} = \left(\int_\Omega \phi(x_k) \phi(x_k)^T dx \right)^{-1} \int_\Omega \phi(x_k) d^T(\phi(x_k), W_{Vi}^T, W_{ui}^T) dx$$

or $$W_{Vi+1}\big|_{m+1} = W_{Vi+1}\big|_m + \beta \phi^T(x_k) \left(-W_{Vi+1}^T\big|_m \phi(x_k) + r(x_k, u_k) + W_{Vi}^T \phi(x_{k+1}) \right)$$

Implicit equation for DT control- use gradient descent for action update

$$W_{ui} = \arg\min_\alpha \left(\begin{array}{l} x_k^T Q x_k + \hat{u}^T(x_k, \alpha) R \hat{u}(x_k, \alpha) + \\ \hat{V}_i(f(x_k) + g(x_k)\hat{u}(x_k, \alpha)) \end{array} \right) \Bigg|_\Omega \Longrightarrow W_{ui(j+1)} = W_{ui(j)} - \alpha \frac{\partial (x_k^T Q x_k + \hat{u}_{i(j)}^T R \hat{u}_{i(j)} + \hat{V}_i(x_{k+1}))}{\partial W_{ui(j)}}$$

$$W_{ui}^{j+1} = W_{ui}^j - \alpha \sigma(x_k)(2R\hat{u}_{i(j)} + g(x_k)^T \frac{\partial \phi^T(x_{k+1})}{\partial x_{k+1}} W_{Vi})^T$$

f(.) is not needed anywhere

Backpropagation- P. Werbos

Adaptive Critic w/ TWO Neural Networks

Use RLS until convergence

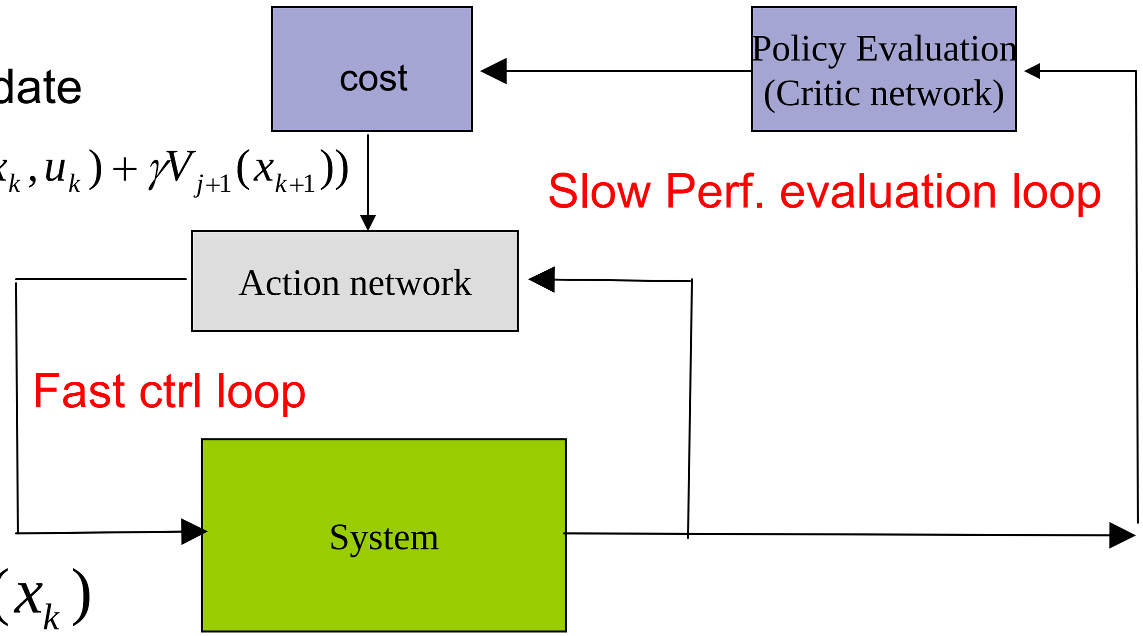
The Adaptive Critic Architecture

Value update

$$V_{j+1}(x_k) = r(x_k, h_j(x_k)) + \gamma V_j(x_{k+1})$$

Control policy update

$$h_{j+1}(x_k) = \arg\min_{u_k}(r(x_k, u_k) + \gamma V_{j+1}(x_{k+1}))$$



$h_j(x_k)$

Control policy

Leads to ONLINE FORWARD-IN-TIME implementation of optimal control

Optimal Adaptive Control -

A 2-time scale DT controller

Interesting Fact for HDP for Nonlinear systems

Linear Case $h_j(x_k) = L_j x_k = -(I + B^T P_j B)^{-1} B^T P_j A x_k$

must know system A and B matrices

NN for control action

Information about A is stored in NN

$$\hat{u}_i(x_k, W_{ui}) = W_{ui}^T \sigma(x_k)$$

Implicit equation for DT control- use gradient descent for action update

$$W_{ui} = \arg\min_{\alpha} \left(\begin{array}{l} x_k^T Q x_k + \hat{u}^T(x_k,\alpha) R \hat{u}(x_k,\alpha) + \\ \hat{V}_i(f(x_k) + g(x_k)\hat{u}(x_k,\alpha)) \end{array} \right) \Bigg|_{\Omega} \Longrightarrow$$

