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Introduction to anti-control of discrete chaos: theory and applications

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In this paper, the notion of anti-control of chaos (or chaotification) is introduced, which means to make an originally non-chaotic dynamical system chaotic or enhance the existing chaos of a chaotic system. The main interest in this paper is to employ the classical feedback control techniques. Only the discrete case is discussed in detail, including both finite-dimensional and infinite-dimensional settings.

Keywords: chaos; chaotification; finite- and infinite-dimensional discrete chaos

1. Introduction

Research on the emerging fields of chaos control and chaos synchronization has seen a very rapid development in the past two decades (e.g. Chen & Dong 1998, and many references cited therein). In particular, the concept of anti-control of chaos (or chaotification), by means of making an originally non-chaotic dynamical system chaotic or enhancing the existing chaos of a chaotic system, has attracted increasing attention in recent years (e.g. Chen 1998, 2001, 2003, and some references therein). This interest seems to be continuously expanding, mainly due to the great potentials of chaos in some non-traditional applications such as those found within the context of electronic, informatic, mechanical, optical and especially biological and medical systems (Chen 1998, 2001; Chen & Dong 1998).

In this paper, the notion of anti-control of discrete chaos is introduced. Discrete maps, i.e. discrete dynamical systems, with chaotic and bifurcating behaviours have been found very useful in some real-world applications particularly in, for instance, encryption (Jakimoski & Kocarev 2001), digital communications (Kocarev *et al.* 2001) and brain science (Schiff *et al.* 1994) as well as heart pathology and analysis (Ditto *et al.* 2000). These provide a strong motivation for the current research on anti-control of chaos, in both continuous and discrete dynamical systems.

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One contribution of 15 to a Theme Issue 'Exploiting chaotic properties of dynamical systems for their control'.

In the pursuit of anti-control of discrete maps, a simple yet mathematically rigorous anti-control method was first developed by Chen & Lai (1996, 1997, 1998) from the engineering state-feedback control approach, which yields chaos in the sense of Devaney (1987) for linear systems, or Wiggins (1990) for nonlinear systems. Afterwards, Wang & Chen (1999, 2000*a*) further showed that the Chen–Lai algorithm for chaotification leads to chaos not only in the sense of Devaney (1987) or Wiggins (1990), but also in the sense of Li–Yorke (Li & Yorke 1975). Lately, Li & Chen (2003*a*) further relaxed the condition for anti-control, showing that the Wang–Chen chaotification theorems established in Wang & Chen (2000*a*) can be somewhat generalized and that the Chen–Lai scheme is indeed a 'universal' anti-control scheme for discrete maps. Recently, the state-feedback anti-control algorithm of Chen–Lai–Wang was improved to be an output-feedback, anti-control algorithm by Zhang & Chen (2004), and meanwhile was simplified to an anti-control scheme with a single-state variable feedback in each dimension, by Zheng *et al.* (2004) and Zheng & Chen (2004), where the generated chaos was also shown to be in the sense of both Devaney and Li–Yorke.

More recently, Shi *et al.* (in press) studied anti-control of chaos for discrete systems in Banach spaces via the state-feedback control. They established several chaotification schemes in general Banach spaces and extended the Chen–Lai and Wang–Chen schemes in finite dimensional spaces (Chen & Lai 1998; Wang & Chen 2000*a*) to a special Banach space, showing that the controlled systems are chaotic in the sense of both Devaney (1987) and Li–Yorke (Li & Yorke 1975). Consequently, the Chen–Lai algorithm and the Wang–Chen algorithm for finite-dimensional chaotification led to chaos not only in the sense of Wiggins (1990) and Li–Yorke (Li & Yorke 1975), but also in the sense of Devaney (1987).

