On the accuracy of the OPC approximation for a symmetric overflow loss model*

Eric W. M. Wong, Bill Moran, Andrew Zalesky, Zvi Rosberg, Moshe Zukerman

Abstract

The overflow priority classification approximation (OPCA) and Erlang's fixed-point approximation (EFPA) are distinct methods for estimating blocking probabilities in overflow loss networks. Mounting numerical evidence has indicated that OPCA provides superior accuracy than EFPA in many circumstances. Furthermore, it has been proven that $P_{EFPA} \leq P_{OPCA}$ for a symmetric overflow loss network called the distributed server model, where P_x is the blocking probability estimate yielded by approximation $x \in \{EFPA, OPCA\}$. The distributed server model is an ideal "proving ground" because the exact blocking probability, P_{exact}, can be calculated with the Erlang B formula, yet the state dependencies caused by mutual overflow are retained. The present paper proves the OPC Conjecture, $P_{OPCA} \leq P_{exact}$. This new result establishes that OPCA is always equally or more accurate relative to EFPA for the distributed server model and suggests OPCA has utility in more general overflow loss networks. The proof of $P_{OPCA} \leq P_{exact}$ turned out to be challenging, since OPCA is remarkably close to the exact solution, requiring delicate inequality arguments on the Maclaurin series of both solutions. Another contribution is the derivation of new tight bounds and limiting regimes for the blocking probabilities yielded by OPCA and EFPA in the case of critical loading. The simple and tight lower bound derived for OPCA can serve as a tight lower bound for the Erlang B blocking probability in the case of critical loading.

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1 Introduction

Loss networks [12, 14] are ubiquitous in telecommunications. Of particular utility to the modeling of alternate routing [2, 4, 18, 19, 31, 33] are *overflow* loss networks, where calls can overflow to an alternative server group, if the primary server group is fully engaged with other calls. In fact, calls may be permitted to successively overflow to many alternative server groups, until an idle server is found or all permissible server groups are attempted. Given that calls arrive randomly and the number of server groups is finite, some calls may overflow to all permissible server groups without finding an idle server. This results in call blocking, where the blocked call is cleared from the network without ever receiving service.

The probability of call blocking, referred to as the *blocking probability*, is one of the most fundamental performance measures for loss networks. Estimating this probability has been a cornerstone of teletraffic modeling and theory. Moment matching approximations [3, 5, 7, 11, 15, 16, 17, 20, 21, 22, 27] are known to be quite accurate if server groups are arranged hierarchically into tiers. With such hierarchically organized networks, the state of a given tier of the hierarchy is independent of all lower tiers, and thus state dependencies between server groups always move from top to bottom of the hierarchy, but not vice versa. The most basic approximation is to treat each tier independently, where the input call arrival process of a given tier has its moments matched to the overflow process of the immediately preceding tier.

Accurate estimation of blocking probabilities in non-hierarchical networks is more challenging, due to complex state dependencies caused by mutual overflow [8]. Congestion within a given server group can cause overflow to other servers, where this overflow congests the other server groups, which in turn results in new calls overflowing to the original server. The stronger dependencies between server groups caused by mutual overflow is known to increase the approximation error associated with treating each server group independently. Despite this, the Erlang's fixed-point approximation (EFPA) proposed in 1964 [5] – a decoupling approximation treating each server group as an independent M/M/N/N queue – remains the analytical approximation of choice for estimating blocking probabilities in overflow loss networks. With EFPA, the total call arrival process offered to each server group (overflow calls + new calls) is pooled together and is assumed to follow a Poisson process with rate equal to the sum of the rates of all individual input streams. In this way, the overflow probability perceived by the traffic offered to a server group is approximated by the Erlang B formula. Let $\mathbf{E}_N(A)$ denote the blocking probability of an M/M/N/N system with N servers and offered traffic A, defined by $A = \lambda/\mu$ where λ is the arrival rate and μ is the service rate per server. Let λ_i be the original rate of the i-th stream. The overflow events of the i-th stream may form an overflow stream of rate $\lambda_i \mathbf{E}_N(\sum_i \lambda_i/\mu)$ to a subsequent server group. This approach inherently gives rise to non-linear equations that entail a fixed-point solution of the overflow probability at each server group. This fixed-point can usually be found by successive substitution. However, it is known that convergence and uniqueness of solutions of these fixed-point techniques are not always guaranteed [26, 33].

The *overflow priority classification approximation* (OPCA) [29] is an alternative to EFPA. OPCA is based on evaluation of the blocking probability of a certain surrogate model which is hierarchical, so mutual dependency between server groups is avoided. Therefore, for the surrogate, treating each server group independently gives a more accurate approximation. Although the surrogate generally does not have the same blocking probability as the original problem, approximating the blocking probability of the surrogate often leads to an accurate approximation for the original problem.

The OPCA surrogate is based on a preemptive priority regime, even though no preemption is present in reality. Calls that have overflowed zero times are given first priority, while calls that have overflowed n times are given nth priority. An n-call can preempt a k-call, $n < k$, resulting in the n-call immediately engaging the server previously occupied by the k-call and causing the k-call to overflow to another server group, where it becomes a $(k + 1)$-call. In OPCA, blocking probabilities of the surrogate are approximated using precisely the same principles as EFPA; namely, each server group is treated independently and the total call arrival process offered to each server group at each priority level is treated as a Poisson process.

In loss systems where calls that do not find a free server in a server group are never allowed to overflow to other server groups, OPCA reduces to EFPA. This applies to the important case of circuit switching networks with fixed routing which EFPA provides exact and unique solutions under asymptotic conditions of large capacity links [13] and high diversity routing [34]. The use of EFPA for large circuit switching network with many circuit per trunks and fixed routing has been demonstrated in [1]. This implies that for these known fixed routing instances where the EFPA results are known to be exact, also OPCA gives exact solution. In other cases, comprehensive numerical evaluation involving a diverse range of overflow loss networks attests to improved accuracy yielded by OPCA [23, 28, 32]. Moreover, OPCA is usually more computationally efficient than EFPA.

While mounting *numerical* evidence points to the superiority of OPCA, this has not been rigorously established, and examples can be found for which EFPA remains the most accurate approximation. The purpose of this paper is to prove the superiority of OPCA, compared to EFPA, in a symmetric overflow loss network for which the exact blocking probability can be calculated using the Erlang B formula.

The overflow loss network on which this entire paper is based comprises N identical single servers, where new calls arrive at each server according to a Poisson process with arrival rate λ. The holding times of all calls are assumed to be exponentially distributed with parameter μ. Let $a = \lambda/\mu$ be offered load (a.k.a offered traffic) of new calls to each server. Then, the offered load of new calls to the entire system is aN. Calls are permitted to overflow to a randomly chosen server until an idle server is found or all servers are attempted. A call is blocked if all servers are attempted and found to be engaged. A call immediately engages an idle server for its service duration and then leaves. We consider here a symmetric distributed overflow loss service system where the arrival processes and service statistics are identical for all servers. This system is henceforth referred to as the *distributed server model.* It is easy to see that the blocking probability of the distributed server model is obtained by the Erlang B formula as $P_{exact} = \mathbf{E}_N(aN)$.

It has been proven [32] that $P_{EFPA} \leq P_{OPCA}$ for all $N \geq 1$ and for all call offered loads, where P_x is the call blocking probability estimate in the distributed server model yielded by approximation x. Moreover, comprehensive numerical evaluation [32] suggests $P_{OPCA} \leq P_{exact}$. This inequality has been referred to as the *OPC Conjecture. The main contribution of this paper is to prove the OPC Conjecture.* Proof of this conjecture verifies that OPCA is at least as accurate as EFPA for the distributed server model.

OPCA and EFPA apply to many overflow systems and networks. In almost all cases, only simulations or experiments can be used to evaluate the accuracy of the approximations and to obtain evidence to which of the approximations is more accurate. Our distributed server model is a rare instance of an overflow system with mutual overflow for which the exact blocking probability is available. Therefore, we consider it an ideal "proving ground" where the accuracy of OPCA and EFPA can be examined analytically.

In addition to proving the OPC Conjecture, this paper makes two further contributions:

- Tight bounds and limiting regimes are derived for the blocking probabilities yielded by OPCA and EFPA in the case of critical loading (Section 5). The simple and tight lower bound derived for OPCA can serve as a tight lower bound for the Erlang B blocking probability in the case of critical loading.

- It is shown that in the case of critical loading, the ratio between the Erlang B and OPCA blocking probabilities is bounded by $\sqrt{2}$ for all $N \geq 1$, while the ratio between Erlang B and EFPA approaches infinity as $N \to \infty$.

Our proof strategy of showing that all coefficients of a suitable polynomial are nonnegative has also been used to prove conjectures related to the Erlang formula [25], but as will become apparent to the reader, the tightness of the inequality, $P_{OPCA} \leq P_{exact}$ makes the challenge here far more formidable. In fact, OPCA is exact for the cases $N = 1, 2$ and approaches the exact blocking probability as $N \to \infty$. The proof of $P_{OPCA} \leq P_{exact}$ given here involves dividing the problem into many cases, and some computer checking for some small integer cases. We emphasize that the computer checking involves only finitely many calculations, and is only there to eliminate many tedious, though entirely feasible, hand calculations. At no stage have we relied on the computer to do simulations or calculations that would require assumptions about the validity of the software or about the representativeness of the simulations. In places where we have used a computer, it is merely as an arbitrary (or at least sufficiently high) precision calculator. In other words, modulo the relatively small number of hand calculations, the descriptions of which are relegated to the appendices in [30], and for which we have used the package Maxima, (`http://maxima.sourceforge.net/index.shtml`) permitting arbitrary precision in arithmetical operations, this proof is mathematically rigorous.

The remainder of the paper is organized as follows. Section 2 defines the distributed server model (Section 2.1) and describes the application of EFPA (Section 2.2) and OPCA (Section 2.3) to this model. Section 3 is devoted to the proof of the OPC Conjecture, while Section 4 shows that OPCA and EFPA approach P_{exact} as $N \to \infty$. Section 5 derives bounds for the case of critical loading.

2 Problem formulation

We begin by defining the distributed server model [32]. We then show how EFPA and OPCA can be applied to evaluate the blocking probability for this model and demonstrate that the Erlang B formula yields the exact blocking probability. This in turn leads to the formulation of the OPC Conjecture.

2.1 The distributed server model

Consider the following model of an overflow loss network [6]. The network comprises N identical servers. Calls are offered to servers according to mutually independent time-homogeneous Poisson processes each with offered load $a > 0$. A call arriving at a busy server overflows to one of the other $N - 1$ servers with equal probability and without delay. A call continues to overflow until it either encounters an idle server, in which case it engages that server until its service period is complete, or has sought to engage all N servers exactly once but found them all busy, in which case it is blocked and never returns. The search for an idle server is assumed to be conducted instantaneously. Service periods are independent and exponentially distributed with normalized unit mean.

An n-call (for $1 \le n \le N - 1$) is defined as a call that overflows (is denied service) n times before engaging the $(n + 1)$th server in its search for an idle server. An N-call is a call that is blocked and cleared and a 0-call is defined as a call initiated by a user (exogenous call).

This distributed-server model can be viewed as an M/M/N/N queue with offered load aN. This allows for exact calculation of blocking probability using the Erlang B formula as

$$P_{\text{exact}} = \mathbf{E}_N(aN).$$

Therefore, $\mathbf{E}_N(aN)$ provides a benchmark for comparison of blocking probability evaluation.

2.2 Erlang's Fixed-Point Approximation

Here we briefly provide background on application of EFPA to the distributed-server model. For more details please refer to [32]. In fact, equations (1) – (4) are equations (4) – (7) in [32].