$$W_{ui(j+1)} = W_{ui(j)} - \alpha \frac{\partial (x_k^T Q x_k + \hat{u}_{i(j)}^T R \hat{u}_{i(j)} + \hat{V}_i(x_{k+1}))}{\partial W_{ui(j)}}$$

$$W_{ui}^{j+1} = W_{ui}^{j} - \alpha \sigma(x_k)(2R\hat{u}_{i(j)} + g(x_k)^T \frac{\partial \phi(x_{k+1})}{\partial x_{k+1}} W_{Vi})^T$$

g(.) is needed

Note that state internal dynamics $f(x_k)$ is NOT needed since:

1. NN Approximation for action is used
2. x_{k+1} is measured

Nonlinear Value Iteration

- Simulation Example
- The linear system – Aircraft longitudinal dynamics

$$A=\begin{bmatrix} 1.0722 & 0.0954 & 0 & -0.0541 & -0.0153 \\ 4.1534 & 1.1175 & 0 & -0.8000 & -0.1010 \\ 0.1359 & 0.0071 & 1.0 & 0.0039 & 0.0097 \\ 0 & 0 & 0 & 0.1353 & 0 \\ 0 & 0 & 0 & 0 & 0.1353 \end{bmatrix} \qquad B=\begin{bmatrix} -0.0453 & -0.0175 \\ -1.0042 & -0.1131 \\ 0.0075 & 0.0134 \\ 0.8647 & 0 \\ 0 & 0.8647 \end{bmatrix}$$

Unstable, Two-input system

- The HJB, i.e. ARE, Solution

$$P=\begin{bmatrix} 55.8348 & 7.6670 & 16.0470 & -4.6754 & -0.7265 \\ 7.6670 & 2.3168 & 1.4987 & -0.8309 & -0.1215 \\ 16.0470 & 1.4987 & 25.3586 & -0.6709 & 0.0464 \\ -4.6754 & -0.8309 & -0.6709 & 1.5394 & 0.0782 \\ -0.7265 & -0.1215 & 0.0464 & 0.0782 & 1.0240 \end{bmatrix} \qquad L=\begin{bmatrix} -4.1136 & -0.7170 & -0.3847 & 0.5277 & 0.0707 \\ -0.6315 & -0.1003 & 0.1236 & 0.0653 & 0.0798 \end{bmatrix}$$

Nonlinear Value Iteration

- Simulation
- The Cost function approximation

$$\hat{V}_{i+1}(x_k, W_{Vi+1}) = W_{Vi+1}^T \phi(x_k)$$

$$\phi^T(x) = \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 & x_1x_4 & x_1x_5 & x_2^2 & x_2x_3 & x_4x_2 & x_2x_5 & x_3^2 & x_3x_4 & x_3x_5 & x_4^2 & x_4x_5 & x_5^2 & \end{bmatrix}$$

$$W_V^T = \begin{bmatrix} w_{V1} & w_{V2} & w_{V3} & w_{V4} & w_{V5} & w_{V6} & w_{V7} & w_{V8} & w_{V9} & w_{V10} & w_{V11} & w_{V12} & w_{V13} & w_{V14} & w_{V15} \end{bmatrix}$$

- The Policy approximation

$$\hat{u}_i = W_{ui}^T \sigma(x_k)$$

$$\sigma^T(x) = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}$$

$$W_u^T = \begin{bmatrix} w_{u11} & w_{u12} & w_{u13} & w_{u14} & w_{u15} \\ w_{u21} & w_{u22} & w_{u23} & w_{u24} & w_{u25} \end{bmatrix}$$

Nonlinear Value Iteration
Critic NN converges to Optimal value

- Simulation

The convergence of the cost

$$W_V^T = [55.5411 \quad 15.2789 \quad 31.3032 \quad -9.3255 \quad -1.4536 \quad 2.3142 \quad 2.9234 \quad -1.6594 \quad -0.2430$$

$$24.8262 \quad -1.3076 \quad 0.0920 \quad 1.5388 \quad 0.1564 \quad 1.0240]$$

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} \\ P_{21} & P_{22} & P_{23} & P_{24} & P_{25} \\ P_{31} & P_{32} & P_{33} & P_{34} & P_{35} \\ P_{41} & P_{42} & P_{43} & P_{44} & P_{45} \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} \end{bmatrix} = \begin{bmatrix} w_{V1} & 0.5w_{V2} & 0.5w_{V3} & 0.5w_{V4} & 0.5w_{V5} \\ 0.5w_{V2} & w_{V6} & 0.5w_{V7} & 0.5w_{V8} & 0.5w_{V9} \\ 0.5w_{V3} & 0.5w_{V7} & w_{V10} & 0.5w_{V11} & 0.5w_{V12} \\ 0.5w_{V4} & 0.5w_{V8} & 0.5w_{V11} & w_{V13} & 0.5w_{V14} \\ 0.5w_{V5} & 0.5w_{V9} & 0.5w_{V12} & 0.5w_{V14} & w_{V15} \end{bmatrix}$$