In an effort to show that the chaos so generated is indeed chaos in a rigorous mathematical sense, the celebrated Li–Yorke theorem (for the one-dimensional case; Li & Yorke 1975) and the Marotto theorem (for the n-dimensional case; Marotto 1978) were usually employed. In retrospect, Li & Yorke introduced the first precise definition of discrete chaos and established a very simple criterion of chaos for one-dimensional maps, i.e. 'period three implies chaos' for brevity (Li & Yorke 1975). After 3 years, Marotto (1978) generalized this result to n-dimensional maps, showing that the existence of a snap-back repeller implies chaos in the sense of Li–Yorke. This theorem is until now the best one for predicting and analysing discrete chaos for higher-dimensional maps. It has been known that there exists an error in the condition of the original Marotto theorem; hence, several authors had tried to correct it in different ways (Chen *et al.* 1998; Lin *et al.* 2002; Li & Chen 2003*b*; Shi & Chen 2004*b*), among which Shi & Chen gave a correct and complete presentation (Shi & Chen 2004*b*). It should be pointed out that it is possible to use other definitions of chaos, which, however, is beyond the scope of the present brief introduction to the subject. Moreover, for two-directional discrete systems, the reader is referred to a recent paper (Chen *et al.* 2004), and for the general continuous setting, to a brief overview (Wang 2003).

This paper is organized as follows. In §2, some necessary and detailed mathematical preliminaries on chaos in the sense of Devaney, Wiggins and Li–Yorke are provided. The central problem of chaotification is formulated and described in §3, where the state-of-the-art achievement and progress in this research area are summarized. In §4, chaotification in Banach spaces is discussed. Finally, §5 gives some concluding remarks.

2. Chaos in the sense of Devaney and Li–Yorke

This section gives some necessary mathematical preliminaries on the concept and criteria of discrete chaos, including chaos in the sense of Devaney and Li–Yorke, providing some rigorous guidelines for chaotification.

(*a*) *Chaos in the sense of Devaney and Wiggins*

A typical textbook definition of discrete chaos was given in Devaney (1987): Let S be a set in a metric space (X, d), and let f^m be the mth-order iteration of a map $f\colon S\to S$, namely, $f^m := f(f^{m-1})$, $m=1,2, \ldots$, with $f^0 =$ identity map.

Definition 2.1. A point $x^* \in S$ is called a *periodic point with period m* (or *m-periodic point*), if $x^* = f^m(x^*)$, but $x^* \neq f^k(x^*)$ for $1 \leq k < m$. If $m=1$, x^* is called a *fixed point*. The point x^* is called *periodic*, or is named a *periodic point*, if it is an m-periodic point for some $m \geq 1$.

Definition 2.2 (Chaos in the sense of Devaney). A map $f\colon S\to S$ is said to be *chaotic*, if:

- (i) the map f has sensitive dependence on initial conditions, in the sense that there exists $\delta > 0$, such that for any $x \in S$ and any neighbourhood $\mathcal{N}$ of x in S, $d(f^m(x), f^m(y)) > \delta$ for some $y \in \mathcal{N}$ and some $m \geq 0$,
- (ii) the map f is topologically transitive, in the sense that for any pair of non-empty open subsets U, $V \subset S$, there exists an integer $m > 0$, such that $f^m(U) \cap V \neq \varnothing$, and
- (iii) the periodic points of the map f are dense in S.

It should be noted that this definition has some redundancy and can be further simplified (e.g. Banks *et al.* 1992; Touhey 1997). If condition (iii) above is dropped, then it is called chaos in the sense of Wiggins (1990).

(*b*) *Chaos in the sense of Li–Yorke*

Let us consider a one-dimensional discrete system:

$$x_{k+1} = f(x_k), \quad x_k \in I \subset R, \quad k = 0, 1, 2, \ldots. \tag{2.1}$$

Li & Yorke (1975) introduced the first mathematical definition of chaos and established a criterion for it—simply called 'period three implies chaos'.

