General Existence Conditions of Homoclinic Trajectories in Dissipative Systems: Lorenz, Shimizu-Morioka, Chen and Lu systems

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Shilnikov theorem. *Suppose, the saddle-focus O has a homoclinic loop Γ and positive saddle value. Then in an arbitrary small neighborhood of Γ there exists an infinite set of saddle periodic orbits.*

Tricomi problem

$$\frac{dx}{dt} = f(X, q), \quad X \in \mathbb{R}^n = \{X\}, \quad q \in \mathbb{R}^m = \{q\}, \qquad (1)$$

where $f(X, q)$ is a smooth vector-function, $\mathbb{R}^n$ is a phase space, $\mathbb{R}^m$ is a parameter space.

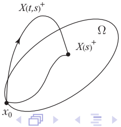
Tricomi problem. Let it be given a smooth path $\gamma(s), \ s \in [0, 1]$ in parameter space $\{q\}$. Does there exist a point $q_0 \in \gamma(s)$ for which system (1) with q_0 has a homoclinic trajectory?

Definition. Trajectory $x(t)$ of system (1) is said to be *homoclinic* if the following relation $\lim\limits_{t\to+\infty} X(t) = \lim\limits_{t\to-\infty} X(t) = X_0$ is satisfied.

$X(t, s)^+$ is a separatrix of saddle point X_0: $\lim\limits_{t\to-\infty} X(t, s)^+ = X_0$, $X(s)^+$ is a point of the first crossing of a separatrix $X(t, s)^+$ with the closed set Ω:

$$X(t, s)^+ \overline{\in} \Omega, \quad t \in (-\infty, T),$$

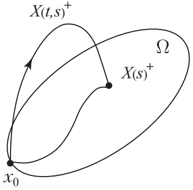
$$X(T, s)^+ = X(s)^+ \in \Omega.$$



Fishing principle: assumptions

$$\frac{dx}{dt} = f(X, q), \quad X \in \mathbb{R}^n = \{X\}, \quad q \in \mathbb{R}^m = \{q\}.$$

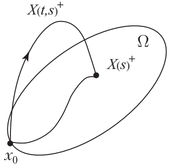
Let for the path $\gamma(s)$ there exist $(n-1)$-dimensional bounded manifold with the piecewise smooth edge $\partial\Omega$ and it has the following properties:

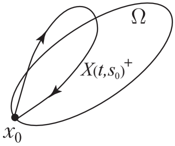


- $\forall\ X \in \Omega \setminus \partial\Omega$ and $s \in [0,1]$ the vector $f(X, \gamma(s))$ is transversal to the manifold Ω;
- $\forall\ s \in [0,1],\ f(X_0, \gamma(s)) = 0$ and the point $x_0 \in \partial\Omega$ is a saddle of the system;
- the inclusion $X(0)^+ \in \Omega \setminus \partial\Omega$ is satisfied;
- the relation $X(1)^+ = \emptyset$ is valid;
- $\forall\ s \in [0,1]$ and $Y \in \partial\Omega \setminus X_0$ $\exists$ a neighborhood $U(Y,\sigma) = \{X \| X - Y| < \sigma\}$ such that $X(0)^+ \overline{\in} U(Y,\sigma)$;
- $\forall\ s:\ \ X(s)^+ \in \Omega$ and $\forall\ t \in (-\infty, T]$ $\exists$ a number $R:\ |X(t,s)^+| \leq R$. Here $X(T,s)^+ = X(s)^+$.

Fishing principle: theorem

Theorem 1. If conditions of the “fishing principle” are satisfied, then there exists $s_0 \in [0, 1]$ such that $X(t, s_0)$ is a homoclinic trajectory of the saddle X_0.





Interpretation. In the left figure at the point X_0 it is placed a fisherman with the rod $X(t, s)^+$. A surface of this lake is the set Ω, a shore of lake is $\partial\Omega$. At $s = 0$ a fisherman hooked a fish. Then $X_0(t, s)^+$, $s \in [0, s_0]$ is a path of a rod with fish to the shore. But the fish can be landed only through the point X_0 (since $\partial\Omega \setminus X_0$ is a prohibited area). Therefore there occurs a situation, shown in the right figure (i.e. a fisherman caught a fish). This corresponds to a homoclinic trajectory.

Lorenz, Lu and Chen system

$$\dot{x} = \sigma(y - x),\quad \dot{y} = rx - dy - xz,\quad \dot{z} = -bz + xy,\quad \sigma > 0, b > 0.$$

$$\Rightarrow \quad \eta = \sigma(y - x),\quad \xi = z - x^2/b$$

$$\dot{x} = \eta,\quad \dot{\eta} = -(\sigma + d)\eta - \sigma x\xi + \varphi(x),\quad \dot{\xi} = -b\xi - \beta x\eta,$$

$$\varphi(x) = \sigma(r - d)x - \frac{\sigma}{b}x^3,\ \beta = \frac{2\sigma - b}{b\sigma}.$$

For Lorenz system $r > 0, d = 1$, for Lu system $d = -c, c > 0, r = 0$ and for Chen system $d = -c, c > 0, r = c - \sigma$.