Actual ARE soln:

$$P = \begin{bmatrix} 55.8348 & 7.6670 & 16.0470 & -4.6754 & -0.7265 \\ 7.6670 & 2.3168 & 1.4987 & -0.8309 & -0.1215 \\ 16.0470 & 1.4987 & 25.3586 & -0.6709 & 0.0464 \\ -4.6754 & -0.8309 & -0.6709 & 1.5394 & 0.0782 \\ -0.7265 & -0.1215 & 0.0464 & 0.0782 & 1.0240 \end{bmatrix}$$

Nonlinear Value Iteration
Action NN converges to Optimal Feedback

- Simulation

  The convergence of the control policy

$$W_u = \begin{bmatrix} 4.1068 & 0.7164 & 0.3756 & -0.5274 & -0.0707 \\ 0.6330 & 0.1005 & -0.1216 & -0.0653 & -0.0798 \end{bmatrix}$$

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} \end{bmatrix} = -\begin{bmatrix} w_{u11} & w_{u12} & w_{u13} & w_{u14} & w_{u15} \\ w_{u21} & w_{u22} & w_{u23} & w_{u24} & w_{u25} \end{bmatrix}$$

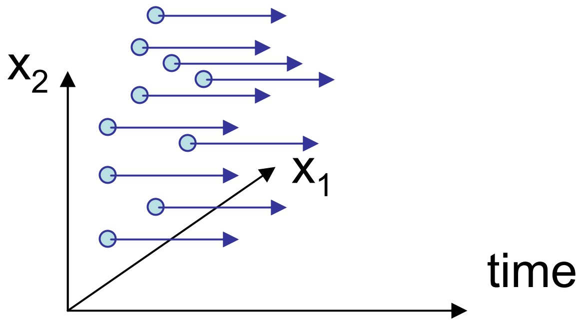
Actual optimal ctrl. $L = \begin{bmatrix} -4.1136 & -0.7170 & -0.3847 & 0.5277 & 0.0707 \\ -0.6315 & -0.1003 & 0.1236 & 0.0653 & 0.0798 \end{bmatrix}$

Note- In this example, internal dynamics matrix A is NOT Needed.

Issues with Nonlinear ADP

LS solution for Critic NN update

$$W_{Vi+1} = \left(\int_{\Omega} \phi(x_k)\phi(x_k)^T dx \right)^{-1} \int_{\Omega} \phi(x_k) d^T(\phi(x_k), W_{Vi}^T, W_{ui}^T) dx$$

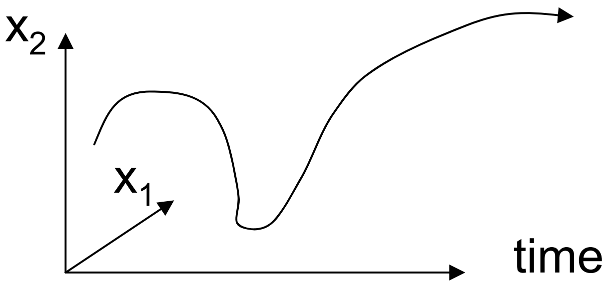


Integral over a region of state-space
Approximate using a set of points

Batch LS

Selection of NN Training Set

$$W_{Vi+1}\big|_{m+1} = W_{Vi+1}\big|_{m} + \beta\phi^T(x_k)\left(-W_{Vi+1}^T\big|_m \phi(x_k) + r(x_k,u_k) + W_{Vi}^T\phi(x_{k+1})\right)$$



Take sample points along a single trajectory

Recursive Least-Squares RLS

Set of points over a region vs. points along a trajectory

For Linear systems- these are the same under PE condition

Exploitation (optimal regulation) vs Exploration

Four ADP Methods proposed by Werbos

Critic NN to approximate:

Heuristic dynamic programming

Value iteration

Value $V(x_k)$

AD Heuristic dynamic programming

(Watkins Q Learning)

Q function $Q(x_k, u_k)$

Dual heuristic programming

Gradient $\frac{\partial V}{\partial x}$

AD Dual heuristic programming

Gradients $\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial u}$

Action NN to approximate the Control

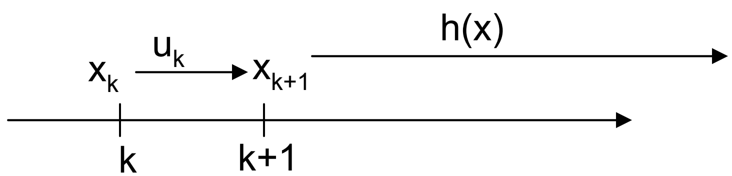
Bertsekas- Neurodynamic Programming

Barto & Bradtke- Q-learning proof (Imposed a settling time)