Theorem 2.1 (Li & Yorke 1975). *Let I be an interval in R and* $f\colon I\to I$ *be a continuous map. Let us assume that there is one point,* $a \in I$, *for which the points* $b = f(a)$, $c = f^2(a)$ *and* $d = f^3(a)$ *satisfy*

$$d \leq a < b < c \quad (\text{or } d \geq a > b > c).$$

Then:

- (i) *for every* $k = 1, 2, \ldots$, *there is a k-periodic point in I;*
- (ii) *there is an uncountable set* $S \subset I$, *containing no periodic points, which satisfies the following conditions:*

(ii$_1$) *for every p, q*$\in S$ *with* $p\neq q$,

$$\overline{\lim}_{n\to\infty}|f^n(p)-f^n(q)|>0$$

and

$$\underline{\lim}_{n\to\infty}|f^n(p)-f^n(q)|=0;$$

(ii$_2$) *for every* $p\in S$ *and periodic point* $q\in I$,

$$\overline{\lim}_{n\to\infty}|f^n(p)-f^n(q)|>0.$$

Motivated by the work of Li & Yorke (1975), Marotto (1978) further generalized this elegant result to the n-dimensional setting.

Let us consider the following n-dimensional system:

$$x_{k+1}=f(x_k),\quad x_k\in R^n,\quad k=0,1,2,\ldots,\tag{2.2}$$

where the map $f\colon R^n\to R^n$ is continuously differentiable. Denote by $B_r^0(x)$ the open ball in R^n of radius r centred at a point $x\in R^n$, and by $B_r(x)$ its closure.

Definition 2.3 (Marotto 1978).

(i) Let f be differentiable in $B_r(x)$. The point $x\in R^n$ is an *expanding fixed point* of f in $B_r(x)$, if $f(x)=x$ and all eigenvalues of $Df(y)$ exceed 1 in absolute value for all $y\in B_r(x)$.

(ii) Let us assume that x is an expanding fixed point of f in $B_r(x)$ for some $r>0$. Then, x is said to be a *snap-back repeller* of f, if there exists a point $x_0\in B_r(x)$ with $x_0\neq x$, such that $f^m(x_0)=x$ and the determinant $|Df^m(x_0)|\neq 0$ for an integer $m>0$.

In the following theorem, $\|x\|_1$ denotes the usual Euclidean norm of $x\in R^n$.

Theorem 2.2 (Marotto 1978). *If f possesses a snap-back repeller, then the system (2.2) is chaotic.*

(i) *There is a positive integer N, such that for each integer* $p\geq N$, *f has a point with period p.*

(ii) *There is a 'scrambled set' of f, namely, an uncountable set S containing no periodic points of f, such that*

(ii$_1$) $f(S)\subset S$,

(ii$_2$) *for every x,* $y\in S$ *with* $x\neq y$,

$$\overline{\lim}_{k\to\infty}\|f^k(x)-f^k(x)\|_1>0,$$

(ii$_3$) *for every* $x\in S$ *and any periodic point y of f,*

$$\overline{\lim}_{k\to\infty}\|f^k(x)-f^k(y)\|_1>0.$$

(iii) *There is an uncountable subset* S_0 *of S, such that for all* $x,y\in S_0$:

$$\underline{\lim}_{k\to\infty}\|f^k(x)-f^k(y)\|_1=0.$$

In the one-dimensional setting, the existence of a snap-back repeller of f is equivalent to the existence of a point of period-3 for the map f^n for some positive integer n, as pointed out in remark 3.1 of Marotto's paper (Marotto 1978).

Conditions (ii$_2$) and (iii) together imply that S_0 contains at most one point x that does not satisfy (ii$_3$) (Zhou 1987). Therefore, condition (ii$_3$) is not essential. Based on this fact, other definitions of scrambled set and chaos in the sense of Li–Yorke were proposed (Zhou 1987; Huang & Ye 2002; Sumi 2003), as stated below.

Definition 2.4 (Chaos in the sense of Li–Yorke). Let X be a metric space and $f: X \to X$ be continuous. A subset S of X is called a *scrambled set* of f, if for any two different points $x, y \in S$,

(i) $\underline{\lim}_{n\to\infty} d(f^n(x), f^n(y)) = 0$;
(ii) $\overline{\lim}_{n\to\infty} d(f^n(x), f^n(y)) > 0$.

A map f is said to be *chaotic in the sense of* Li–Yorke, if there exists an uncountable scrambled set S of f.

Under some conditions, chaos in the sense of Devaney is stronger than that in the sense of Li–Yorke (Huang & Ye 2002).