We make the following simplifying assumptions:

1. *Independence* — states of servers are mutually independent;

2. *Poisson* — arrivals of n-calls to any server follow a Poisson process,

and write b for the probability that a server is busy. As mentioned above, we assume that a is the offered load of new calls at a server. Also let a_n be the offered load of n-calls at a server. Then, from the independence assumption,

$$a_n = ab^n, \quad n = 0, 1, 2 \ldots, N-1, \tag{1}$$

and from the Poisson assumption, the probability of a server being busy gives rise to an M/M/1/1 model for which the probability is

$$b = \frac{\sum_{n=0}^{N-1} a_n}{1 + \sum_{n=0}^{N-1} a_n}. \tag{2}$$

Substituting (1) into (2), we obtain

$$b = a(1 - b^N), \tag{3}$$

which forms a fixed-point equation that can be solved for b, giving an EFPA blocking probability approximation

$$P_{EFPA} = b^N. \tag{4}$$

2.3 OPCA

As in the EFPA case, we assume that events related to the N different servers are mutually independent and that the servers are statistically equivalent. Also as in the EFPA case, we let $a > 0$ be the offered load of new calls at a server and a_n the offered load of n-calls at a server. Thus, $a_0 = a$. The stream formed by n-calls, $N-1 \geq n \geq 0$, arriving at a server, is assumed to follow a Poisson stream with rate μa_n (so that the offered load is a_n). The preemptive priority regime defined by OPCA gives priority to "junior" n_h-calls over "senior" n_l-calls for any $0 \leq n_h < n_l \leq N-1$. Accordingly, the a_n values can be obtained recursively by

$$a_{n+1} = \mathbf{E}_1\left(\sum_{i=0}^{n} a_i\right)\sum_{i=0}^{n} a_i - \sum_{i=1}^{n} a_i, \tag{5}$$

for all $n = 0, \ldots, N-1$, where a_N is defined as the offered load of the stream formed by calls that are blocked and cleared. Note that $\sum_{i=0}^{n} a_i$ is the total offered load per server for 0-call up to n-call, and $\mathbf{E}_1(\sum_{i=0}^{n} a_i)$ is the blocking probability of those arriving calls. Their product is the overflow traffic per server for 1-call up to $n+1$-call, which is in turn the total offered load per server for 1-call up to $n+1$-call, i.e. $\sum_{i=1}^{n+1} a_i$. Subtracting $\sum_{i=1}^{n} a_i$, we obtain a_{n+1}. See [32] for further details. Also note that Eq. (5) is equivalent to Eq. (11) in [32].

The OPCA blocking probability approximation is given by

$$P_{\text{OPCA}} = \frac{a_N}{a}. \tag{6}$$

Eq. (6) has a clear physical interpretation: the a_N/a ratio is the proportion of calls that are promoted to N-calls (blocked calls) which is the OPCA blocking probability approximation.

Graphs are shown in Fig. 1 demonstrating the tightness of P_{OPCA} and P_{EFPA} as lower bounds for P_{exact}.

We are now able to state the main theorem of this paper.

Theorem 1.

$$P_{EFPA} \leq P_{OPCA} \leq P_{exact}. \tag{7}$$

The first inequality, namely $P_{\text{EFPA}} \leq P_{\text{OPCA}}$, was proved in [32], where the second inequality is stated as the *the OPC Conjecture.* It is a key purpose of this paper to provide a mathematical proof of the OPC conjecture.

By the Erlang B formula, we have

$$P_{\text{exact}} = \frac{\frac{(aN)^N}{N!}}{\sum_{i=0}^{N} \frac{(aN)^i}{i!}}. \tag{8}$$

Therefore, by (6) – (8), the proof of Theorem 1 will be completed when we have proved the

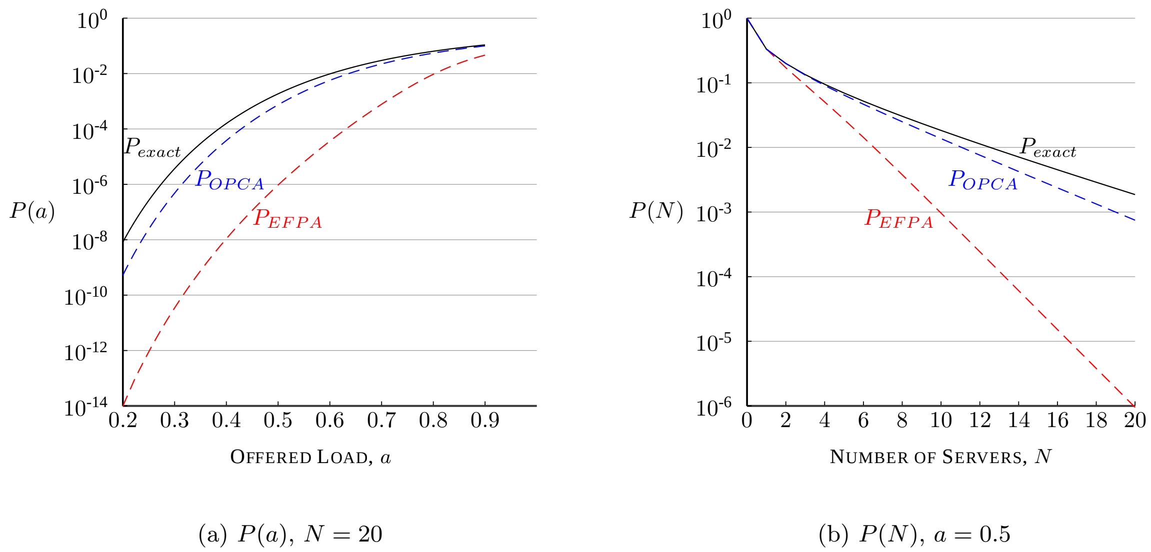

Figure 1: Numerical Comparison of OPCA and EFPA

following inequality for all $a > 0$ and $N = 1, 2, 3, \ldots$

$$\frac{a_N}{a} \leq \frac{\frac{(aN)^N}{N!}}{\sum_{i=0}^{N} \frac{(aN)^i}{i!}}. \tag{9}$$

Before we commence the proof we note that it is easy to see that P_{EFPA} and P_{OPCA} approach 0 as $a \to 0$. It is also easy to see that $P_{OPCA} = P_{exact}$ for $a > 0$ in the trivial cases of $N = 1$ and $N = 2$.

Most of the remainder of this paper is given over to establishing this innocuous looking and quite generic inequality between the Erlang B formula and the OPCA system.

2.4 System Interpretations and Intuitive Arguments

A discussion of systems for which the approximations we consider are exact will help in understanding the implications of the approximations. To this end consider the following four systems.

The actual system: In the present case, the actual system is our distributed server system.

EFPA: A system for which EFPA is exact can be constructed by considering the actual system with a small modification, where an arbitrarily large exponentially distributed delay is introduced at each overflow between the moment a call is denied service and the time it attempts a new server. This large delay will induce Poisson independent arrivals in line with EFPA assumptions.

The surrogate system: This system is the actual system with the following modification. An arriving junior call that finds the server busy serving a senior call will preempt the senior call and will be served by the server. Assuming it has not visited all servers already, the senior call then immediately attempts another server that it has not visited before.

OPCA: OPCA is constructed by introducing the independence and Poisson assumptions to the surrogate system. This is equivalent to introducing arbitrarily long exponentially distributed delay to any overflow before it attempts a new server in a similar way that EFPA applies to the actual system.

As discussed in [32], the blocking probability of the surrogate system is higher than that of the actual system because overflowed senior calls that are being preempted by junior calls will visit only servers that they have not visited before. Such servers may be free at the time that the preemption takes place. This leads to a slight inefficiency which somewhat increases the blocking probability.

Then the blocking probability incurred by OPCA is less than that of the surrogate system because it eliminates all the dependencies and traffic variability. For the same reason EFPA underestimates the blocking probability of the actual system.

Since OPCA underestimates the blocking probability of the surrogate which incurs higher blocking than the actual system, it in fact cancels out error introduced by the surrogate and provides a better blocking probability approximation of the actual system than EFPA.

Another interesting fact to observe is that the OPCA system is the one most different from the actual system. It has both the preemptive priority structure as well as arbitrarily large delay which do not exist in the actual system. Yet it provides a very accurate approximation to the actual system.

3 Proof of the OPC Conjecture

We begin the proof by recalling the definition of Appell polynomials

$$S_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!}, \tag{10}$$

the truncated Maclaurin series of the exponential function, which will play a key role in the proof of the conjecture. At this point, we make the convention that the symbol N is taken to range over all positive integers unless otherwise stated, so that any statement made about N without further constraints is assumed to hold for $N = 1, 2, 3, \ldots$.

Let $A = aN$ be the offered load to the entire N-server system. The following function of A and N will be key in the proof of the OPC conjecture.

$$\begin{aligned} T_N(A) \equiv \left(\frac{A}{N} - 2\right)\left(\frac{N}{N+1}\right)^N S_N(A) S_N\left(A\left(1+\frac{1}{N}\right)\right) + S_N(A)^2 - \frac{A^{N+1}}{NN!} S_N(A) \\ + \left(1 - \frac{A}{N}\right)\left(\frac{N}{N+1}\right)^N \frac{A^N}{N!} S_N\left(A\left(1+\frac{1}{N}\right)\right) + \frac{A^{2N+1}}{N(N!)^2}. \end{aligned} \tag{11}$$

To prove the OPC conjecture, we begin by defining a sufficient condition, described by (13) below, for the OPC condition to hold (Lemma 1). Then we show that (13) is equivalent to $T_N(A) \geq 0$, so the latter becomes a sufficient condition for the OPC condition to hold (Proposition 1). In Appendix B, we prove that $T_N(A) \geq 0$. This completes the proof of the OPC conjecture.

Proposition 1. *If $T_N(A) \geq 0$ for all $A \geq 0$ and all integer N, then the OPC conjecture holds.*

By (5) we have

$$\begin{aligned} a_n &= \frac{(\sum_{i=0}^{n-1} a_i)^2}{1 + \sum_{i=0}^{n-1} a_i} - \sum_{i=1}^{n-1} a_i = \frac{\sum_{i=0}^{n-1} a_i + (\sum_{i=0}^{n-1} a_i)^2}{1 + \sum_{i=0}^{n-1} a_i} - \frac{\sum_{i=0}^{n-1} a_i}{1 + \sum_{i=0}^{n-1} a_i} - \sum_{i=1}^{n-1} a_i \\ &= \sum_{i=0}^{n-1} a_i - \frac{1 + \sum_{i=0}^{n-1} a_i}{1 + \sum_{i=0}^{n-1} a_i} + \frac{1}{1 + \sum_{i=0}^{n-1} a_i} - \sum_{i=1}^{n-1} a_i = a - 1 + \frac{1}{1 + \sum_{i=0}^{n-1} a_i}. \end{aligned} \tag{12}$$

For the purposes of this section, we write down the following inequality

$$\frac{1}{aP_{\text{exact}}(N)+1-a}+aP_{\text{exact}}(N) \geq \frac{1}{aP_{\text{exact}}(N+1)+1-a}. \tag{13}$$

Lemma 1. *If* (13) *holds, then the OPC conjecture is true.*

Proof. We define

$$\nu_n = \begin{cases} 1, & n=-1, \\ 1+\sum_{i=0}^{n} a_i, & n>-1, \end{cases} \tag{14}$$

and note

$$a_n = \nu_n - \nu_{n-1} \qquad n = 0,1,2,.... \tag{15}$$

As a result,

$$a_n = a-1+\frac{1}{1+\sum_{i=0}^{n-1} a_i} = a-1+\frac{1}{\nu_{n-1}}. \tag{16}$$

It then follows from equations (16), (9), and (8) that the OPC conjecture is equivalent to

$$\frac{1}{\nu_{N-1}} \leq aP_{\text{exact}}(N)+1-a. \tag{17}$$

It is easy to show that both sides of (17) are positive. Therefore,

$$\nu_{N-1} \geq \frac{1}{aP_{\text{exact}}(N)+1-a} \geq 1. \tag{18}$$

We now begin an inductive proof by induction on N and use equation (18) as the statement to be proved by induction. For the case $N=1$, $\nu_0 = 1+a$ and

$$aP_{\text{exact}}(N)+1-a = \frac{a^2}{1+a}+1-a = \frac{1}{1+a} = \frac{1}{\nu_0}, \tag{19}$$

so the inequality is in fact an equality for this case.

Assume that (18) is true for N. Using the fact that the function $x+(1/x)$ is increasing for $x \geq 1$, we have

$$\nu_{N-1}+\frac{1}{\nu_{N-1}} \geq \frac{1}{aP_{\text{exact}}(N)+1-a}+aP_{\text{exact}}(N)+1-a. \tag{20}$$

Adding $a-1$ to both sides and combining with (16) we obtain

$$\nu_N \geq \frac{1}{aP_{\text{exact}}(N)+1-a} + aP_{\text{exact}}(N). \tag{21}$$

From (18) we see that to complete the proof of the induction step it is enough to show (13).