Theorem 2. Let be $\sigma + d > 0, r > d, \beta < 0$. Then a separatrix of the saddle $x = \eta = \xi = 0$ of the system doesn't intersect the plane $\{x = 0\}$ and any solution of the system tends to a certain equilibrium as $t \to +\infty$.

$$V(x, \eta, \xi) = \eta^2 - \frac{\sigma}{\beta}\xi^2 - 2\int_0^x \varphi(s)ds, \qquad \dot{V}(x, \eta, \xi) = -2(\sigma + d)\eta(t)^2 + \frac{2\sigma b}{\beta}\xi(t)^2.$$

Lorenz, Lu and Chen system: case $2\sigma > b$, $r > d$

$$\dot{x} = \sigma(y - x), \quad \dot{y} = rx - dy - xz, \quad \dot{z} = bz + xy, \quad \sigma > 0, b > 0.$$

$$\Rightarrow \quad \eta = \sigma(y - x), \quad Q = z - x^2/2\sigma$$

$$\dot{x} = \eta, \quad \dot{\eta} = -(\sigma + d)\eta - \sigma x Q + \sigma(r - d)x - \frac{1}{2}x^3, \quad \dot{Q} = -bQ + \left(1 - \frac{b}{2\sigma}\right)x^2.$$

Consider the separatrix $x(t)^+, \eta(t)^+, Q(t)^+$ of a zero saddle equilibrium, $x(t)^+ > 0, \ \forall t \in (-\infty, T)$.

Lemma 1. The estimate $Q(t)^+ \geq 0, \quad \forall\, t \in (-\infty, +\infty)$ is valid.

Lemma 2. The estimate
$\eta(t)^+ \leq Lx(t)^+, \quad \forall\, t \in (-\infty, T), \quad L = -\frac{\sigma+d}{2} + \sqrt{\frac{(\sigma+d)^2}{4} + \sigma(r - d)}$ is valid.

Lemma 3. Let the inequality $3\sigma - (2b + d) < \frac{2b(2\sigma - b)}{2L+b}$ be valid. Then
$\dot{V}(x(t)^+, \eta(t)^+, Q(t)^+) + (\sigma + d)V(x(t)^+, \eta(t)^+, Q(t)^+) < 0, \ \forall t \in (-\infty, T)$.

Lorenz, Lu and Chen system: case $2\sigma > b$, $r > d$

Theorem 3. If the inequality $3\sigma - (2b+d) < \frac{2b(2\sigma-b)}{2L+b}$ be valid, then $x(t)^+ > 0$, $\forall\, t \in (-\infty, +\infty)$ and the separatrix $x(t)^+, \eta(t)^+, Q(t)^+$ doesn't tend to zero as $t \to +\infty$.

$$\begin{aligned} \dot{x} &= \eta, \\ \dot{\eta} &= -(\sigma+d)\eta - \sigma x\xi + \varphi(x), \\ \dot{\xi} &= -b\xi - \beta x\eta, \\ \varphi(x) &= \sigma(r-d)x - \frac{\sigma}{b}x^3 \end{aligned}$$

$$\begin{aligned} \varepsilon &= (r-d)^{-1/2}, \\ t_1 &\to \frac{\sqrt{\sigma}t}{\varepsilon}, \\ x_1 &\to \frac{\varepsilon x}{\sqrt{2\sigma}}, \\ \eta_1 &\to \frac{\varepsilon^2\eta}{\sigma\sqrt{2}}, \\ \xi_1 &\to \varepsilon^2\xi \end{aligned}$$

$$\begin{aligned} \dot{x}_1 &= \eta_1, \\ \dot{\eta}_1 &= -\mu\eta_1 - x_1\xi_1 - \psi(x_1), \\ \dot{\xi}_1 &= -\alpha\xi_1 - \nu x_1\eta_1. \end{aligned}$$

Here $\psi(x_1) = -x_1 - 2\sigma x_1^3/b$, $\mu = \frac{\varepsilon(\sigma+d)}{\sqrt{\sigma}}$, $\alpha = \frac{\varepsilon b}{\sqrt{\sigma}}$, $\nu = 2(2\sigma-b)/b$.