As mentioned in §1, to verify that the generated chaos is indeed a chaos in a rigorous mathematical sense, the celebrated Li–Yorke theorem for the one-dimensional case and the Marotto theorem for the n-dimensional case were usually employed. The Marotto theorem is the best one in predicting and analysing discrete chaos in higher-dimensional difference equations to date. As also mentioned, there exists an error in the condition of the original Marotto theorem (Marotto 1978), which has been corrected recently and a modified version of this important theorem is given by Shi & Chen (2004*b*), as follows:

Theorem 2.3 (A Modified Version of the Marotto Theorem) (Shi & Chen 2004*b*; Theorem 4.5). *Let $f: R^n \to R^n$ be a map with a fixed point $z \in R^n$. Let us assume that*

(1) *f is continuously differentiable in some neighbourhood of z and all the eigenvalues of $Df(z)$ have absolute values larger than 1, which implies that there exists a positive constant r and a norm $\|\cdot\|$ in R^n, such that f is expanding in $B_r(z)$ in norm $\|\cdot\|$, and*
(2) *z is a snap-back repeller of f with $f^m(x_0) = z$ for some $x_0 \in B_r^0(z)$, $x_0 \neq z$ and some positive integer m. Furthermore, f is continuously differentiable in some neighbourhoods of $x_0, x_1, \ldots, x_{m-1}$, respectively, and $\det Df(x_j) \neq 0$ for $0 \le j \le m-1$, where $x_j = f(x_{j-1})$, $0 \le j \le m-1$.*

Then, all the results of the Marotto Theorem hold.

(*c*) *Generalizations of the Marotto theorem*

The two concepts, expanding fixed point and snap-back repeller, introduced by Marotto for continuously differentiable maps in R^n, were extended recently to maps in metric spaces by Shi & Chen (2004*a*). Moreover, generalizations of the Marotto theorem in Banach spaces and complete metric spaces were also established (Shi *et al.* in press; Shi & Chen 2004*a*,*b*). These new results are introduced here in this section.

Let (X, d) be a metric space and $f: X \to X$ be a map.

(1) A point $z \in X$ is called an *expanding fixed point* of f in $B_r(z)$ for some constant $r > 0$, if $f(z) = z$ and there exists a constant $\lambda > 1$, such that

$$d(f(x), f(y)) \geq \lambda d(x, y), \quad \forall x,\ y \in B_r(z).$$

The constant λ is called an *expanding coefficient* of f in $B_r(z)$. Furthermore, z is called a *regular expanding fixed point* of f in $B_r(z)$, if z is an interior point of $f(B_r(z))$.

(2) Let us assume that z is an expanding fixed point of f in $B_r(z)$ for some $r > 0$. Then, z is said to be a *snap-back repeller* of f, if there exists a point $x_0 \in B_r^0(z)$ with $x_0 \neq z$ and $f^m(x_0) = z$ for some positive integer m. Furthermore, z is said to be a *non-degenerate snap-back repeller* of f, if there exist positive constants μ and r_0, such that $B_{r_0}^0(x_0) \subset B_r^0(z)$ and

$$d(f^m(x), f^m(y)) \geq \mu d(x, y), \quad \forall x,\ y \in B_{r_0}(x_0);$$

z is called a *regular snap-back repeller* of f, if $f(B_r^0(z))$ is open and there exists a positive constant δ_0, such that $B_{\delta_0}^0(x_0) \subset B_r^0(z)$ and z is an interior point of $f^m(B_\delta^0(x_0))$ for all positive constants $\delta \leq \delta_0$.

Several remarks are given below in order to understand the above-introduced concepts (see Shi & Chen (2004*a*,*b*) for more details):

(1) Let $(X, \|\cdot\|)$ be a Banach space. If f is continuously Frechét differentiable in the neighbourhood of a fixed point $z \in X$, then z is an expanding fixed point of f if, and only if,

$$\|Df(z)\|^0 > 1,$$

where

$$\|D f(z)\|^0 := \inf\{\|D f(z)x\| : x \in X \text{ with } \|x\| = 1\}.$$

Furthermore, z is a regular expanding fixed point of f if $D f(z)$ is an invertible linear map, i.e. $D f(z)$ has a bounded linear inverse map. In the special case of $X = R^n$, if f is continuously differentiable in the neighbourhood of a fixed point z, then z is an expanding fixed point of f in some norm if, and only if, all the eigenvalues of the Jacobi matrix $D f(z)$ have absolute values strictly larger than 1. In this case, z is a regular expanding fixed point of f. Hence, if f satisfies the conditions in (i) of the Marotto definitions (see §2*b*), then z is a regular expanding fixed point of f in some norm (but generally not in the usual Euclidean norm).