This completes the proof of Lemma 1. $\square$

Proof of Proposition 1. Our strategy here is to prove that if $T_N(A) \geq 0$ for all $A \geq 0$, then (13) holds. We recall that

$$P_{\text{exact}}(N+1) = P_{\text{exact}} = \frac{\frac{(a(N+1))^{N+1}}{(N+1)!}}{\sum_{i=0}^{N+1} \frac{(a(N+1))^i}{i!}}, \tag{22}$$

and write

$$p_N = \frac{\frac{(a(N+1))^{N}}{N!}}{\sum_{i=0}^{N} \frac{(a(N+1))^i}{i!}}. \tag{23}$$

Then

$$P_{\text{exact}}(N+1) = \frac{\frac{a(a(N+1))^{N}}{N!}}{\frac{(a(N+1))^{N+1}}{(N+1)!} + \sum_{i=0}^{N} \frac{(a(N+1))^i}{i!}} = \frac{ap_N}{1+ap_N}. \tag{24}$$

We make this substitution for $P_{\text{exact}}(N+1)$ in (13) to see that it is enough to prove

$$\frac{1}{aP_{\text{exact}}(N)+1-a} + aP_{\text{exact}}(N) \geq \frac{1}{a\frac{ap_N}{1+ap_N}+1-a}; \tag{25}$$

in other words, we have to prove

$$\frac{aP_{\text{exact}}(N)(aP_{\text{exact}}(N)+1-a)+1}{aP_{\text{exact}}(N)+1-a} \geq \frac{1+ap_N}{1-a+ap_N}. \tag{26}$$

This reduces to

$$P_{\text{exact}}(N)^2\big(1-a+ap_N\big) + P_{\text{exact}}(N)(a-ap_N-2) + p_N \geq 0. \tag{27}$$

Dividing by $P_{\text{exact}}(N)^2 p_N$, we require

$$\frac{a-2}{P_{\text{exact}}(N)p_N} + \frac{1}{P_{\text{exact}}(N)^2} - \frac{a}{P_{\text{exact}}(N)} + \frac{1-a}{p_N} + a \geq 0. \tag{28}$$

Next we convert to the Appell polynomials referred to earlier by noting that

$$\frac{1}{P_{\text{exact}}(N)} = S_N(aN)\frac{N!}{(aN)^N}, \tag{29}$$

and

$$\frac{1}{p_N} = S_N(a(N+1))\frac{N!}{(a(N+1))^N}. \tag{30}$$

After substitution of these two into (28), it is enough to prove that

$$\frac{(a-2)(N!)^2}{(aN)^N(a(N+1))^N}S_N(aN)S_N(a(N+1)) + \frac{(N!)^2}{(aN)^{2N}}S_N(aN)^2 \\ - \frac{aS_N(aN)N!}{(aN)^N} + \frac{(1-a)N!}{(a(N+1))^N}S_N(a(N+1)) + a \geq 0. \tag{31}$$

Now we multiply through by $\frac{(aN)^{2N}}{(N!)^2}$ and set $A = aN$ to obtain that

$$T_N(A) = \left(\frac{A}{N}-2\right)\left(\frac{N}{N+1}\right)^N S_N(A)S_N\left(A\left(1+\frac{1}{N}\right)\right) + S_N(A)^2 - \frac{(A)^{N+1}}{NN!}S_N(A) \\ + \left(1-\frac{A}{N}\right)\left(\frac{N}{N+1}\right)^N \frac{A^N}{N!}S_N\left(A\left(1+\frac{1}{N}\right)\right) + \frac{A^{2N+1}}{N(N!)^2} \geq 0 \tag{32}$$

means that (13) holds and by Lemma 1 this means the OPC conjecture is true. This proves Proposition 1. □

We are now in position to deconstruct $T_N(A)$. We note that $T_N(A)$ is a polynomial of degree $2N+1$. We write it then as its (finite) Maclaurin series:

$$T_N(A) = T_N(0) + T_N'(0)A + \frac{T_N''(0)}{2!}A^2 + \frac{T_N^{(3)}(0)}{3!}A^2 + \cdots + \frac{T_N^{(2N+1)}(0)}{(2N+1)!}A^{2N+1}. \tag{33}$$

In Appendix B we show that $T_N(A) \geq 0$ for all $A \geq 0$. Therefore, by Proposition 1, the proof of the OPC conjecture is complete.

4 The Case: $N \to \infty$

Now that the proof is complete, the following corollary establishes that OPCA (as well as EFPA) approaches the exact solution for a large N.

Corollary

Both P_{EFPA} and P_{OPCA} approach $P_{\text{exact}} = \mathbf{E}_N(aN)$ as N tends to infinity with a fixed.

Proof. We know from page 498 of [24] that $\mathbf{E}_N(aN)$ approaches zero for $a \leq 1$ and approaches $1 - \frac{1}{a}$ for $a > 1$ as N tends to infinity with a fixed.

For all $a > 0$, P_{EFPA} is obtained by finding the non-negative unique solution b of (3) and then the blocking probability is b^N. In all cases the result is trivially non-negative. Since the EFPA value for the blocking probability never exceeds the exact result given by the Erlang B formula, it follows immediately from the results of [24] and our proof of the OPC Conjecture that both P_{EFPA} and P_{OPCA} tend to zero when $a \leq 1$. It follows that, to show that both P_{EFPA} and P_{OPCA} have the same limit as P_{exact}, we only need to consider the case $a > 1$ for P_{EFPA}.

Suppose b does not tend to 1. Then, for any fixed M, $b < \left(1 - \frac{M}{N}\right)$ for infinitely many N. For these N, by (3),

$$b = a - ab^N > a\left[1 - \left(1 - \frac{M}{N}\right)^N\right] \rightarrow a(1 - e^{-M})$$

as $N \rightarrow \infty$. Now choose M so that $a(1 - e^{-M}) > 1$, and this gives a contradiction. Hence $b \rightarrow 1$ as $N \rightarrow \infty$. Therefore, by (3), the blocking probability is

$$b^N = 1 - \frac{b}{a} \rightarrow 1 - \frac{1}{a},$$

which coincides with the result in the exact case. $\square$

5 New Bounds and Asymptotic Results under Critical Loading

A system where the offered traffic load is equal to the number of servers, namely $a = 1$, is called *critically loaded* [10]. It is known that for a critically loaded Erlang B system, as the number of servers and the offered load approach infinity, the blocking probability approaches zero. These conditions are important because they reflect the exponential increase of capacity and traffic in telecommunications networks in recent times.

5.1 The bounds and asymptotic behavior of OPCA for $a = 1$

Let $A_n = \sum_{i=0}^{n} a_i$. Then, from Eq. (5) and for $a_0 = a = 1$, we have

$$A_{n+1} = 1 + \frac{A_n^2}{1 + A_n}. \tag{34}$$

Moreover from Eq. (5), we have

$$a_{n+1} = \frac{A_n^2}{1 + A_n} - A_n + 1 = \frac{1}{A_n + 1}. \tag{35}$$

Substituting into Eq. (6), we have for $N \geq 1$

$$P_{\text{OPCA}} = \frac{1}{A_{N-1} + 1}. \tag{36}$$

By (34), we have

$$A_{n+1} = 1 + \frac{1 + 2A_n + A_n^2}{1 + A_n} - \frac{1 + 2A_n}{1 + A_n} = 2 + A_n - 1 - \frac{A_n}{1 + A_n} = 1 + A_n - \frac{A_n}{1 + A_n}.$$

Therefore,

$$A_{n+1}^2 = 1 + A_n^2 + \left(\frac{A_n}{1 + A_n}\right)^2 + 2A_n - 2\frac{A_n^2}{1 + A_n} - 2\frac{A_n}{1 + A_n} = 1 + A_n^2 + \left(\frac{A_n}{1 + A_n}\right)^2. \tag{37}$$

Since $A_n \geq 1$, we obtain

$$\frac{5}{4} + A_n^2 \leq A_{n+1}^2 < 2 + A_n^2, \tag{38}$$

and iteration, noting that $A_0 = 1$ provides bounds for A_n.

$$\frac{5}{4}n + 1 \leq A_n^2 < 2n + 1. \tag{39}$$

Now from (37) and (39)

$$A_{n+1}^2 = 1 + A_n^2 + \left(1 - \frac{1}{1+A_n}\right)^2 \geq 1 + A_n^2 + \left(1 - \frac{1}{1+\sqrt{\frac{5}{4}n}}\right)^2. \tag{40}$$

Now we sum this equation from $n = 0$ to $N-1$, replacing A_0, which equals 1, by $\left(1 - \frac{1}{1+\sqrt{\frac{5}{4}N}}\right)^2$ to obtain

$$A_N^2 \geq N + \sum_{n=1}^{N} \left(1 - \frac{1}{1+\sqrt{\frac{5}{4}n}}\right)^2 \geq 2N - 2\sum_{n=1}^{N}\left(\frac{1}{\sqrt{\frac{5}{4}n}}\right) = 2N - \frac{4}{\sqrt{5}}\sum_{n=1}^{N}\frac{1}{\sqrt{n}} \geq 2N - \frac{8}{\sqrt{5}}\sqrt{N}. \tag{41}$$

Similarly,

$$A_N^2 \leq N + 1 + \sum_{n=0}^{N-1}\left(1 - \frac{1}{1+\sqrt{2n+1}}\right)^2. \tag{42}$$

For $N \geq 3$

$$A_N^2 \leq N + 3 + \sum_{n=3}^{N-1}\left(1 - \frac{1}{1+\sqrt{2n+1}}\right)^2 \leq 2N. \tag{43}$$

Combining equations (41) and (43), we obtain

$$2N - \frac{8}{\sqrt{5}}\sqrt{N} \leq A_N^2 \leq 2N. \tag{44}$$

Note that there is an opportunity here to modify (44) to make it apply to both $N = 1$ and $N = 2$, but this will weaken the inequality, and we refrain from doing it. It follows from (36) that

$$\frac{1}{1+\sqrt{2N-2}} \leq P_{\text{OPCA}} = \frac{1}{A_{N-1}+1} \leq \frac{1}{\sqrt{2(N-1) - \frac{4\sqrt{2}}{\sqrt{5}}\sqrt{2(N-1)}}}. \tag{45}$$

We write $M = 2(N-1)$ and calculate as follows

$$\begin{aligned} 0 &\leq P_{\text{OPCA}} - \frac{1}{1+\sqrt{M}} \leq \frac{1}{\sqrt{M - \frac{4\sqrt{2}}{\sqrt{5}}\sqrt{M}}} - \frac{1}{1+\sqrt{M}} \\ &= \frac{1}{(1+\sqrt{M})(\sqrt{M - \frac{4\sqrt{2}}{\sqrt{5}}\sqrt{M}})} + \frac{\frac{4\sqrt{2}}{\sqrt{5}}\sqrt{M}}{(1+\sqrt{M})(\sqrt{M - \frac{4\sqrt{2}}{\sqrt{5}}\sqrt{M}})(\sqrt{M} + \sqrt{M - \frac{4\sqrt{2}}{\sqrt{5}}\sqrt{M}})}. \end{aligned} \tag{46}$$

Noting that $\sqrt{M - \frac{4\sqrt{2}}{\sqrt{5}}\sqrt{M}} \geq \sqrt{\frac{M}{2}}$ provided $M \geq 26$, we have

$$0 \leq P_{\text{OPCA}} - \frac{1}{1+\sqrt{M}} \leq \frac{\sqrt{2}}{M} + \frac{8/\sqrt{5}}{M(1+\frac{1}{\sqrt{2}})} = \sqrt{2}\left(1 + \frac{8}{\sqrt{5}(1+\sqrt{2})}\right)\frac{1}{M}. \tag{47}$$

Thus we obtain for $N \geq 14$

$$0 \leq P_{\text{OPCA}} - \frac{1}{1+\sqrt{2N-2}} \leq B\frac{1}{N-1}, \tag{48}$$

where

$$B = \left(1 + \frac{8}{\sqrt{5}(1+\sqrt{2})}\right)\frac{1}{\sqrt{2}}. \tag{49}$$

These result in the following bounds for P_{OPCA}:

$$\frac{1}{1+\sqrt{2N-2}} \leq P_{\text{OPCA}} \leq \frac{1}{1+\sqrt{2N-2}} + B\frac{1}{N-1}, \tag{50}$$

where the upper bound starts from $N = 14$ and lower bound starts from $N = 3$. An important observation to make here is that for a large N, the blocking probability of OPCA is of the same order as that of the Erlang B formula, namely $1/\sqrt{N}$. This explains the closeness of OPCA to the exact solution (i.e. the Erlang B formula) shown in Figure 2. As observed from (50) and verified by Figure 2 the lower bound of OPCA is a tight bound for large N and OPCA approaches to this lower bound as N approaches infinity.

It is interesting to note that our lower bound of OPCA can serve as a tight lower bound of the Erlang B formula because OPCA is itself a tight lower bound of the Erlang B formula, although it is worth noting that the bound of [9] is slightly tighter than our lower bound of OPCA for $N \geq 5$. Note also that our bound applies only for the case of critical loading while that of [9] is not so constrained.