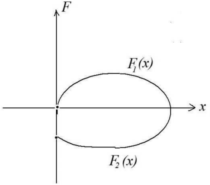
Lorenz, Lu and Chen system: case $2\sigma > b$, $r > d$

$$\begin{aligned} \dot{x}_1 &= \eta_1, \\ \dot{\eta}_1 &= -\mu\eta_1 - x_1\xi_1 - \psi(x_1), \\ \dot{\xi}_1 &= -\alpha\xi_1 - \nu x_1\eta_1. \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \frac{dF}{dx} &= \frac{-\mu F - Px - \psi(x)}{F}, \\ \frac{dP}{dx} &= \frac{-\alpha P - \nu F x}{F}. \end{aligned}$$

The right system is equivalent ro the left one on the sets $\{x \geq 0, \eta > 0\}$ and $\{x \geq 0, \eta < 0\}$.

$F_1(0) = 0,\ F_1(x) > 0,\ \forall\, x \in (0, x_1),\ F_1(x_1) = 0.$

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - dy - xz, \quad (*) \\ \dot{z} &= bz + xy. \end{aligned}$$



For any compact set of parameters of the system $(*)$ the separatrices $x(t)^+, y(t)^+, z(t)^+$ are uniformly bounded on $(-\infty, T]$:

$$\sqrt{(x(t)^+)^2 + (y(t)^+)^2 + (z(t)^+)^2} \leq R,\ \forall\, t \in (-\infty, T].$$

$$
\begin{aligned}
\dot{x} &= \sigma(y-x),\\
\dot{y} &= rx - dy - xz,\\
\dot{z} &= -bz + xy.
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
x &\to \sigma x,\\
y &\to \sigma y,\\
z &\to \sigma z,\\
t &\to \sigma^{-1} t
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\dot{x} &= y - x,\\
\dot{y} &= r_1 x - d_1 y - xz,\\
\dot{z} &= -b_1 z + xy,\\
r_1 &= r/\sigma, d_1 = d/\sigma, b_1 = b/\sigma.
\end{aligned}
$$

Lu: $r_1 = 0, d_1 = c/a, b_1 = b/a,$

Chen: $r_1 = \dfrac{c}{a} - 1, d_1 = c/a, b_1 = b/a.$

Proposition 1. Without loss of generality, for Lu nd Chen systems it can be assumed that $a = 1$.

Lemma 4. Let be $a = 1, b = 0, c \in (0.101, 1)$. The a separatrix of Lu system crosses the plane $\{x = 0\}$.

Lemma 5. Let be $a = 1, b = 0, c \in (0.509, 1)$. The a separatrix of Chen system crosses the plane $\{x = 0\}$.

Lorenz, Lu and Chen system: fisherman principle

$$\dot{x} = \sigma(y - x),$$
$$\dot{y} = rx - dy - xz,$$
$$\dot{z} = -bz + xy.$$

$$\Omega = \{x = 0, y \leq 0, y^2 + z^2 \leq R^2\}$$
$$x(t) \equiv y(t) \equiv 0, z(t) = z(0) \exp -bt$$

Theorem 4. Let the numbers b, σ, d be given. For the existence of a number $r > 1$ such that system has a homoclinic trajectory, it is necessary and sufficient that $2\sigma + d < 3\sigma$.

Theorem 5. Let be $r = 0, \sigma = 1, d = -c, c \in (0.101, 1)$. Then there exists $b \in (0, 2)$ such that the system has a homoclinic trajectory.

Theorem 6. Let be $r = c - 1, \sigma = 1, d = -c, c \in (0.509, 1)$. Then there exists $b \in (0, 2)$ such that the system has a homoclinic trajectory.

Fisherman principle: numerical results

Lorenz system: $\sigma(s) \equiv 10, b(s) \equiv 8/3, d(s) \equiv 1, r(0) = 28, r(1) = 2,$ $r \in [13.92, 13.93]$

Lu system: $\sigma(s) \equiv 35, d(s) \equiv -28, r(s) \equiv 0,$ $b \in [44.963, 44.974]$

Chen system: $\sigma(s) \equiv 35, d(s) \equiv -28, r(s) \equiv -7,$ $b \in [40.914, 44.935]$

Shimizu-Morioka system: $\dot{x} = y,\ \dot{y} = x - ay - xz,\ \dot{z} = -bz + x^2$
$b(s) = 0.8,\ a \in [0.93, 0.94]$

$$\begin{aligned} \dot{x} &= \sigma(y-x), \\ \dot{y} &= rx - dy - xz, \\ \dot{z} &= -bz + xy. \end{aligned} \quad \cong \quad \begin{aligned} \dot{x} &= \sigma(y-x) + p(x,y,z)(L(K_1x - y) + Mz), \\ \dot{y} &= rx - dy - xz + p(x,y,z)K_2(L(K_1x - y) + Mz), \\ \dot{z} &= -bz + xy + p(x,y,z)(y - K_1x). \end{aligned}$$

$$p(x,y,z) \begin{cases} = 1,\ \forall\, x,y,z : |y - K_2x| \le \varepsilon,\ (y - K_1x)^2 + z^2 \le \varepsilon^2, \\ \le 1,\ \forall\, x,y,z : |y - K_2x| \le 2\varepsilon,\ (y - K_1x)^2 + z^2 \le 4\varepsilon^2, \\ = 0, \text{for all rest } x,y,z. \end{cases}$$