(2) In the case of $X = R^n$, if z is a snap-back repeller of f with x_0, m, r specified as in the above definition, and if f is continuous in $B_r(z)$ and continuously differentiable in some neighbourhoods of $x_0, x_1, \ldots, x_{m-1}$, respectively, with $\det\ D f(x_j) \neq 0$ for $0 \leq j \leq m-1$, where $x_j = f(x_{j-1})$ $(0 \leq j \leq m-1)$, then z is a regular and non-degenerate snap-back repeller of f. Obviously, $\det\ Df^m(x_0) = \det Df(x_{m-1}) \cdot \det Df(x_{m-2}) \cdots \det Df(x_0)$. Hence, z is a regular and non-degenerate snap-back repeller of f, if f satisfies the conditions in (ii) of the Marotto definitions (see §2*b*).

The following is a generalization of the Marotto Theorem in Banach spaces and in complete metric spaces.

Theorem 2.4 (A generalization of the Marotto Theorem in Banach Spaces) (Shi *et al.* in press; Theorem 2.1). *Let $(X, \|\cdot\|)$ be a Banach space and let $f: X \to X$ be a map with a fixed point $z \in X$. Let us assume that*

(1) *f is continuously differentiable in $B^0_{r_0}(z)$ for some $r_0 > 0$ and $Df(z)$ is an invertible linear map satisfying*

$$\|Df(z)\|^0 > 1,$$

which is equivalent to that there exists a positive constant $r \leq r_0$, such that z is a regular expanding fixed point of f in $B_r(z)$, and

(2) *z is a snap-back repeller of f with $f^m(x_0) = z$ for some $x_0 \in B^0_r(z)$, $x_0 \neq z$, and some positive integer m. Furthermore, f is continuously differentiable in some neighbourhoods of $x_1, \ldots, x_{m-1}$, respectively, satisfying that $Df(x)$ is an invertible linear map for all $x \in B^0_r(z)$, $x = x_j$ $(1 \leq j \leq m-1)$ and $\|Df(x_j)\|^0 > 0$ for $1 \leq j \leq m-1$, where $x_j = f(x_{j-1})$, $1 \leq j \leq m-1$.*

Then, for any neighbourhood U of z, there exists a positive integer $n > m$ and a Cantor set $\Lambda \subset U$, such that $f^n: \Lambda \to \Lambda$ is topologically conjugate to the symbolic dynamical system $\sigma: \sum_2^+ \to \sum_2^+$. Consequently, there exists a compact and perfect invariant set $S \subset X$ containing a Cantor set, such that f is chaotic on S in the sense of Devaney as well as Li–Yorke, and has a dense orbit on S.

Theorem 2.5 (A generalization of the Marotto Theorem in Complete Metric Spaces) (Shi & Chen 2004*a*; Theorem 4.1). *Let (X, d) be a complete metric space and $f: X \to X$ be a map. Let us assume that*

(1) *f has a regular and non-degenerate snap-back repeller $z \in X$, i.e. there exist positive constants r_1 and $\lambda_1 > 1$, such that $f(B^0_{r_1}(z))$ is open and*

$$d(f(x), f(y)) \geq \lambda_1 d(x, y), \quad \forall x, y \in B_{r_1}(z),$$

and there exists a point $x_0 \in B^0_{r_1}(z)$, $x_0 \neq z$, a positive integer m, and positive constants δ and λ_2, such that $f^m(x_0) = z$, $B^0_\delta(x_0) \subset B^0_{r_1}(z)$, z is an interior point of $f^m(B^0_\delta(x_0))$, and

$$d(f^m(x), f^m(y)) \geq \lambda_2 d(x, y), \quad \forall x, y \in B_\delta(x_0), \text{ and}$$