5.2 The bounds and asymptotic behavior of EFPA for $a = 1$

In this section we analyze the asymptotic behavior of the EFPA for $a = 1$ and for large N. Let $P_N \equiv P_{EFPA}$ for $N \geq 1$. From Eqs. (3) and (4) and with $a = 1$, we have

$$1 - P_N - P_N^{\frac{1}{N}} = 0. \tag{51}$$

Let

$$f_N(x) \equiv 1 - x - x^{\frac{1}{N}}. \tag{52}$$

Note that $f_N(x)$ is a monotonically decreasing function of x in the interval $[0, 1]$, that $f_N(0) = 1$ and that $f_N(1) < 0$ and so $f_N(x)$ has exactly one zero in that interval, namely P_N.

We first show that P_{EFPA} is upper bounded by $\frac{\log N}{N}$ for $N \geq 2$. Let $y_N \equiv \frac{\log N}{N}$. Put y_N into (52), to obtain

$$h(N) = f_N(\frac{\log N}{N}) = 1 - \frac{\log N}{N} - \Big(\frac{\log N}{N}\Big)^{1/N} \tag{53}$$

Differentiation with respect to N yields

$$\begin{aligned}
h'(N) &= \frac{1}{N^2 \log N}\Big[\Big(\frac{\log N}{N}\Big)^{1/N}\Big(\big(\log N\big)\log\Big(\frac{\log N}{N}\Big) + \log N - 1\Big) + \big(\log N\big)^2 - \log N\Big] \\
&= \frac{1}{N^2 \log N}\Big[\Big(\frac{\log N}{N}\Big)^{1/N}\Big(\log N \log\log N - \big(\log N\big)^2 + \log N - 1\Big) + \big(\log N\big)^2 - \log N\Big] \\
&\geq \frac{1}{N^2 \log N}\Big[1 - \Big(\frac{\log N}{N}\Big)^{1/N}\Big]\Big(\big(\log N\big)^2 - \log N\Big)
\end{aligned}$$

provided $N \geq 6$ since in that case $\log N \log\log N > 1$. It follows that $h(N)$ is strictly increasing for $N \geq 6$ and numerical calculations show that this property extends down to $N \geq 2$.

Now consider

$$\begin{aligned}
\lim_{N\to\infty} h(N) &= \lim_{N\to\infty} 1 - \frac{\log N}{N} - \Big(\frac{\log N}{N}\Big)^{\frac{1}{N}} \\
&= 1 - \lim_{N\to\infty}\Big(\frac{\log N}{N}\Big)^{1/N} = 0
\end{aligned}$$

If follows that $h(N) < 0$ for $N \geq 2$ and hence that $P_{EFPA} < (\log N)/N$ for all $N \geq 2$.

Now, we try to find an asymptotic lower bound for P_{EFPA}. Let $z_N = \frac{d\log(N)}{N}$ for some choice of $1 > d > 0$ to be made later and put it into (52), to obtain

$$f_N(z_N) = 1 - \frac{d\log(N)}{N} - \Big(\frac{d\log(N)}{N}\Big)^{\frac{1}{N}}. \tag{54}$$

Observe that by the Mean Value Theorem, for $0 < \beta < 1$

$$e^x \leq 1 + \beta x \text{ for } \log\beta < x \leq 0, \tag{55}$$

so that, if we apply this inequality to

$$x = \frac{\log\left(\frac{d\log N}{N}\right)}{N}$$

for sufficiently large N to achieve $\log\beta < x < 0$, we obtain

$$\Big(\frac{d\log N}{N}\Big)^{\frac{1}{N}} < 1 + \frac{\beta\log\left(\frac{d\log N}{N}\right)}{N}. \tag{56}$$

Putting it into (54), we have

$$f_N(z_N) > -\frac{d\log N}{N} - \frac{\beta\log\left(\frac{d\log N}{N}\right)}{N}. \tag{57}$$

In order that z_N be a lower bound, we need $f(z_N) > 0$ which by (57) follows from:

$$-\frac{d\log N}{N} - \frac{\beta\log\left(\frac{d\log N}{N}\right)}{N} > 0. \tag{58}$$

This is equivalent to

$$N^{(1-\frac{d}{\beta})} > d\log N. \tag{59}$$

For any $d < \beta$ the above inequality holds for arbitrarily large N.

Therefore, for all $N \geq 2$, we have

$$P_{EFPA} < \frac{\log N}{N} \tag{60}$$

and for any $d < \beta < 1$, there is an $N(d)$ such that inequality (59) holds for $N > N(d)$, so

$$\frac{d \log N}{N} < P_{EFPA}. \tag{61}$$

Since, for arbitrarily large N, we can choose d to be very close to β such that $d < \beta$, and we can also choose β very close to 1 that satisfies $\beta < 1$, as N approaches infinity, we have

$$\frac{\log N}{N} > P_{EFPA} \asymp \frac{\log N}{N}. \tag{62}$$

Recall that the upper bound of P_{EFPA} starts from $N = 2$.

Note that these results are different from those obtained in [10] where a different limit regime has been considered. The authors of [10] considered a limited availability case where the traffic can overflow only a limited number of times, and this limit stays fixed although the total number of servers approaches infinity. In the present case there is no such limit.

These bounds signify the improvement achieved by OPCA over EFPA under critical loading. We demonstrate numerically in Figure 2 that these bounds are quite tight and we also observe from the numerical results that the significant difference between the upper bound of EFPA and the lower bound of OPCA starts from $N = 3$.

We capture the results of this section in Table 1 and recall that for $N = 1$ and $N = 2$, OPCA gives the same results as the Erlang B formula. It shows that the ratio of the upper bound of Erlang B formula [9] to the lower bound of OPCA is bounded by $\sqrt{2}$ for any $N \geq 1$, while the ratio of the lower bound of the Erlang B formula [9] to the upper bound of EFPA approaches infinity as $N \to \infty$. This OPCA error bound provides further evidence for the accuracy of OPCA in the present case and helps in understanding why the OPC Conjecture is difficult to prove.

6 Conclusions

By completing the proof of the OPC Conjecture, we have established that for the symmetric distributed server system under consideration, OPCA is always at least as accurate as EFPA. Furthermore, $P_{\text{OPCA}} = P_{\text{exact}}$ for the cases $N = 1$ and $N = 2$, and we have now established that

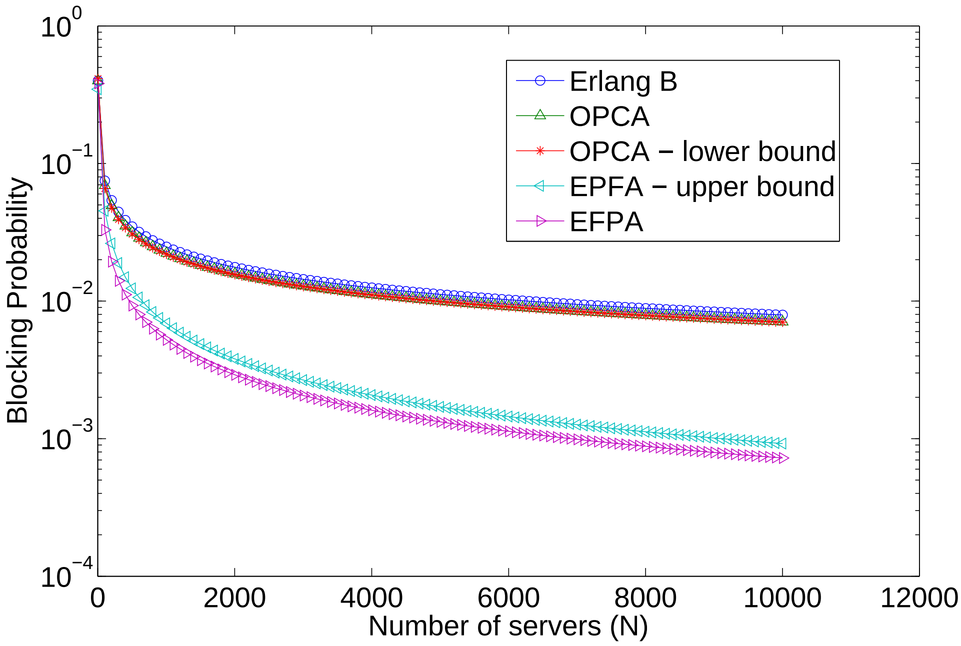

Figure 2: Blocking probability versus number of servers for Erlang B, OPCA and EFPA.

Table 1: Summaries of bounds for Erlang B, OPCA and EFPA for $N \geq 3$.

Erlang B Upper Bound [9]	$\frac{1}{1+\sqrt{N}}$
Erlang B Lower Bound [9]	$\frac{1}{1+\sqrt{\frac{\pi}{2}N}}$
OPCA Lower Bound	$\frac{1}{1+\sqrt{2N-2}}$
EFPA Upper Bound	$\frac{\log N}{N}$

this is also true at the limit of large N. (Note that the latter applies also to EFPA.)

Furthermore, we have provided new results of tight bounds and asymptotic behavior for both OPCA and EFPA under the important case of critical loading that show the closeness of OPCA to the exact solution and the significant improvement achieved by OPCA over EFPA. We have also shown that the ratio of the Erlang B formula to OPCA is bounded by $\sqrt{2}$ for any $N \geq 1$, while the ratio of the Erlang B formula to EFPA approaches infinity as $N \to \infty$.

The results of this work establish a close relationship between two seemingly unrelated models: the Erlang B and the priority queue model OPCA and demonstrate a potential of improving accuracy by surrogate-based approaches for many other overflow loss model applications.

Appendix

A Existence and Uniqueness of the Solutions for EFPA and OPCA

A1 Existence and Uniqueness of the Fixed Point Solution for EFPA

Eq. (3), namely $b = a(1 - b^N)$, is the fixed-point equation for EFPA. Rearranging, we obtain: $0 = a(1 - b^N) - b$.

Define: $f(x) = a(1 - x^N) - x$.

To prove existence and uniqueness, we show that $f(x) = 0, 0 \leq x \leq 1$, has a unique solution. To establish solution existence, we observe that $f(x)$ is continuous and for $x = 0$, $f(x) > 0$ and for $x = 1$, $f(x) < 0$. Therefore, a solution exists by the intermediate-value theorem.

To establish solution uniqueness, suppose $f(x_1) = f(x_2) = 0$ for $0 < x_1 < x_2 < 1$. The mean-value theorem requires the existence of a y satisfying $f(x_2) - f(x_1) = f'(y)(x_2 - x_1)$ or $f'(y) = 0$, where $x_1 \leq y \leq x_2$. This provides us with a contradiction because simple calculations reveal that $f'(x) < 0$. Therefore, $x_1 = x_2$.

A2 Existence and Uniqueness of the Solution for OPCA

From Eq. (5), we observe that a unique a_{n+1} is recursively obtained from $a_0, a_1, \ldots, a_n$. And, it takes N steps before we obtain a_N (the last an value). Then, we can obtain a unique P_{OPCA} which is given by a_N/a.

B Proof that $T_N(A) \geq 0$ for all $A \geq 0$

We show that $T_N(A)$ is positive on the whole positive real line and we do that by showing that each of its derivatives at 0 is positive. In other words, we divide the original problem into several sub-problems, which are relatively easier to solve, and solve them individually. Then some of the sub-problems are further subdivided to enable solution. We continue to apply this "divide and conquer" approach to each sub-problem until they are all solved.

It will simplify the formulae involving the derivatives of $T_N(A)$ if we define $\alpha = \frac{N}{N+1}$. Then we have

$$\begin{aligned} T_N(A) = {} & \frac{A}{N}\alpha^N S_N(A) S_N\left(\frac{A}{\alpha}\right) - 2\alpha^N S_N(A) S_N\left(\frac{A}{\alpha}\right) + S_N(A)^2 \\ & - \frac{A^{N+1}}{NN!} S_N(A) + \frac{\alpha^N A^N}{N!} S_N\left(\frac{A}{\alpha}\right) - \frac{\alpha^N A^{N+1}}{NN!} S_N\left(\frac{A}{\alpha}\right) + \frac{A^{2N+1}}{N(N!)^2}. \end{aligned} \tag{B1}$$

Now we approach the task of calculating the derivatives of each of the terms in this sum. The following equations will help here.

$$S'_n(x) = S_{n-1}(x) \qquad n = 0, 1, \ldots, \tag{B2}$$

where $S_{-1}(x)$ is identically zero, and so

$$S_n(0) = \begin{cases} 0 & n = -1 \\ 1 & \text{otherwise.} \end{cases} \tag{B3}$$

We record as a series of lemmas the derivatives of each of the terms in $T_N(A)$. In proving these lemmas, we will rely on (B2) in addition to Leibnitz rule in the guise

$$\frac{d^n}{dx^n}[x^c f(ax) g(bx)] = \sum_{i=0}^{n} \binom{n}{i} c(c-1)\cdots(c-i+1) x^{c-i} \frac{d^{n-i}}{dx^{n-i}}[f(ax)g(bx)] \tag{B4}$$

where

$$\frac{d^n}{dx^n}[f(ax)g(bx)] = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} g^{(i)}(ax) f^{(n-i)}(bx),$$

and $f^{(n)}(x)$ is shorthand for $d^n f(x)/dx^n$ and c is integer.