Modification of suggested systems

$$\begin{aligned} \dot{x} &= \sigma(y-x), \\ \dot{y} &= rx - dy - xz, \\ \dot{z} &= -bz + xy. \end{aligned} \quad \cong \quad \begin{aligned} \dot{x} &= \sigma(y-x) + p(x,y,z)(L(K_1x - y) + Mz), \\ \dot{y} &= rx - dy - xz + p(x,y,z)K_2(L(K_1x - y) + Mz), \\ \dot{z} &= -bz + xy + p(x,y,z)(y - K_1x). \end{aligned}$$

$$K_1 = \frac{1}{2\sigma}(\sigma - d + \sqrt{(\sigma-d)^2 + 4\sigma r})$$

$$K_2 = \frac{1}{2\sigma}(\sigma - d - \sqrt{(\sigma-d)^2 + 4\sigma r})$$

$$(b + \lambda_2 + L(K_1 - K_2))^2 \leq 4M(K_1 - K_2)$$

$$b - \lambda_2 - 2\lambda_1 < L(K_1 - K_2) < b - \lambda_2$$

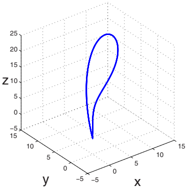
$$\lambda_1 = \frac{1}{2}(-\sigma - d + \sqrt{(\sigma-d)^2 + 4\sigma r})$$

$$\lambda_2 = \frac{1}{2}(-\sigma - d - \sqrt{(\sigma-d)^2 + 4\sigma r})$$

$$\lambda_1 > 0$$

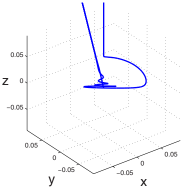
Lemma 6. Linear approximation of stable and unstable manifolds of the saddle $x = y = z = 0$ of these two systems coincide.

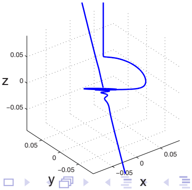
$\sigma = 10$, $b = 8/3$, $d = 1$, $\varepsilon = 0.03$,
$M = 1000$, $L = 3$.



$$r = 13.9265576714,$$
$$K_1 = 1.712994761327219,$$
$$K_2 = -0.812994761327219,$$
$$\lambda_1 = 7.129947613272194,$$
$$\lambda_2 = -18.129947613272194.$$

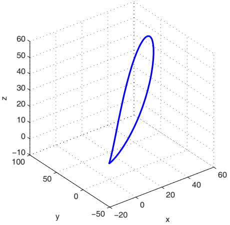
$$r = 13.9265576715,$$
$$K_1 = 1.712994761331179,$$
$$K_2 = -0.812994761331178,$$
$$\lambda_1 = 7.129947613311783,$$
$$\lambda_2 = -18.129947613311785.$$



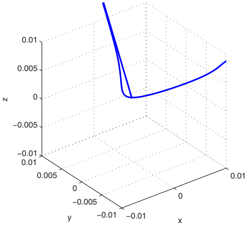


Numerical results: Lu system

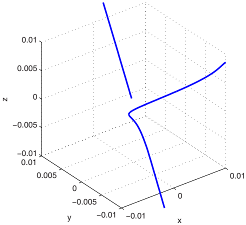
$\sigma = 35,\ d = -28,\ r = 0,\ \varepsilon = 0.01,$
$M = 1000,\ L = 14,\ K_1 = 1.8,\ K_2 = 0,$
$\lambda_1 = 28,\ \lambda_2 = -35.$



$b = 44.963209.$

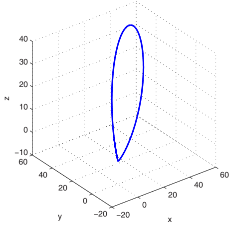


$b = 44.963208.$

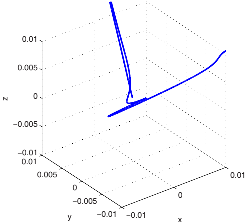


Numerical results: Chen system

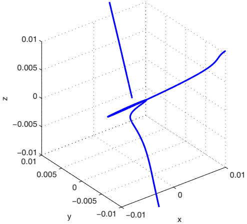
$\sigma = 35,\ d = -28,\ r = -7,\ \varepsilon = 0.01,$
$M = 1500,\ L = 35,$
$K_1 = 1.681024967590665,$
$K_2 = 0.118975032409335,$
$\lambda 1 = 23.835873865673289,$
$\lambda 2 = -30.835873865673289.$



$b = 40.89290999.$



$b = 40.89290998.$



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