(2) *there exists a positive constant μ_1, such that*

$$d(f(x), f(y)) \leq \mu_1 d(x, y), \quad \forall x, y \in B_{r_1}(z), \text{ and}$$

(3) *there exists a positive constant μ_2, such that*

$$d(f^m(x), f^m(y)) \leq \mu_2 d(x, y), \quad \forall x, y \in B_\delta(x_0).$$

Then, for each neighbourhood U of z, there exists a positive integer $n > m$ and a Cantor set $\Lambda \subset U$, such that $f^n: \Lambda \to \Lambda$ is topologically conjugate to the symbolic dynamical system $\sigma: \sum_2^+ \to \sum_2^+$. Consequently, f^n is chaotic on Λ in the sense of Devaney and also f is chaotic in the sense of Li–Yorke.

(*d*) *Chaos in complete metric spaces*

Motivated by some basic properties of the logistic map, which produces chaos, Shi & Chen (2004*a*) established a criterion of chaos for discrete maps in complete metric spaces.

Theorem 2.6 (Shi & Chen 2004*a*; Theorems 3.1 and 3.2). *Let* (X, d) *be a complete metric space and* V_1, V_2 *be non-empty, closed and bounded subsets of* X *with* $d(V_1, V_2)>0$. *If a continuous map* $f: V_1\cup V_2\to X$ *satisfies the following conditions:*

(1) $f(V_j)\supset V_1\cup V_2$ *for* $j=1,2$,
(2) f *is expanding in* V_1 *and* V_2, *respectively, i.e. there exists a constant* $\lambda_0>1$, *such that*

$$d(f(x),f(y))\geq \lambda_0 d(x,y), \quad \forall x,y\in V_1 \text{ and } \forall\, x,y\in V_2, \text{ and}$$

(3) *there exists a constant* $\mu_0>0$, *such that*

$$d(f(x),f(y))\leq \mu_0 d(x,y), \forall x,y\in V_1 \text{ and} \forall\, x,y\in V_2,$$

then, there exists a Cantor set $\Lambda\subset V_1\cup V_2$, *such that* $f:\Lambda\to\Lambda$ *is topologically conjugate to the symbolic dynamical system* $\sigma:\sum_2^+\to\sum_2^+$. *Consequently,* f *is chaotic on* Λ *in the sense of both Devaney and Li–Yorke. In the special case where* V_1 *and* V_2 *are compact subsets of* X, *the above results still hold, with the assumption* (3) *removed.*

3. Chaotification: problem description and basic schemes

Now, let us consider a general finite-dimensional discrete-time dynamical system, originally neither chaotic or complex, nor ill-behaved or unstable, in the following form:

$$x_{k+1}=f_k(x_k), \quad x_0\in R^n \quad \text{is given,} \tag{3.1}$$

where $f_k(\cdot)$ is only assumed to be continuously differentiable, at least locally in a region of interest. In other words, the given system can be linear or nonlinear, time-invariant or time-varying and stable or unstable.

The objective is to design a control input sequence, $\{u_k\}$, such that the output of the controlled system,

$$x_{k+1}=f_k(x_k)+u_k, \tag{3.2}$$

is chaotic, in the sense of Devaney, Wiggins or Li–Yorke defined above.

An important remark is that in a practical design, a meaningful controller should be simple in structure, as simple as possible and preferably simpler than the given system, such that the goal of control (here, chaotification) can be achieved. The main reason is that although many things can be done mathematically, a designed controller should be simple, cheap, user-friendly and implementable in engineering applications; therefore, practically it does not make sense to come out with a controller that is more complex and more expensive than the given system to be controlled. In real life, these kind of controllers will not be used anywhere. The discussion below tries to follow this basic engineering principle, thereby designing some very simple and implementable chaotifiers (anti-controllers).