Lemma 2.

$$\frac{d^n}{dA^n}\left[\frac{A}{N}\alpha^N S_N(A) S_N\left(\frac{A}{\alpha}\right)\right] = \frac{\alpha^N}{N}\left[n\sum_{i=0}^{n-1}\binom{n-1}{i} S_{N-i}\left(\frac{A}{\alpha}\right) S_{N-n+1+i}(A)\left(\frac{1}{\alpha}\right)^i \right.$$
$$\left. + A\sum_{i=0}^{n}\binom{n}{i} S_{N-i}\left(\frac{A}{\alpha}\right) S_{N-n+i}(A)\left(\frac{1}{\alpha}\right)^i\right],$$
$$\frac{d^n}{dA^n}\left[\frac{A}{N}\alpha^N S_N(A) S_N\left(\frac{A}{\alpha}\right)\right]_{A=0} = \begin{cases} \frac{\alpha^N}{N} n \sum_{i=0}^{n-1}\binom{n-1}{i}\left(\frac{1}{\alpha}\right)^i, & 0 \leq n \leq N+1, \\ \frac{\alpha^N}{N} n \sum_{i=n-N-1}^{N}\binom{n-1}{i}\left(\frac{1}{\alpha}\right)^i, & N+2 \leq n \leq 2N+1, \\ 0, & n > 2N+1. \end{cases} \tag{B5}$$

Proof. The first equality follows immediately by rewriting (B4). In particular, we set $f = g = S_N$, $a = 1/\alpha$, $b = 1$ and $c = 1$.

We evaluate at $A = 0$. The second term is multiplied by A and is thus immediately eliminated. The form of the first term depends on n. Three cases arise, namely, those stated in Eq. (B5) For $0 \leq n \leq N+1$, the product $S_{N-i}(0)S_{N-n+1+i}(0) = 1$, as a consequence of (B2). The rest is straightforward.

□

Lemma 3.

$$\frac{d^n}{dA^n}\left[-2\alpha^N S_N(A) S_N\left(\frac{A}{\alpha}\right)\right] = -2\alpha^N \sum_{i=0}^{n}\binom{n}{i} S_{N-i}\left(\frac{A}{\alpha}\right) S_{N-n+i}(A)\left(\frac{1}{\alpha}\right)^i \tag{B6}$$
$$\frac{d^n}{dA^n}\left[-2\alpha^N S_N(A) S_N\left(\frac{A}{\alpha}\right)\right]_{A=0} = \begin{cases} -2\alpha^N \sum_{i=0}^{n}\binom{n}{i}\left(\frac{1}{\alpha}\right)^i, & 0 \leq n \leq N, \\ -2\alpha^N \sum_{i=n-N}^{N}\binom{n}{i}\left(\frac{1}{\alpha}\right)^i, & N+1 \leq n \leq 2N. \\ 0, & n > 2N. \end{cases}$$

Proof. The first equality follows immediately from a rewriting of (B4). In particular, we set $f = g = S_N$, $a = 1/\alpha$, $b = 1$ and $c = 0$. Again the rest is straightforward.

□

Lemmas 4-8 involve differentiation of the general form given by (B4). This can be accomplished in the same way as detailed in the proofs of Lemmas 2 and 3. We provide a summary in Table B1, but omit the proofs of Lemmas 4–8, which involve straightforward calculations.

Table B1: Form of (B4) for Lemmas 2-8

	f	g	a	b	c
Lemma 2	S_N	S_N	$1/\alpha$	1	1
Lemma 3	S_N	S_N	$1/\alpha$	1	0
Lemma 4	S_N	S_N	1	1	0
Lemma 5	S_N	1	1	1	$N+1$
Lemma 6	S_N	1	$1/\alpha$	1	N
Lemma 7	S_N	1	$1/\alpha$	1	$N+1$
Lemma 8	1	1	1	1	$2N+1$

Lemma 4.

$$\frac{d^n}{dA^n}\left[S_N(A)S_N(A)\right] = \sum_{i=0}^{n}\binom{n}{i} S_{N-i}(A)S_{N-n+i}(A) \tag{B7}$$

$$\frac{d^n}{dA^n}\left[S_N(A)S_N(A)\right]_{A=0} = \begin{cases} \sum_{i=0}^{n}\binom{n}{i}, & 0 \le n \le N, \\ \sum_{i=n-N}^{N}\binom{n}{i}, & N < n \le 2N \\ 0, & n > 2N. \end{cases}$$

Lemma 5.

$$\frac{d^n}{dA^n}\left[-\frac{1}{NN!}A^{N+1}S_N(A)\right] = -\frac{1}{NN!}\sum_{i=0}^{n}\binom{n}{i}\frac{(N+1)!}{(N+1-i)!}A^{N+1-i}S_{N-n+i}(A) \tag{B8}$$

$$\frac{d^n}{dA^n}\left[-\frac{1}{NN!}A^{N+1}S_N(A)\right]_{A=0} = \begin{cases} 0, & 0 \le n \le N, \\ -\frac{(N+1)!}{NN!}\binom{n}{N+1}, & N+1 \le n \le 2N+1 \\ 0, & n > 2N+1. \end{cases}$$

Lemma 6.

$$\frac{d^n}{dA^n}\left[\frac{\alpha^N}{N!}A^N S_N\left(\frac{A}{\alpha}\right)\right] = \frac{\alpha^N}{N!}\sum_{i=0}^{n}\binom{n}{i}\frac{N!}{(N-i)!}A^{N-i}S_{N-n+i}\left(\frac{A}{\alpha}\right)\left(\frac{1}{\alpha}\right)^{n-i}$$

$$\frac{d^n}{dA^n}\left[\frac{\alpha^N}{N!}A^N S_N\left(\frac{A}{\alpha}\right)\right]_{A=0} = \begin{cases} 0, & 0 \le n \le N-1, \\ \alpha^N \binom{n}{N}\left(\frac{1}{\alpha}\right)^{n-N} & N \le n \le 2N \\ 0 & n > 2N. \end{cases} \tag{B9}$$

Lemma 7.

$$\frac{d^n}{dA^n}\left[\frac{-\alpha^N}{NN!}A^{N+1} S_N\left(\frac{A}{\alpha}\right)\right] = \frac{-\alpha^N}{NN!}\sum_{i=0}^{n}\binom{n}{i}\frac{(N+1)!}{(N+1-i)!}A^{N+1-i}S_{N-n+i}\left(\frac{A}{\alpha}\right)\left(\frac{1}{\alpha}\right)^{n-i}$$

$$\frac{d^n}{dA^n}\left[\frac{-\alpha^N}{NN!}A^{N+1} S_N\left(\frac{A}{\alpha}\right)\right]_{A=0} = \begin{cases} 0, & 0 \le n \le N, \\ -\alpha^N \frac{N+1}{N}\binom{n}{N+1}(\frac{1}{\alpha})^{n-N-1} & N+1 \le n \le 2N+1 \\ 0 & n > 2N+1. \end{cases} \tag{B10}$$

Lemma 8.

$$\frac{d^n}{dA^n}\left[\frac{A^{2N+1}}{N(N!)^2}\right]_{A=0} = \begin{cases} \frac{(2N+1)!}{N(N!)^2}, & n = 2N+1 \\ 0, & \text{otherwise}. \end{cases} \tag{B11}$$

The next step is to collect all of these terms together to obtain $T_N^{(n)}(0)$. The formula is different for different ranges of n, as the previous lemmas indicate, and the ranges depend on N.

Lemma 9.

$$T_N^{(n)}(0) =$$
$$\begin{cases} \frac{\alpha^N}{N} n\left(1+\frac{1}{\alpha}\right)^{n-1} - 2\alpha^N\left(1+\frac{1}{\alpha}\right)^n + 2^n & 0 \le n < N \\ \frac{\alpha^N}{N} n\left(1+\frac{1}{\alpha}\right)^{n-1} - 2\alpha^N\left(1+\frac{1}{\alpha}\right)^n + 2^n + \alpha^N & n = N \\ \frac{\alpha^N}{N}(N+1)\left(1+\frac{1}{\alpha}\right)^N - 2\alpha^N \sum_{i=1}^N \binom{N+1}{i}\left(\frac{1}{\alpha}\right)^i + \sum_{i=1}^N \binom{N+1}{i} & \\ \quad -\frac{N+1}{N} + \alpha^{N-1}(N+1) - \alpha^N \frac{N+1}{N} & n = N+1 \\ \frac{\alpha^N}{N} n \sum_{i=n-N-1}^N \binom{n-1}{i}\left(\frac{1}{\alpha}\right)^i - 2\alpha^N \sum_{i=n-N}^N \binom{n}{i}\left(\frac{1}{\alpha}\right)^i + \sum_{i=n-N}^N \binom{n}{i} & \\ \quad -\frac{N+1}{N}\binom{n}{N+1} + \alpha^N \binom{n}{N}\left(\frac{1}{\alpha}\right)^{n-N} - \alpha^N \frac{N+1}{N}\binom{n}{N+1}\left(\frac{1}{\alpha}\right)^{n-N-1} & N+2 \le n \le 2N \\ 0 & \text{otherwise.} \end{cases} \tag{B12}$$

B1 The Easy Cases

We now have to verify that each of the derivatives of $T_N(A)$ is non-negative to complete the proof of the OPC conjecture. The cases are

$$\{0 \le n < N\}, \{N\}, \{N+1\}, \{N+2 \le n \le 2N\}. \tag{B13}$$

The most difficult case turns out to be the last of these: $\{N+2 \le n \le 2N\}$, and we leave it till the next and subsequent sections. We deal with all of the three other cases here. It simplifies formulae a little to consider $n!T_N^{(n)}$ in each case.

Lemma 10.

$$\frac{\alpha^N}{N} n \left(1+\frac{1}{\alpha}\right)^{n-1} - 2\alpha^N \left(1+\frac{1}{\alpha}\right)^n + 2^n \ge 0 \qquad n = 1, 2, ..., N-1. \tag{B14}$$

Proof. We divide by 2^n, replace α by $\frac{N}{N+1}$ and rearrange to obtain in place of the left side of

(B14) that

$$1+\frac{n}{2N}\frac{1}{\left(1+\frac{1}{N}\right)^N}\left(1+\frac{1}{2N}\right)^{n-1}-2\frac{1}{\left(1+\frac{1}{N}\right)^N}\left(1+\frac{1}{2N}\right)^{n}$$
$$=1-\frac{1}{\left(1+\frac{1}{N}\right)^N}\left(1+\frac{1}{2N}\right)^{n-1}\left[2-\frac{n-2}{2N}\right]. \quad \text{(B15)}$$

For this to be non-negative for $0 \leq n < N$ we observer that the logarithm (to base e) of the second term is less than 0; that is,

$$-N\log\left(1+\frac{1}{N}\right)+(n-1)\log\left(1+\frac{1}{2N}\right)+\log 2-\log\left(\frac{1}{1-\frac{(n-2)}{4N}}\right)<0. \quad \text{(B16)}$$

Using the inequalities

$$x-\frac{x^2}{2}<\log(1+x)<x, \quad \log\left(\frac{1}{1-x}\right)>x \text{ for } 0\leq x\leq 1. \quad \text{(B17)}$$

we see that the expression in (B16) does not exceed

$$-N\left(\frac{1}{N}-\frac{1}{2N^2}\right)+\frac{(n-1)}{2N}+\log 2-\frac{(n-2)}{4N}\leq(\log 2-1)+\frac{n+2}{4N}\leq\log 2-1+\frac{N+1}{4N}.$$

which is clearly negative for $N \geq 10$. The remaining small collection of cases of the original inequality can be checked by hand. We have used Maxima to check the cases $N < 10$. □

Lemma 11.

$$\alpha^N\left(1+\frac{1}{\alpha}\right)^{N-1}-2\alpha^N\left(1+\frac{1}{\alpha}\right)^N+2^N+\alpha^N\geq 0. \quad \text{(B18)}$$

Proof. For $N \geq 2$, $\alpha^N < \frac{1}{2}$, so that

$$\alpha^N\left(1+\frac{1}{\alpha}\right)^{N-1}-2\alpha^N\left(1+\frac{1}{\alpha}\right)^N+2^N+\alpha^N=2^N+\alpha^N-2^N\left(\frac{3N+2}{2N+1}\right)\left(1-\frac{1}{2(N+1)}\right)^N$$
$$\geq 2^N\left[1-\left(\frac{3N+2}{2N+1}\right)\left(1-\frac{1}{2(N+1)}\right)^N\right]. \quad \text{(B19)}$$