Q Learning - Action Dependent ADP

Value function recursion for given policy $h(x_k)$

$$V_h(x_k) = r(x_k, h(x_k)) + \gamma V_h(x_{k+1})$$



Define Q function

$$Q_h(x_k, \underline{u_k}) = r(x_k, \underline{u_k}) + \gamma V_h(x_{k+1})$$

u_k arbitrary

policy *h(.)* used after time k

Note $\quad Q_h(x_k, h(x_k)) = V_h(x_k)$

Optimal Q function $\quad Q^*(x_k, u_k) = r(x_k, u_k) + \gamma V^*(x_{k+1}))$

1. Simple expression of Bellman's Opt. principle

$$V^*(x_k) = \min_{u_k}(r(x_k, u_k) + \gamma V^*(x_{k+1})) \qquad V^*(x_k) = \min_{u_k}(Q^*(x_k, u_k))$$

$$h^*(x_k) = \arg\min_{u_k}(Q^*(x_k, u_k)) \quad = \quad \frac{\partial}{\partial u_k} Q^*(x_k, u_k)$$

Q Learning does not need to know $f(x_k)$ or $g(x_k)$

For LQR $$V(x) = W^T \varphi(x) = x^T P x$$ V is quadratic in x

---

Q is quadratic in x and u

$$Q_h(x_k, u_k) = r(x_k, u_k) + V_h(x_{k+1})$$

$$= x_k^T Q x_k + u_k^T R u_k + (Ax_k + Bu_k)^T P (Ax_k + Bu_k)$$

$$= \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} Q + A^T PA & A^T PB \\ B^T PA & R + B^T PB \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \equiv \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T H \begin{bmatrix} x_k \\ u_k \end{bmatrix} = \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

Control update is found by $$0 = \frac{\partial Q}{\partial u_k} = 2[B^T PAx_k + (R + B^T PB)u_k] = 2[H_{ux} x_k + H_{uu} u_k]$$

so $$u_k = -(R + B^T PB)^{-1} B^T PAx_k = -H_{uu}^{-1} H_{ux} x_k = L_{j+1} x_k$$

2. Control found only from Q function
A and B not needed

How to find Q function online?

$$Q(x_k,u_k)=\begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} Q+A^T PA & A^T PB \\ B^T PA & R+B^T PB \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

$$Q(x_k,u_k)=\begin{bmatrix} x_k \\ u_k \end{bmatrix}^T H \begin{bmatrix} x_k \\ u_k \end{bmatrix} = \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

Q Learning - Action Dependent ADP

Identify H matrix from input/state data

Q Learning does not need to know $f(x_k)$ or $g(x_k)$

Optimal Adaptive Control - for unknown DT systems

Q Function Recursion

Define Q function

$$Q_h(x_k, \underline{u_k}) = r(x_k, \underline{u_k}) + \gamma V_h(x_{k+1})$$

u_k arbitrary

policy *h(.)* used after time k

Note $\quad Q_h(x_k, h(x_k)) = V_h(x_k)$

Bellman's eq
Recursion for V

$$V_h(x_k) = r(x_k, h(x_k)) + \gamma V_h(x_{k+1})$$

Recursion for Q

$$Q_h(x_k, u_k) = r(x_k, u_k) + \gamma Q_h(x_{k+1}, h(x_{k+1}))$$

Define temporal difference error

$$e_k = -Q_h(x_k, u_k) + r(x_k, u_k) + \gamma Q_h(x_{k+1}, h(x_{k+1}))$$

Q Function Definition

Specify a control policy $u_j = h(x_j); \quad j = k, k+1, \ldots$

Define Q function

$$Q_h(x_k, \underline{u_k}) = r(x_k, \underline{u_k}) + \gamma V_h(x_{k+1})$$

$\left\{ \begin{array}{l} u_k \text{ arbitrary} \\ \text{policy } h(.) \text{ used after time k} \end{array} \right.$

Note $Q_h(x_k, h(x_k)) = V_h(x_k)$

Bellman's eq $V_h(x_k) = r(x_k, h(x_k)) + \gamma V_h(x_{k+1})$

Recursion for Q $Q_h(x_k, u_k) = r(x_k, u_k) + \gamma Q_h(x_{k+1}, h(x_{k+1}))$

Optimal Q function $Q^*(x_k, u_k) = r(x_k, u_k) + \gamma V^*(x_{k+1}))$

$$Q^*(x_k, u_k) = r(x_k, u_k) + \gamma Q^*(x_{k+1}, h^*(x_{k+1}))$$

Optimal control solution

$$V^*(x_k) = Q^*(x_k, h^*(x_k)) = \min_h (Q_h(x_k, h(x_k))) \qquad h^*(x_k) = \arg\min_h (Q_h(x_k, h(x_k)))$$

Simple expression of Bellman's principle

$$V^*(x_k) = \min_{u_k} (Q^*(x_k, u_k)) \qquad h^*(x_k) = \arg\min_{u_k} (Q^*(x_k, u_k))$$

Q Function ADP – Action Dependent ADP

Q function for any given control policy $h(x_k)$ satisfies the recursion

$$Q_h(x_k, u_k) = r(x_k, u_k) + \gamma Q_h(x_{k+1}, h(x_{k+1}))$$

Recursive solution – POLICY ITERATION with Q function

Pick stabilizing initial control policy

Find Q function

$$Q_{j+1}(x_k, u_k) = r(x_k, u_k) + \gamma Q_{j+1}(x_{k+1}, h_j(x_{k+1}))$$

Update control

$$h_{j+1}(x_k) = \arg\min_{u_k}(Q_{j+1}(x_k, u_k)) \quad = \quad \frac{\partial}{\partial u_k} Q_{j+1}(x_k, u_k)$$