(*a*) *A generic chaotification algorithm*

Chen & Lai (1997, 1998) developed the first rigorous anti-control algorithm based on feedback control with mod-operation, described by

$$x_{n+1} = f(x_n) + u_n(\mathrm{mod}\,1), \tag{3.3}$$

with

$$u_n = \mu x_n, \tag{3.4}$$

where $f(0)=0$; f is continuously differentiable at least locally in a region near $x^*=0$, in the usual k-dimensional real space R^k, and satisfies

$$\|Df(x)\|_1 \leq L, \tag{3.5}$$

for all $x \in R^k$ or for all x in some region containing $x^*=0$, and for some constant $L>0$, with $\|C\|_1$ being the spectral norm of any $k\times k$ matrix $C=(c_{ij})$, $\mu=L+\mathrm{e}^c$, $c>0$ is a parameter, and k is a positive integer.

It is noted that $\|C\|_1 \leq \|C\|$, where $\|C\| = \max\{\sum_{j=1}^{k}|c_{ij}| : 1\leq i\leq k\}$ is the operator norm of a $k\times k$ matrix $C=(c_{ij})$. It is shown by Chen & Lai (1998) that for any $c>0$, the controlled systems (3.3) and (3.4) are chaotic in the sense of Devaney in the linear case of $f(x)=Ax$, where A is a $k\times k$ real matrix, and is chaotic in the sense of Wiggins in the nonlinear case. Later, Wang & Chen (1999) showed that the controlled systems (3.3) and (3.4) are chaotic in the sense of Li–Yorke by using the Marotto theorem. It is known that chaos in the sense of Devaney is stronger than that in the sense of Wiggins and also Li–Yorke under some conditions, as mentioned above. Recently, this chaotification algorithm with the mod-operation has been further extended to a general metric space Y_k (Shi *et al.* in press), which will be discussed later.

Let us assume that $f(0)=0$ and there exist positive constants r and L, such that f is continuously differentiable in $B_r(0)$ and satisfies

$$\|Df(x)\| \leq L, \quad \forall x \in B_r(0). \tag{3.6}$$

Let us consider the controlled system

$$x_{n+1} = f(x_n) + \mu x_n(\mathrm{mod}\ r), \tag{3.7}$$

where the mod-operation is component-wise. It is proved by Shi *et al.* (in press) that for each constant μ satisfying

$$\mu > \mu_0 := \max\{5(1+L), 10L\}.$$

The controlled system (3.7) is chaotic on a Cantor set in the sense of Devaney, as well as both Li–Yorke and Wiggins.

It is well known that if f is continuously differentiable in $B_1(0)$ in the case of $k<\infty$, then both $\|Df(x)\|$ and $\|Df(x)\|_1$ are bounded in $B_1(0)$. Hence, by taking $r=1$ and $k<\infty$, the controlled system (3.3) with controller (3.4) is chaotic on a Cantor set in the sense of Devaney, as well as both Li–Yorke and Wiggins.

In summary, the Chen–Lai anti-control algorithm via state-feedback control with the mod-operation leads to chaos in the sense of Devaney, Li–Yorke and Wiggins.

Some simplification and modification of the above-described basic Chen–Lai anti-control algorithm were also developed (Wang & Chen 2000*b*; Zhang & Chen 2004; Zheng & Chen 2004; Zheng *et al.* 2004).

(*b*) *Chaotification for continuous maps*

It has been noted that in most publications concerning chaotifying discrete dynamical systems, it is assumed that the maps, corresponding to the original system and the designed controller, are continuously differentiable in the domain of interest, and the map, corresponding to the original system, has at least one fixed point in the domain. It is clear that these conditions may not be satisfied in some physical models.

Recently, chaotification of discrete dynamical systems governed by continuous maps was studied. If the map f, corresponding to the original system (3.1), is continuous and monotonic on two disjoint closed intervals in the one-dimensional case, and is continuous and satisfies the Lipschitz condition on two disjoint closed rectangular regions in the higher-dimensional case, then some simple state-feedback controllers can be designed, such that the controlled system,

$$x_{n+1} = f(x_n) + \mu g(x_n), \tag{3.8}$$

is chaotic in the sense of Devaney for all sufficiently large values of μ (Shi & Chen 2005). For example, g can be taken as the sine function or a piecewise linear function component-wise. In these results, it is not required that f has a fixed point and is continuously differentiable in the domain, and it is also not required that g is continuously differentiable in its domain. This makes it possible to design some very simple controllers for a large family of system models.