As in the previous lemma, we do this by showing that the logarithm of $\left(\frac{3N+2}{2N+1}\right)\left(1-\frac{1}{2(N+1)}\right)^N$

is negative using the inequalities given in (B17). This reduces to

$$\log\frac{3}{2}+\log\Big(1+\frac{1}{6N+3}\Big)+N\log\Big(1-\frac{1}{2(N+1)}\Big)\le\log\frac{3}{2}-\frac{1}{2}+\frac{1}{2(N+1)}+\frac{1}{6N+3}. \quad \text{(B20)}$$

which is negative for $N \geq 7$. The remaining few cases can be checked by hand (on the original inequality (B18)), and we have used Maxima to do this. It is an indication of the tightness of this inequality that the first four cases all give the value zero. $\square$

Lemma 12.

$$\frac{\alpha^N}{N}(N+1)\Big(1+\frac{1}{\alpha}\Big)^N-2\alpha^N\sum_{i=1}^{N}\binom{N+1}{i}\Big(\frac{1}{\alpha}\Big)^i+\sum_{i=1}^{N}\binom{N+1}{i}-\frac{N+1}{N}$$
$$+\alpha^N\binom{N+1}{N}\Big(\frac{1}{\alpha}\Big)-\alpha^N\frac{N+1}{N}\geq 0. \quad \text{(B21)}$$

Proof. We begin by rewriting the expression as

$$\alpha^{N-1}\Big(1+\frac{1}{\alpha}\Big)^N-2\alpha^N\Bigg[\Big(1+\frac{1}{\alpha}\Big)^{N+1}-1-\frac{1}{\alpha^{N+1}}\Bigg]+$$
$$2^{N+1}-2-\frac{1}{\alpha}+\alpha^{N-1}(N+1)-\alpha^{N-1}$$
$$=2^{N+1}-\alpha^{N-1}\Big(1+\frac{1}{\alpha}\Big)^N\Big[1+2\alpha\Big]+(N+3)\alpha^N+\frac{1}{\alpha}-2. \quad \text{(B22)}$$

Now we use the inequality $\alpha^N > \frac{1}{e}$ to obtain

$$2^{N+1}-\alpha^{N-1}\Big(1+\frac{1}{\alpha}\Big)^N\Big[1+2\alpha\Big]+(N+3)\alpha^N+\frac{1}{\alpha}-2$$
$$\geq 2^{N+1}-(\alpha+1)^N\Big[\frac{1}{\alpha}+2\Big]+\frac{N+3}{e}-2\geq 2^N\Bigg[2-\Big(\frac{19}{6}\Big)\Big(1-\frac{1}{2(N+1)}\Big)^N\Bigg] \quad \text{(B23)}$$

for $N \geq 6$ and the final term is positive for $N \geq 9$, because $\Big(1-\frac{1}{2(N+1)}\Big)^N$ is decreasing in N and is less than $\frac{12}{19}$ for $N = 9$. Again the remaining few terms can be checked by hand, and again we have done this using Maxima. $\square$

B2 The Final Term

The proof of the OPC conjecture will be complete when we have proved the following lemma. This proof will occupy the remainder of the paper.

Lemma 13.

$$\frac{\alpha^N}{N} n \sum_{i=0}^{n-1} \binom{n-1}{i} \left(\frac{1}{\alpha}\right)^i - 2\alpha^N \sum_{i=n-N}^{N} \binom{n}{i} \left(\frac{1}{\alpha}\right)^i + \sum_{i=N-n}^{N} \binom{n}{i} - \frac{N+1}{N} \binom{n}{N+1}$$
$$+\alpha^N \binom{n}{N} \left(\frac{1}{\alpha}\right)^{n-N} - \alpha^N \frac{N+1}{N} \binom{n}{N+1} \left(\frac{1}{\alpha}\right)^{n-N-1} \geq 0 \qquad \text{(B24)}$$

for $n = N+2, N+3, ..., 2N$.

First we replace n by $n - N - 1$ to obtain for the left side of the above inequality:

$$\frac{n+1+N}{N} \alpha^N \sum_{i=n}^{N} \binom{n+N}{i} \alpha^{-i} - 2\alpha^N \sum_{i=n+1}^{N} \binom{n+N+1}{i} \alpha^{-i} + \sum_{i=n+1}^{N} \binom{n+N+1}{i}$$
$$- \alpha^{-1} \binom{n+N+1}{N+1} + \alpha^{N-n-1} \left[\binom{n+N+1}{N} - \binom{n+N+1}{N+1} \right] \qquad \text{(B25)}$$

where now we have to prove this non-negative for $1 \leq n \leq N-1$, and as before $\alpha = \frac{N}{N+1}$.

We fix $n + N + 1 = R$ and define

$$\Omega(n, N) = \frac{n+1+N}{N} \alpha^N \sum_{i=n}^{N} \binom{n+N}{i} \alpha^{-i} - 2\alpha^N \sum_{i=n+1}^{N} \binom{n+N+1}{i} \alpha^{-i} + \sum_{i=n+1}^{N} \binom{n+N+1}{i}$$
$$- \alpha^{-1} \binom{n+N+1}{N+1} + \alpha^{N-n-1} \left[\binom{n+N+1}{N} - \binom{n+N+1}{N+1} \right]. \qquad \text{(B26)}$$

The idea of the proof here is to show that for the "diagonal" this expression is zero, and that when we move away from the diagonal the value increases. The first assertion is (more precisely) expressed in the following lemma.

Lemma 14.

$$\Omega(K-1, K) = \Omega(K-1, K+1) = 0, \qquad K = 1, 2, \ldots. \qquad \text{(B27)}$$

Proof. This is a straightforward calculation:

$$\Omega(K-1, K) = \binom{2K}{K} \left[\frac{K}{K+1} - 1 \right] + \binom{2K}{K} \left(1 - \frac{K}{K+1} \right) = 0. \qquad \text{(B28)}$$

$$\Omega(K-2,K) = \binom{2K-1}{K}\left(\frac{K}{K+1}-1\right) + \binom{2K-1}{K-1}\left[\left(\frac{K-1}{K+1}-2\right)\frac{K}{K+1}+1\right] \qquad \text{(B29)}$$
$$+ \frac{K}{K+1}\binom{2K-1}{K}\left(1-\frac{K-1}{K+1}\right) = 0.$$

□

Now let $\Upsilon(n,N) = \Omega(n-1,N+1) - \Omega(n,N)$. The remainder of the proof is concerned with the following lemma. Once it is completed, combined with Lemma 14, it gives the required result.

Lemma 15. *For* $0 \leq n < N$ $\Upsilon(n,N) \geq 0$.

The proof of this result in itself will require several lemmas. Indeed, its proof appears to be non-trivial, although elementary. To simplify the expressions, we let $\beta = \frac{N+1}{N+2}$. Then a straightforward and omitted calculation shows that

$$\Upsilon(n,N) = \binom{N+n+1}{N+1}\left(\beta-\beta^{N-n+1}\right) - \binom{N+n+1}{N}\left(1-\frac{n+1}{N+1}\right)\alpha^{N-n-1} \qquad \text{(B30)}$$
$$+ \sum_{i=0}^{N-n-1}\binom{N+n+1}{N-i}\Big[\big(\frac{N-i}{N+2}-2\big)\beta^{i+1} - \big(\frac{N-i}{N+1}-2\big)\alpha^{i}\Big].$$

Now we note that the sum in this expression for $\Upsilon(n,N)$ is (the details are provided in Appendix I of [30]):

$$\sum_{i=0}^{N-n-1}\binom{N+n+1}{N-i}\Big[\big(\frac{N-i}{N+2}-2\big)\beta^{i+1} - \big(\frac{N-i}{N+1}-2\big)\alpha^{i}\Big]$$
$$= \frac{1}{2}\Big\{\sum_{i=0}^{N-n-1}\binom{N+n+1}{N-i}\Big[\big(\frac{N-i}{N+2}-2\big)\beta^{i+1} - \big(\frac{N-i}{N+1}-2\big)\alpha^{i}\Big]$$
$$+ \sum_{i=0}^{N-n-1}\binom{N+n+1}{n+1+i}\Big[\big(\frac{n+1+i}{N+2}-2\big)\beta^{N-n-i} - \big(\frac{n+1+i}{N+1}-2\big)\alpha^{N-n-1-i}\Big]\Big\}. \qquad \text{(B31)}$$

This yields

$$\Upsilon(n,N) = \binom{N+n+1}{N+1}\left(\beta-\beta^{N-n+1}\right) - \binom{N+n+1}{N}\left(1-\frac{n+1}{N+1}\right)\alpha^{N-n-1}$$
$$+ \sum_{i=0}^{N-n-1}\binom{N+n+1}{N-i}\frac{S(i)}{2} \qquad \text{(B32)}$$

where

$$S(i) = \left[\left(\frac{N-i}{N+2}-2\right)\beta^{i+1} - \left(\frac{N-i}{N+1}-2\right)\alpha^{i}\right] \\ + \left[\left(\frac{i+n+1}{N+2}-2\right)\beta^{N-n-i} - \left(\frac{i+n+1}{N+1}-2\right)\alpha^{N-n-1-i}\right]. \quad \text{(B33)}$$

Lemma 16. *$S(i)$ is increasing in i for $0 \leq i \leq \frac{1}{2}(N-n-1)$.*

Proof. Let

$$Q(i) = \left[\left(\frac{N-i}{N+2}-2\right)\beta^{i+1} - \left(\frac{N-i}{N+1}-2\right)\alpha^{i}\right]. \quad \text{(B34)}$$

Then

$$S(i) = Q(i) + Q(N-n-1-i). \quad \text{(B35)}$$

Differentiating with respect to i we have

$$S'(i) = Q'(i) - Q'(N-n-1-i). \quad \text{(B36)}$$

To show, then, that $S(i)$ is increasing for $0 \leq i \leq \frac{1}{2}(N-n-1)$, we shall show that $Q'(i)$ is decreasing for $0 \leq i \leq N-n-1$. We observe that $\beta^{i+1} \leq \alpha^i$ for all real i in the range in question, and hence that

$$\begin{aligned} Q''(i) &= -\frac{2}{N+2}\beta^{i+1}\ln\beta + \left(\frac{N-i}{N+2}-2\right)\beta^{i+1}(\ln\beta)^2 + \frac{2}{N+1}\alpha^i\ln\alpha - \left(\frac{N-i}{N+1}-2\right)\alpha^i(\ln\alpha)^2 \\ &= \left[2-(N+4+i)\ln\left(\frac{N+2}{N+1}\right)\right]\frac{\beta^{i+1}}{N+2}\ln\left(\frac{N+2}{N+1}\right) \\ &\quad - \left[2-(N+2+i)\ln\left(\frac{N+1}{N}\right)\right]\frac{\alpha^i}{N+1}\ln\left(\frac{N+1}{N}\right) \\ &\leq -\left[(N+4+i)\ln\left(\frac{N+2}{N+1}\right) - (N+2+i)\ln\left(\frac{N+1}{N}\right)\right]\frac{\alpha^i}{N+1}\ln\left(\frac{N+1}{N}\right) \leq 0, \end{aligned} \quad \text{(B37)}$$

which is enough to complete the proof. □

The next lemma now follows from the preceding one.

Lemma 17.

$$\Upsilon(n,N) \geq \binom{N+n+1}{N+1}\left(\beta-\beta^{N-n+1}\right) - \binom{N+n+1}{N}\left(1-\frac{n+1}{N+1}\right)\alpha^{N-n-1} + \binom{N+n+1}{N}S_0 + \frac{S_1}{2}\sum_{i=1}^{N-n-2}\binom{N+n+1}{N-i}. \tag{B38}$$

Now define

$$U_n = \sum_{i=1}^{\lfloor\frac{N-n-1}{2}\rfloor}\binom{N+n+1}{N-i}. \tag{B39}$$

Then we have the following lemma.