Now $f(x_k,u_k)$ not needed

Bradtke & Barto (1994) proved convergence for LQR

Implementation- DT Q Function Policy Iteration

For LQR

Q function update for control $u_k = L_j x_k$ is given by

$$Q_{j+1}(x_k, u_k) = r(x_k, u_k) + \gamma Q_{j+1}(x_{k+1}, L_j x_{k+1})$$

Assume measurements of u_k, x_k and x_{k+1} are available to compute u_{k+1}

QFA – Q Fn. Approximation

$$Q(x,u) = W^T \varphi(x,u)$$

Now u is an input to the NN- Werbos- Action dependent NN

Then

regression matrix

$$W_{j+1}^T \left[\varphi(x_k, u_k) - \gamma \varphi(x_{k+1}, L_j x_{k+1}) \right] = r(x_k, L_j x_k)$$

Since x_{k+1} is measured, do not need knowledge of f(x) or g(x) for value fn. update

Solve for weights using RLS or backprop.

For LQR case

$$\varphi(x) = \left[x_1^2, \ldots, x_1 x_n, x_2^2, \ldots, x_2 x_n, \ldots, x_n^2 \right]'.$$

Model-free policy iteration

Q Policy Iteration

$$Q_{j+1}(x_k, u_k) = r(x_k, u_k) + \gamma Q_{j+1}(x_{k+1}, L_j x_{k+1})$$

Bradtke, Ydstie, Barto

$$W_{j+1}^T\left[\varphi(x_k, u_k) - \gamma\varphi(x_{k+1}, L_j x_{k+1})\right] = r(x_k, L_j x_k)$$

Control policy update

Stable initial control needed

$$h_{j+1}(x_k) = \arg\min_{u_k}(Q_{j+1}(x_k, u_k)) \qquad u_k = -H_{uu}^{-1} H_{ux} x_k = L_{j+1} x_k$$

Greedy Q Fn. Update - Approximate Dynamic Programming
ADP Method 3. Q Learning
Action-Dependent Heuristic Dynamic Programming (ADHDP)

Paul Werbos

Greedy Q Update

Model-free HDP

Stable initial control NOT needed

$$Q_{j+1}(x_k, u_k) = r(x_k, u_k) + \gamma Q_j(x_{k+1}, h_j(x_{k+1}))$$

$$W_{j+1}^T \varphi(x_k, u_k) = r(x_k, L_j x_k) + W_j^T \gamma\varphi(x_{k+1}, L_j x_{k+1}) \equiv \text{target}_{j+1}$$

Update weights by RLS or backprop.

Q learning actually solves the Riccati Equation
WITHOUT knowing the plant dynamics

Model-free ADP

Direct OPTIMAL ADAPTIVE CONTROL

Works for Nonlinear Systems

Proofs?
Robustness?
Comparison with adaptive control methods?

Discrete-Time Zero-Sum Games

- Consider the following continuous-state and action spaces discrete-time dynamical system

$$x_{k+1} = Ax_k + Bu_k + Ew_k \qquad x \in R^n \quad u_k \in R^{m_1}$$

$$y_k = x_k, \qquad y \in R^p \quad w_k \in R^{m_2}$$

with quadratic cost

$$V(x_k) = \sum_{i=k}^{\infty} \left[x_i^T Q x_i + u_i^T u_i - \gamma^2 w_i^T w_i \right]$$

- The zero-sum game problem can be formulated as follows:

$$V(x_k) = \min_u \max_w \sum_{i=k}^{\infty} \left[x_i^T Q x_i + u_i^T u_i - \gamma^2 w_i^T w_i \right]$$

- The goal is to find the optimal strategies (State-feedback)

$$u^*(x) = Lx \qquad w^*(x) = Kx$$

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DT Game
Heuristic Dynamic Programming: Forward-in-time Formulation

- An Approximate Dynamic Programming Scheme (ADP) where one has the following incremental optimization

$$V_{i+1}(x_k) = \min_{u_k} \max_{w_k} \left\{ x_k^T Q x_k + u_k^T u_k - \gamma^2 w_k^T w_k + V_i(x_{k+1}) \right\}$$

which is equivalently written as

$$V_{i+1}(x_k) = x_k^T Q x_k + u_i^T(x_k) u_i(x_k) - \gamma^2 w_i^T(x_k) w_i(x_k) + V_i(x_{k+1})$$

Game Algebraic Riccati Equation

- Using Bellman optimality principle "Dynamic Programming"

$$V^*(x_k) = \min_{u_k}\max_{w_k}(x_k^T Q x_k + u_k^T u_k - \gamma \wedge 2 w_k^T w_k + V^*(x_{k+1}))$$

$$x_k^T P x_k = \min_{u_k}\max_{w_k}(r(x_k, u_k, w_k) + x_{k+1}^T P x_{k+1}).$$

- The Game Algebraic Riccati equation GARE

$$P = A^T PA + Q - [A^T PB \quad A^T PE]\begin{bmatrix} I + B^T PB & B^T PE \\ E^T PA & E^T PE - \gamma^2 I \end{bmatrix}^{-1}\begin{bmatrix} B^T PA \\ E^T PA \end{bmatrix}$$

- The condition for saddle point are

$$I + B^T PB > 0$$
$$I - \gamma^{-2} E^T PE > 0$$

Game Algebraic Riccati Equation

The optimal policies for control and disturbance are

$$L = (I + B^T PB - B^T PE(E^T PE - \gamma^2 I)^{-1} E^T PB)^{-1} \times (B^T PE(E^T PE - \gamma^2 I)^{-1} E^T PA - B^T PA).$$

$$K = (E^T PE - \gamma^2 I - E^T PB(I + B^T PB)^{-1} B^T PE)^{-1} \times (E^T PB(I + B^T PB)^{-1} BPA - E^T PA).$$

Linear Quadratic case- V and Q are quadratic

Asma Al-Tamimi

$$V^*(x_k) = x_k^T P x_k$$

Q learning for H-infinity Control

$$Q^*(x_k,u_k,w_k) = r(x_k,u_k,w_k) + V^*(x_{k+1})$$
$$= \begin{bmatrix} x_k^T & u_k^T & w_k^T \end{bmatrix} H \begin{bmatrix} x_k^T & u_k^T & w_k^T \end{bmatrix}^T$$

Q function update

$$Q_{i+1}(x_k,\hat{u}_i(x_k),\hat{w}_i(x_k)) = x_k^T R x_k + \hat{u}_i(x_k)^T \hat{u}_i(x_k) - \gamma^2 \hat{w}_i(x_k)^T \hat{w}_i(x_k) + Q_i(x_{k+1},\hat{u}_i(x_{k+1}),\hat{w}_i(x_{k+1}))$$

$$[x_k^T \; u_k^T \; w_k^T] H_{i+1} [x_k^T \; u_k^T \; w_k^T]^T = x_k^T R x_k + u_k^T u_k - \gamma^2 w_k^T w_k + [x_{k+1}^T \; u_{k+1}^T \; w_{k+1}^T] H_i [x_{k+1}^T \; u_{k+1}^T \; w_{k+1}^T]^T$$

Control Action and Disturbance updates

$$\begin{bmatrix} H_{xx} & H_{xu} & H_{xw} \\ H_{ux} & H_{uu} & H_{uw} \\ H_{wx} & H_{wu} & H_{ww} \end{bmatrix}$$

$$u_i(x_k) = L_i x_k, \qquad w_i(x_k) = K_i x_k$$

$$L_i = (H_{uu}^i - H_{uw}^i H_{ww}^{i\;-1} H_{wu}^i)^{-1} (H_{uw}^i H_{ww}^{i\;-1} H_{wx}^i - H_{ux}^i),$$
$$K_i = (H_{ww}^i - H_{wu}^i H_{uu}^{i\;-1} H_{uw}^i)^{-1} (H_{wu}^i H_{uu}^{i\;-1} H_{ux}^i - H_{wx}^i).$$

A, B, E NOT needed ☺

Compare to Q function for H_2 Optimal Control Case

$$Q_h(x_k,u_k) = r(x_k,u_k) + V_h(x_{k+1})$$

$$= x_k^T Q x_k + u_k^T R u_k + (Ax_k + Bu_k)^T P(Ax_k + Bu_k)$$

$$= \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} Q + A^T PA & A^T PB \\ B^T PA & R + B^T PB \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \equiv \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T H \begin{bmatrix} x_k \\ u_k \end{bmatrix} = \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

H-infinity Game Q function

$$H_{i+1} = \begin{bmatrix} A^T P_i A + R & A^T P_i B & A^T P_i E \\ B^T P_i A & B^T P_i B + I & B^T P_i E \\ E^T P_i A & E^T P_i B & E^T P_i E - \gamma^2 I \end{bmatrix}.$$

Quadratic Basis set is used to allow on-line solution

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$$\hat{Q}(\bar{z}, h_i) = z^T H_i z = h_i^T \bar{z}$$ where $z = \begin{bmatrix} x^T & u^T & w^T \end{bmatrix}^T$ and $\bar{z} = (z_1^2, \ldots, z_1 z_q, z_2^2, z_2 z_3, \ldots, z_{q-1} z_q, z_q^2)$

Quadratic Kronecker basis

Q function update

$$Q_{i+1}(x_k, \hat{u}_i(x_k), \hat{w}_i(x_k)) = x_k^T R x_k + \hat{u}_i(x_k)^T \hat{u}_i(x_k) - \gamma^2 \hat{w}_i(x_k)^T \hat{w}_i(x_k) + Q_i(x_{k+1}, \hat{u}_i(x_{k+1}), \hat{w}_i(x_{k+1}))$$

Solve for 'NN weights' - the elements of kernel matrix H

$$h_{i+1}^T \bar{z}(x_k) = x_k^T R x_k + \hat{u}_i(x_k)^T \hat{u}_i(x_k) - \gamma^2 \hat{w}_i(x_k)^T \hat{w}_i(x_k) + h_i^T \bar{z}(x_{k+1})$$