Lemma 18.

$$\frac{U_n}{\binom{N+n+1}{N}} \geq \Big(\frac{N+3n+5}{2N-2n-2}\Big)\Big[\Big(\frac{3N+n+3}{N+3n+5}\Big)^{\frac{N-n}{2}}-1\Big]-1. \tag{B40}$$

In other words, the following is true.

$$\begin{aligned}\Upsilon(n,N) \geq& \binom{N+n+1}{N+1}(\beta-\beta^{N-n+1}) - \binom{N+n+1}{N}(1-\frac{n+1}{N+1})\alpha^{N-n-1} \\ &+\binom{N+n+1}{N}S_0 \\ &+\binom{N+n+1}{N}\Big[\Big(\frac{N+3n+5}{2N-2n-2}\Big)\Big[\Big(\frac{3N+n+3}{N+3n+5}\Big)^{\frac{N-n}{2}}-1\Big]-1\Big]S_1.\end{aligned} \tag{B41}$$

Proof. This is proved as follows:

$$\begin{aligned}\frac{U_n}{\binom{N+n+1}{N}} &= \sum_{i=1}^{\lfloor\frac{N-n-1}{2}\rfloor}\prod_{k=1}^{i}\Big(\frac{N+1-k}{n+1+k}\Big) = \sum_{i=1}^{\lfloor\frac{N-n-1}{2}\rfloor}\Big(\prod_{k=1}^{i}\frac{(N+1-k)(N+1+k-i-1)}{(n+1+k)(n+1-k+i+1)}\Big)^{\frac{1}{2}} \\ &\geq \sum_{i=1}^{\lfloor\frac{N-n-1}{2}\rfloor}\left(\prod_{k=1}^{i}\frac{(N+1)^2-(i+1)(N+1)}{(n+1)^2+(i+1)(n+1)}\right)^{\frac{1}{2}} \geq \sum_{i=1}^{\lfloor\frac{N-n-1}{2}\rfloor}\left(\frac{(N+1-\frac{i+1}{2})}{(n+1+\frac{i+1}{2})}\right)^{i} \\ &= \sum_{i=1}^{\lfloor\frac{N-n-1}{2}\rfloor}\left(\frac{2N-i+1}{2n+i+3}\right)^{i} \geq \sum_{i=1}^{\lfloor\frac{N-n-1}{2}\rfloor}\left(\frac{2N-\frac{N-n-1}{2}+1}{2n+\frac{N-n-1}{2}+3}\right)^{i} \\ &= \sum_{i=1}^{\lfloor\frac{N-n-1}{2}\rfloor}\left(\frac{3N+n+3}{N+3n+5}\right)^{i} \geq \left(\frac{N+3n+5}{2N-2n-2}\right)\left(\left(\frac{3N+n+3}{N+3n+5}\right)^{\frac{N-n}{2}}-1\right)-1.\end{aligned} \tag{B42}$$

□

B3 Final Estimates

At this point we change from using N and n as the variables to using $R = N + n + 1$ and $D = N - n - 1$, so that $N = \frac{R+D}{2}$ and $n + 1 = \frac{R-D}{2}$. The right side of inequality (B41) can be rewritten as

$$\mathcal{Q}(D, R) = \frac{R - D}{R + 4 + D}\big(1 - \beta^{D+1}\big) - \frac{2 + 2D}{R + 2 + D}\alpha^D + (\mathcal{L}_0 + \mathcal{R}_0) + \mathcal{M}(\mathcal{L}_1 + \mathcal{R}_1) \tag{B43}$$

where

$$\mathcal{M} = \Big(\frac{2R + 2 - D}{2D}\Big)\Big(\Big(\frac{2R + 2 + D}{2R + 2 - D}\Big)^{\frac{D+1}{2}} - 1\Big) - 1 \tag{B44}$$

$$\mathcal{L}_0 = \Big(\frac{N}{N + 2} - 2\Big)\beta^1 - \Big(\frac{N}{N + 1} - 2\Big)\alpha^0 \tag{B45}$$

$$= \Big(2 - \frac{R + D}{R + 2 + D}\Big) - \Big(2 - \frac{R + D}{R + 4 + D}\Big)\beta^1 \tag{B46}$$

$$\mathcal{R}_0 = \Big(\frac{n + 1}{N + 2} - 2\Big)\beta^{N-n} - \Big(\frac{n + 1}{N + 1} - 2\Big)\alpha^{N-n-1} \tag{B47}$$

$$= \Big(2 - \frac{R - D}{R + 2 + D}\Big)\alpha^D - \Big(2 - \frac{R - D}{R + 4 + D}\Big)\beta^{D+1} \tag{B48}$$

$$\mathcal{L}_1 = \Big(\frac{N - 1}{N + 2} - 2\Big)\beta^2 - \Big(\frac{N - 1}{N + 1} - 2\Big)\alpha^1 \tag{B49}$$

$$= \Big(2 - \frac{R - 2 + D}{R + 2 + D}\Big)\alpha^1 - \Big(2 - \frac{R - 2 + D}{R + 4 + D}\Big)\beta^2 \tag{B50}$$

$$\mathcal{R}_1 = \Big(\frac{n + 2}{N + 2} - 2\Big)\beta^{N-n-1} - \Big(\frac{n + 2}{N + 1} - 2\Big)\alpha^{N-n-2} \tag{B51}$$

$$= \Big(2 - \frac{R + 2 - D}{R + 2 + D}\Big)\alpha^{D-1} - \Big(2 - \frac{R + 2 - D}{R + 4 + D}\Big)\beta^D. \tag{B52}$$

where $\alpha = \frac{N}{N+1} = \frac{R+D}{R+2+D}$ and $\beta = \frac{N+1}{N+2} = \frac{R+2+D}{R+4+D}$.

We state, as a lemma, the final requirement for our proof.

Lemma 19. *$\mathcal{Q}(D, R) \geq 0$ for $0 \leq D \leq R - 4$ and $R \geq 4$.*

We note that the cases $D = 0$ and $D = 1$ are already covered by equation (B27). We also note that $\mathcal{L}_0, \mathcal{L}_1, \mathcal{R}_0, \mathcal{R}_1$ are non-negative, as is the first term in $Q(D, R)$. Moreover the negative (second) term does not exceed 1 in absolute value.

It will be convenient to write

$$\mathcal{W} = \frac{R - D}{R + 4 + D}\big(1 - \beta^{D+1}\big) - \frac{2 + 2D}{R + 2 + D}\alpha^D. \tag{B53}$$

It is a remarkable (and frustrating) fact that all of the non-negative terms are needed to make the inequality $\mathcal{Q}(D,R)$ valid in this range. We begin by replacing each of them by somewhat simpler forms. We write

$$\rho = \frac{2}{2+R+D}. \tag{B54}$$

Lemma 20.

$$\mathcal{L}_0 = \rho - \Big(\frac{\rho}{\rho+1}\Big) + 2\Big(\frac{\rho}{\rho+1}\Big)^2, \tag{B55}$$

$$\mathcal{W} = (1-\rho)^D - \frac{2}{(1+\rho)^{D+1}} + 1 - \frac{\rho}{\rho+1}(D+2), \tag{B56}$$

$$\mathcal{L}_1 = \Big(\frac{\rho^2}{(\rho+1)^3}\Big)(4-2\rho-5\rho^2-\rho^3), \tag{B57}$$

$$\mathcal{R}_1 = (1+D\rho)(1-\rho)^{D-1} - \Big(1+(D+1)\frac{\rho}{\rho+1}\Big)\Big(1-\frac{\rho}{\rho+1}\Big)^D, \tag{B58}$$

$$\mathcal{M} = \frac{1}{u}\Big((1+u)^{(D+1)/2} - 1\Big) - 1 \tag{B59}$$

where

$$\frac{1}{u} = \frac{2R+2-D}{2D} = \frac{2}{D}\Big(\frac{1}{\rho} - \frac{1}{2}\Big) - \frac{3}{2}. \tag{B60}$$

The following lemma is obtained by the judicious use of the binomial expansion. We aim to have a common denominator of $1/(1+\rho)^D$ in the various summands in the required inequality.

Lemma 21. *Let*

$$\mathcal{CL}_0 = \frac{\rho^2}{(1+\rho)^{D+1}}\Big(1+(D-1)\rho+\frac{1}{2}(D-1)(D-2)\rho^2\Big)(3+\rho) \tag{B61}$$

$$\mathcal{CL}_1 = \frac{\rho^2}{(1+\rho)^{D+1}}\Big(1+(D-2)\rho+\frac{1}{2}(D-2)(D-3)\rho^2\Big)(4-2\rho-5\rho^2-\rho^3), \tag{B62}$$

$$\mathcal{CW} = \frac{-1}{(\rho+1)^{D+1}}\Big(\frac{5}{2}D\rho^2+\frac{1}{2}D^2\rho^2+\frac{1}{2}D\rho^3+\frac{1}{2}D^3\rho^3\Big) \tag{B63}$$

$$\mathcal{CR}_1 = \frac{\rho^2}{(1+\rho)^{D+1}}\big((D+2)-\rho(D^2-2)-\rho^2(D-1)(1+2D)-\rho^3(D-1)D\big) \tag{B64}$$

$$\mathcal{CM} = \frac{D-1}{2} + \frac{D(D^2-1)\rho}{16}\Big(1+\frac{1}{2}(D+2)\rho\Big)\Big(1+\frac{\rho}{4}(D-2)\big(1+\frac{1}{2}(D+2)\rho\big)\Big). \tag{B65}$$

Then for $D \geq 3$

$$\mathcal{L}_0 \geq \mathcal{CL}_0,\ \mathcal{L}_1 \geq \mathcal{CL}_1,\ \mathcal{M} \geq \mathcal{CM},\ \mathcal{W} \geq \mathcal{CW},\ \mathcal{R}_1 \geq \mathcal{CR}_1. \tag{B66}$$

The details of the proof are given in Appendix II of [30]. In view of this lemma it is enough to show that

$$\mathcal{CL}_0 + \mathcal{CW} + \mathcal{CM}(\mathcal{CL}_1 + \mathcal{CR}_1) \geq 0, \tag{B67}$$

for $R \geq 6$ and $3 \leq D \leq R - 4$.

The next two lemmas can be proved by straightforward calculations (See Appendices III and IV of [30]) based on what we know about the ranges of ρ and D. We note that, for $D \geq 5$, $\rho \leq \frac{2}{13}$.

Lemma 22.

$$\begin{aligned}
&\mathcal{CL}_0 + \mathcal{CW} + \mathcal{CM}(\mathcal{CL}_1 + \mathcal{CR}_1) \geq \\
&\quad - \left[D^8(2\rho^5) + D^7(6\rho^4) + D^6(16\rho^3) + D^5(44\rho^5 + 338\rho^4 - 128\rho^3 + 8\rho^2)\right. \\
&\quad + D^4(96\rho^4 + 496\rho^3 - 288\rho^2 - 32\rho) + D^3(576\rho^3 + 152\rho^2 + 320\rho) \\
&\quad \left. + D^2(2336\rho^2 - 1248\rho) + D(1984\rho)\right].
\end{aligned} \tag{B68}$$

Lemma 23. *For $D \geq 5$,*

$$\begin{aligned}
&\mathcal{CL}_0 + \mathcal{CW} + \mathcal{CM}(\mathcal{CL}_1 + \mathcal{CR}_1) \geq \\
&\quad - \left[2D^8\rho^5 + 6D^7\rho^4 + 16D^6\rho^3) - (128\rho^3 - 8\rho^2)D^5\right. \\
&\quad \left. - (106\rho^2 + 32\rho)D^4 + 320D^3\rho + D^2(2336\rho^2 - 851\rho)\right].
\end{aligned} \tag{B69}$$

Now we define

$$\begin{aligned}
W = &- 2336\rho + 851 - 320D + (106\rho + 32)D^2 \\
&+ (128\rho^2 - 8\rho)D^3 - 16\rho^2 D^4 - 6\rho^3 D^5 - 2\rho^4 D^6,
\end{aligned} \tag{B70}$$

so that by Lemma 23

$$\mathcal{CL}_0 + \mathcal{CM}(\mathcal{CL}_1 + \mathcal{CR}_1) \geq \rho D^2 W \tag{B71}$$

for $D \geq 5$, and, leaving aside the special case where this fails, it is enough to show that W is non-negative. For the moment we shall assume that $D \geq 5$, and return to the special cases excluded by this at the end.

Now we rewrite W in terms of R and D, reversing the substitution $\rho = \frac{2}{R+D+2}$, and simplify to obtain

$$\begin{aligned} W =&(32D^2 - 320D + 851)R^6 + (112D^3 - 812D^2 + 844D + 2136)R^3 \\ &+ (80D^4 - 100D^3 - 534D^2 - 1272D - 7608)R^2 \\ &+ (-96D^5 + 700D^4 + 1660D^3 - 5384D^2 - 25456D - 28832)R \\ &- 128D^6 + 212D^5 + 1931D^4 - 56D^3 - 15640D^2 - 33952D - 23760. \end{aligned} \tag{B72}$$

Next we split the argument into two cases: $R/D \geq 10$ and $R/D < 10$. For the former, we note that the sum of the R^3 and R terms in W exceeds

$$\begin{aligned} 11200D^5 - 81200D^4 + 84400D^3 + 213600D^2 - 95D^5 + 700D^4 \\ + 1660D^3 - 5384D^2 - 25456D - 28832 \\ = 11105D^5 - 80500D^4 + 86060D^3 + 208216D^2 - 25456D - 28832, \end{aligned} \tag{B73}$$

which is positive for $D \geq 5$. The sum of the R^6, R^2 and constant terms in R and in W (B70) exceeds

$$\begin{aligned} 320000D^6 - 3200000D^5 + 8510000D^4 + 8000D^6 - 10000D^5 - 53400D^4 - 127200D^3 - 760800D^2 \\ - 128D^6 + 212D^5 + 1931D^4 - 56D^3 - 15640D^2 - 33952D - 23760 \\ \geq 327D^6 - 3210D^5 + 8450D^4 - 128D^3 - 780D^2 - 34D - 24 \geq 550D^4 - 128D^3 - 780D^2 - 35D - 24, \end{aligned} \tag{B74}$$

which is again positive in the range $D \geq 5$.

Now we consider the case $R/D < 10$. We rewrite W in terms of R/D and split it into two parts:

$$P_1 = \left[32\left(\frac{R}{D}\right)^4 + 112\left(\frac{R}{D}\right)^3 + 80\left(\frac{R}{D}\right)^2 - 96\left(\frac{R}{D}\right) - 128\right] D^6 \tag{B75}$$
$$+ \left[-320\left(\frac{R}{D}\right)^4 - 812\left(\frac{R}{D}\right)^3 - 100\left(\frac{R}{D}\right)^2 + 700\left(\frac{R}{D}\right) + 212\right] D^5$$
$$+ \left[851\left(\frac{R}{D}\right)^4 + 844\left(\frac{R}{D}\right)^3 - 534\left(\frac{R}{D}\right)^2 + 1660\left(\frac{R}{D}\right) + 1931\right] D^4.$$

$$P_2 = \left[2136\left(\frac{R}{D}\right)^3 - 1272\left(\frac{R}{D}\right)^2 - 5384\left(\frac{R}{D}\right) - 56\right] D^3$$
$$+ \left[-7608\left(\frac{R}{D}\right)^2 - 25456\left(\frac{R}{D}\right) - 15640\right] D^2 + \left[-28832\left(\frac{R}{D}\right) - 33952\right] D - 23760. \tag{B76}$$

It is simple to check that P_1 is positive for all $R/D < 10, R \geq 9, D \geq 5$, and that P_2 is positive provided in addition $R/D > 3.3$. Thus W is positive provided $3.3 < R/D < 10, R \geq 9, D \geq 5$. The details are given in Appendix V of [30]. Apart from a few values that can be checked by hand, then, it remains to show the result for $R/D < 3.3$.

We next write

$$W = Q_1 D^6 + Q_2 D^5 + Q_3 D^4 + Q_4 D^3 + Q_5 D^2 + Q_6 D - 23760, \tag{B77}$$

where

$$\begin{aligned}
Q_1 &= 32\left(\frac{R}{D}\right)^4 + 112\left(\frac{R}{D}\right)^3 + 80\left(\frac{R}{D}\right)^2 - 96\left(\frac{R}{D}\right) - 128;\\
Q_2 &= -320\left(\frac{R}{D}\right)^4 - 812\left(\frac{R}{D}\right)^3 - 100\left(\frac{R}{D}\right)^2 + 700\left(\frac{R}{D}\right) + 212;\\
Q_3 &= 851\left(\frac{R}{D}\right)^4 + 844\left(\frac{R}{D}\right)^3 - 534\left(\frac{R}{D}\right)^2 + 1660\left(\frac{R}{D}\right) + 1931;\\
Q_4 &= 2136\left(\frac{R}{D}\right)^3 - 1272\left(\frac{R}{D}\right)^2 - 5384\left(\frac{R}{D}\right) - 56;\\
Q_5 &= -7608\left(\frac{R}{D}\right)^2 - 25456\left(\frac{R}{D}\right) - 15640;\\
Q_6 &= -28832\left(\frac{R}{D}\right) - 33952.
\end{aligned} \tag{B78}$$

We note that Q_1, Q_3, Q_4 are increasing functions and that Q_2, Q_5, Q_6 are decreasing functions of R/D. In turn we replace R/D by each of $1.1, 1.2, 1.3, \ldots, 3.2$ in the increasing functions and respectively, by $1.2, 1.3, 1.4 \ldots, 3.3$ in the decreasing ones to give worst case estimates over, respectively, each of the intervals

$$[1.1, 1.2],\ [1.2, 1.3],\ [1.3, 1.4], \ldots, [3.2, 3.3]. \tag{B79}$$

It can be shown that, in each case, W given by (B77) is increasing (and therefore clearly positive) by repeated differentiation and checking for positivity of the derivatives at the endpoint $D = 14$. The details are given in Appendix VI of [30].

Finally we consider the case when $R/D < 1.1$. We rewrite W in terms of R and $y = D/R$ as follows

$$\begin{aligned} W = {} & 32y^2R^8 - 320yR^7 + (851 + 112y^3 + 80y^4 - 96y^5 - 128y^6)R^6 \\ & + (-812y^2 - 100y^3 + 700y^4 + 212y^5)R^5 + (844y - 534y^2 + 1660y^3 + 1931y^4)R^4 \\ & + (2136 - 1272y - 5384y^2 - 56y^3)R^3 + (-7608 - 25456y - 15640y^2)R^2 \\ & + (-28832 - 33952y)R - 23760. \end{aligned} \tag{B80}$$

For each coefficient of a power of R, we again consider the worst case, by replacing y by 1 if that coefficient is decreasing in y over the interval $[1/1.1, 1]$, and by $1/1.1$ if that coefficient is to increasing over that interval. We collect the first two terms (R^7 and R^8) together and note that this is increasing in y for any $R \geq 8$. The result is that

$$W \geq \frac{32}{(1.1)^2}R^8 - \frac{320}{1.1}R^7 + 800R^6 - 136.5R^5 + 2892R^4 - 4576R^3 - 48704R^2 - 62784R - 23760. \tag{B81}$$

It can be shown that this is increasing (and therefore clearly positive) by repeated differentiation and checking for positivity of the derivatives at the endpoint $1/1.1$ The details are given in Appendix VII of [30].

There only remain a fairly small finite number of cases for small values of R and D. Each of these have been checked using Maxima. We give the details in Appendix VII of [30] for all of the special cases, $D = 2, 3, 4$. This completes the proof!

References

[1] V. Abramov, S. Li, M. Wang, E. W. M. Wong, and M. Zukerman. Computation of blocking probability for large circuit switched networks. *IEEE Communications Letters*. to appear.

[2] G. R. Ash and B. D. Huang. An analytical model for adaptive routing networks. *IEEE Transaction on Communiations*, 41(11):1748–1759, November 1993.

[3] E. Brockmeyer. The simple overflow problem in the theory of telephone traffic. *Teleteknik*, 5:361–374, 1954.

[4] S. P. Chung, A. Kashper, and K. W. Ross. Computing approximate blocking probabilities for large loss networks with state-dependent routing. *IEEE/ACM Transactions on Networking*, 1(1):105–115, 1993.

[5] R. B. Cooper and S. Katz. Analysis of alternate routing networks with account taken of nonrandomness of overflow traffic. Memo., Bell Telephone Lab., 1964.

[6] D. Deloddere, W. Verbiest, and H. Verhille. Interactive video on demand. *IEEE Comunications Magazine*, 32(5):82–88, May 1994.

[7] A. A. Fredericks. Congestion in blocking systems – a simple approximation technique. *Bell Syst. Tech. J.*, 59(6):805–827, 1980.

[8] A. Girard. *Routing and Dimensioning in Circuit-Switched Networks.* Addison-Wesley, 1990.

[9] A. Harel. Sharp and simple bounds for the Erlang delay and loss formulae. *Queueing Systems*, 64:119–143, 2010.

[10] P. J. Hunt and F. P. Kelly. On critically loaded loss networks. *Advances in Applied Probability*, 21:831–841, 1989.

[11] D. L. Jagerman. Methods in traffic calculations. *AT&T Bell Laboratories Technical Journal*, 63(7):1283–1301, Sep. 1984.

[12] F. P. Kelly. *Reversibility and Stochastic Networks.* Wiley, Chichester, 1979.

[13] F. P. Kelly. Blocking probabilities in large circuit-switched networks. *Advances in Applied Probability*, 18:473–505, June 1986.

[14] F. P. Kelly. Loss networks. *The Annals of Applied Probability*, 1(3):319–378, August 1991.

[15] L. Kosten. On the blocking probability of graded multiples. *Nachrichtentech. Elek.*, 14:5–12, Jan. 1937.

[16] A. Kuczura. The interrupted Poisson process as an overflow process. *Bell Syst. Tech. J.*, 52(3):437–448, 1973.

[17] A. Kuczura and D. Bajaj. A method of moments for the analysis of a switched communication network's performance. *IEEE Transactions on Communications*, COM-25(2):185–193, February 1977.

[18] D. Mitra. Asymptotic analysis and computational methods for a class of simple circuit-switched networks with blocking. *Advances in Applied Probability*, 19:219–239, March 1987.

[19] D. Mitra, J. A. Morrison, and K. G. Ramakrishnan. ATM network design and optimization: a multirate loss network framework. *IEEE/ACM Transactions on Networking*, 4(4):531–543, August 1996.

[20] C. Pearce and R. M. Potter. Some formulae old and new for overflow traffic in telephony. *Australian Telecommunication Research*, 11:92–97, 1977.

[21] R. M. Potter. Explicit formulae for all overflow moments in Kosten and Brockmeyer system with renewal input. *Australian Telecommunication Research*, 13:39–49, 1980.

[22] K.-T. Ko Q. Huang and V. B. Iversen. Approximation of loss calculation for hierarchical networks with multiservice overflows. *IEEE Trans. Commun.*, 56(3):466–473, Mar. 2008.

[23] G. Raskutti, A. Zalesky, E. W. M. Wong, and M. Zukerman. Enhanced blocking probability evaluation method for circuit-switched trunk reservation networks. *IEEE Communications Letters*, 11(6):543–545, June 2007.

[24] R. Syski. *Introduction to Congestion Theory in Telephone Systems*, volume 4 of *North Holland Studies in Telecommunications*. Elsevier Science Publishers B. V., 2nd edition, 1986.

[25] E. A. van Doorn and A. T. K. Ta. Proofs for some conjectures of Rajaratnam and Takawira on the peakedness of handoff traffic. *IEEE Transactions on Vehicular Technology*, 52(4):953–957, 2003.

[26] W. Whitt. Blocking when service is required from several facilities simultaneously. *AT&T Tech. J.*, 64(8):1807–1856, Oct. 1985.

[27] R. I. Wilkinson. Theories of toll traffic engineering in the U.S.A. *The Bell System Technical Journal*, 35(2):421–514, March 1956.

[28] E. W. M. Wong, J. Baliga, M. Zukerman, A. Zalesky, and G. Raskutti. A new method for blocking probability evaluation in OBS/OCS networks with deflection routing. *Journal of Lightwave Technology*, 27(23):5335–5347, 2009.

[29] E. W. M. Wong, M. Y. M. Chiu, Z. Rosberg, M. Zukerman, S. Chan, and A. Zalesky. A novel method for modeling and analysis of distributed video on demand systems. In *Proceedings of IEEE ICC 2005*, volume 1, pages 88–92, May 2005.

[30] E. W. M. Wong, B. Moran, A. Zalesky, Z. Rosberg, and M. Zukerman. *A Proof of the OPC Conjecture (extended version).* Avaiable online: <http://www.ee.cityu.edu.hk/∼ewong/opc/proof_of_the_opc_conjecture.pdf>.

[31] E. W. M. Wong and T. S. Yum. Maximum free circuit routing in circuit-switched networks. In *Proceedings of IEEE INFOCOM '90*, volume 3, pages 934–937, June 1990.

[32] E. W. M. Wong, A. Zalesky, Z. Rosberg, and M. Zukerman. A new method for approximating blocking probability in overflow loss networks. *Computer Networks*, 51(11):2958–2975, 2007.

[33] A. Zalesky, H. L. Vu, Z. Rosberg, E. W. M. Wong, and M. Zukerman. Stabilizing deflection routing in optical burst switched networks. *IEEE Journal on Selected Areas in Communications, supplement on optical communications and networking*, 25(6):3–19, August 2007.

[34] I. B. Ziedins and F. P. Kelly. Limit theorems for loss networks with diverse routing. *Advances in Applied Probability*, 21(4):804–830, December 1989.