Use batch LS or online RLS

Control and Disturbance Updates

$$\hat{u}_i(x) = L_i x \qquad \hat{w}_i(x) = K_i x$$

Probing Noise injected to get Persistence of Excitation

$$\hat{u}_{ei}(x_k) = L_i x_k + n_{1k} \qquad \hat{w}_{ei}(x_k) = K_i x_k + n_{2k}$$

Proof- Still converges to exact result

Lemma 1 Iterating on equations (20), and (34) is equivalent to

$$H_{i+1} = G + \begin{bmatrix} A & B & E \\ L_iA & L_iB & L_iE \\ K_iA & K_iB & K_iE \end{bmatrix}^T H_i \begin{bmatrix} A & B & E \\ L_iA & L_iB & L_iE \\ K_iA & K_iB & K_iE \end{bmatrix} . \quad (35)$$

Lemma 2 The matrices H_{i+1}, L_{i+1} and K_{i+1} can be written

$$H_{i+1} = \begin{bmatrix} A^T P_i A + R & A^T P_i B & A^T P_i E \\ B^T P_i A & B^T P_i B + I & B^T P_i E \\ E^T P_i A & E^T P_i B & E^T P_i E - \gamma^2 I \end{bmatrix}. \quad (36)$$

$$L_{i+1} = (I + B^T P_i B - B^T P_i E(E^T P_i E - \gamma^2 I)^{-1} E^T P_i B)^{-1} \times (B^T P_i E(E^T P_i E - \gamma^2 I)^{-1} E^T P_i A - B^T P_i A), \quad (37)$$

$$K_{i+1} = (E^T P_i E - \gamma^2 I - E^T P_i B(I + B^T P_i B)^{-1} B^T P_i E)^{-1} \times (E^T P_i B(I + B^T P_i B)^{-1} B^T P_i A - E^T P_i A). \quad (38)$$

where P_i is given as

$$P_i = \begin{bmatrix} I & L_i^T & K_i^T \end{bmatrix} H_i \begin{bmatrix} I & L_i^T & K_i^T \end{bmatrix}^T . \quad (39)$$

Lemma 3: Iterating on H_i is similar to iterating on P_i as

$$P_{i+1} = A^T P_i A + R - [A^T P_i B \quad A^T P_i E] \begin{bmatrix} I + B^T P_i B & B^T P_i E \\ E^T P_i A & E^T P_i E - \gamma^2 I \end{bmatrix}^{-1} \begin{bmatrix} B^T P_i A \\ E^T P_i A \end{bmatrix} \quad (40)$$

with P_i defined as in (39).

Theorem 1: Assume that the linear quadratic zero-sum game is solvable and has a value under the state feedback information structure. Then, iterating on equation(35) in Lemma 1, with $H_0 = 0$, $L_0 = 0$ and $K_0 = 0$ converges with $H_i \to H$. where H is corresponds to $Q^*(x_k, u_k, w_k)$ as in (10) and (12) with corresponding P solving the GARE (5).

ADHDP Application for Power system

- System Description

$$x(t) = [\Delta f(t) \quad \Delta P_g(t) \quad \Delta X_g(t) \quad \Delta F(t)]^T$$

$$A = \begin{bmatrix} -1/T_p & K_p/T_p & 0 & 0 \\ 0 & -1/T_T & 1/T_T & 0 \\ -1/RT_G & 0 & -1/T_G & -1/T_G \\ K_E & 0 & 0 & 0 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0 & 0 & 1/T_G & 0 \end{bmatrix}$$

$$E^T = \begin{bmatrix} 1 - K_p/T_p & 0 & 0 & 0 \end{bmatrix}$$

$$1/T_p \in [0.033, 0.1]$$

$$K_p/T_p \in [4,12]$$

$$1/T_T \in [2.564, 4.762]$$

$$1/T_G \in [9.615, 17.857]$$

$$1/RT_G \in [3.081, 10.639]$$

- The Discrete-time Model is obtained by applying ZOH to the CT

ADHDP Application for Power system

- The system state

  Δf _incremental frequency deviation (Hz)

  ΔP_g _incremental change in generator output (p.u. MW)

  ΔX_g _incremental change in governor position (p.u. MW)

  ΔF _incremental change in integral control.

  ΔP_d _is the load disturbance (p.u. MW); and

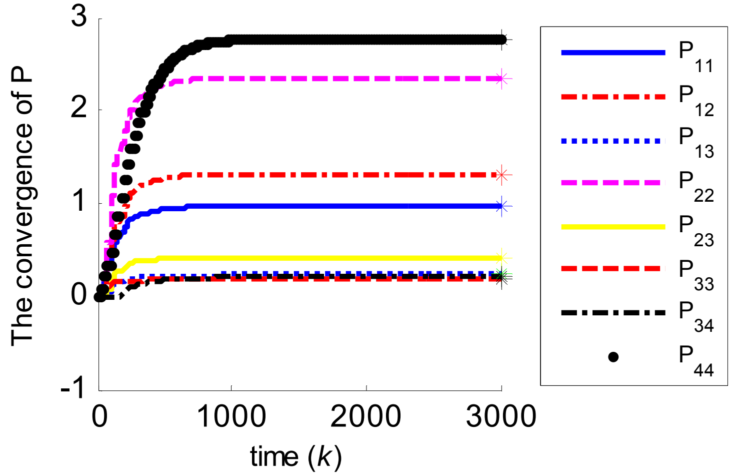
- The system parameters are:

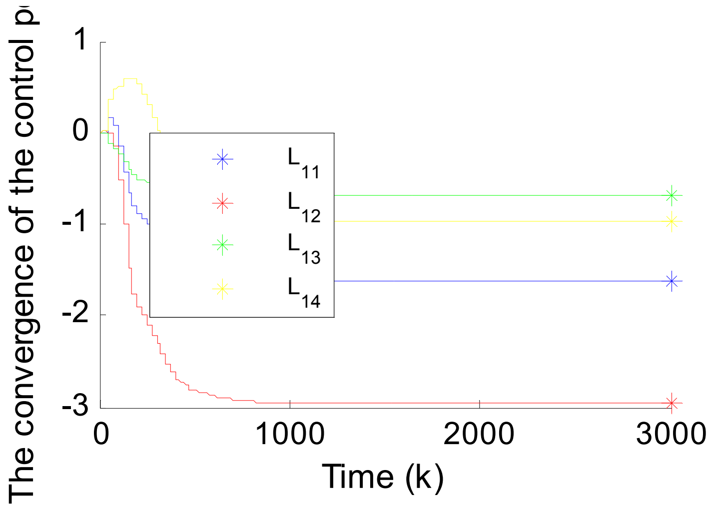
  T_G _the governor time constant

- T_T _turbine time constant
- T_P _plant model time constant
- K_p _planet model gain
- R _speed regulation due to governor action
- K_E_ integral control gain.

ADHDP Application for Power system

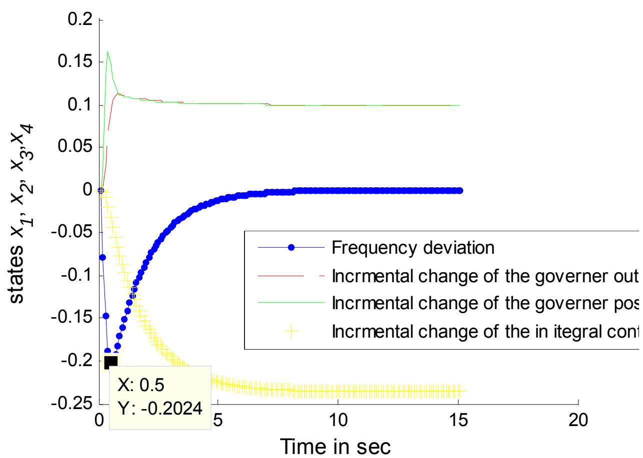
- ADHDP policy tuning



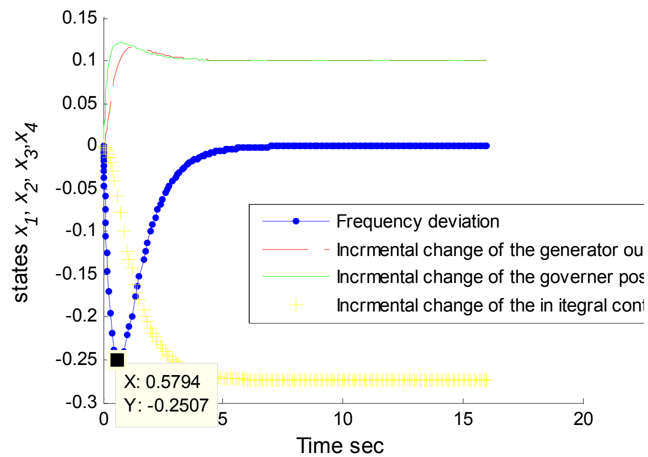


ADHDP Application for Power system

- *Comparison*



The ADHDP controller design



The design from [1]

- The maximum frequency deviation when using the ADHDP controller is improved by 19.3% from the controller designed in [1]

- [1] Wang, Y., R. Zhou, C. Wen, “Robust load-frequency controller design for power systems”, IEE Proc.-C, Vol. 140, No. I , 1993

Discrete-time nonlinear HJB solution using Approximate dynamic programming : Convergence Proof

- Problem Formulation

$$x_{k+1} = f(x_k) + g(x_k)u_k \qquad V^*(x_k) = \min_{u_k} \sum_{i=k}^{\infty} x_i Q x_i + u_i R u_i$$

- requires solving the DT HJB

$$\begin{aligned} V^*(x_k) &= \min_{u_k} \left[x_k^T Q x_k + u_k^T R u_k + V^*(x_{k+1}) \right] \\ &= \min_{u_k} \left[x_k^T Q x_k + u_k^T R u_k + V^*\left(f(x_k) + g(x_k)u_k \right) \right] \end{aligned}$$

$$u^*(x_k) = -\frac{1}{2} R^{-1} g(x_k)^T \frac{dV^*(x_{k+1})}{dx_{k+1}}$$