4. Chaotification in Banach spaces

Recently, there were some attempts in chaotifying discrete dynamical systems in Banach spaces, with some fundamental results established by Shi *et al.* (in press), as follows.

Let us consider the following time-invariant discrete dynamical system:

$$x_{n+1} = f(x_n), \quad n \geq 0, \tag{4.1}$$

where $f: D \subset X \to X$ is a map and $(X, \|\cdot\|)$ is a Banach space.

The objective is to design a (simple) control input sequence, $\{u_n\}$, such that the output of the controlled system

$$x_{n+1} = f(x_n) + u_n, \quad n \geq 0, \tag{4.2}$$

is chaotic in the sense of Devaney. The controller to be designed is in the form of

$$u_n = g(\mu x_n), \tag{4.3}$$

or in the form

$$u_n = \mu g(x_n), \tag{4.4}$$

where μ is a positive parameter, and the map $g: D' \to X$ is expected to be (very) simple with D' being a suitable subset of X.

(*a*) *Chaotification in general Banach spaces*

Let us assume that the map f, corresponding to the original system (4.1), satisfies $f(0)=0$ and there exists positive constants r and L, such that f is continuous in $B_r(0)$ and continuously differentiable in $B_r^0(0)$, satisfying

$$\|D f(x)\| \leq L, \quad \forall x \in B_r^0(0), \tag{4.5}$$

where $D f(x)$ denote the Frechét derivative of f at x, $\|D f(x)\|$ is the norm of a bounded linear operator $Df(x)$, i.e.

$$\|Df(x)\| := \sup\{\|Df(x)y\| : y \in X \text{ with } \|y\| = 1\},$$

while $B_r(0) = \{x \in X : \|x\| \leq r\}$ and $B_r^0(0) = \{x \in X : \|x\| < r\}$ are the closed and open balls of radium r centred at $x=0$, respectively.

First, let us consider controller (4.3). If the map g in controller (4.3) satisfies the following conditions:

(i) g is continuous in $B_r(0) \cup \Omega$ and continuously differentiable in $B_r^0(0) \cup \Omega^0$, where $\Omega = \{x \in X : a \leq \|x\| \leq b\}$ and Ω^0 is the interior of Ω with $r < a < b$,
(ii) $x^* = 0$ is a fixed point of g and there exists a point $\xi \in \Omega^0$, such that $g(\xi) = 0$, and
(iii) $Dg(x)$ is an invertible linear operator for each $x \in B_r^0(0) \cup \Omega^0$ and there exists a positive constant N, such that

$$\|g(x) - g(y)\| \geq N\|x - y\|, \quad \forall x, y \in B_r(0) \text{ and } \forall\, x, y \in \bar{\Omega},$$

then, for each constant μ satisfying

$$\mu > \mu_0 := \max\left\{\frac{b}{r}, \frac{Lr + b}{Nr}, \frac{Lb}{N(\|\xi\| - a)}, \frac{Lb}{N(b - \|\xi\|)}\right\},$$

and for any neighbourhood U of $x^* = 0$, there exists a positive integer $n > 2$ and a Cantor set $\Lambda \subset U$, such that $F_\mu^n : \Lambda \to \Lambda$ is topologically conjugate to the symbolic dynamical system $\sigma : \sum_2^+ \to \sum_2^+$, where $F_\mu(x) = f(x) + g(\mu x)$. Consequently, as stated in Theorem 2.4 above, F_μ is chaotic on S in the sense of Devaney as well as Li–Yorke.

Here, g can be designed to be simple and satisfies assumptions (i)–(iii). For example, g can be taken as one of the following four simple functions:

$$g_1(x) = \begin{cases} \pm x, & \text{if } \|x\| \leq r \\ \text{abitrary}, & r < \|x\| < a \;, \\ \pm(x - \xi), & \text{if } a \leq \|x\| \leq b, \end{cases}$$

where $0 < r < a < b$ and $\xi \in X$ is a fixed point satisfying $a < \|\xi\| < b$.

Next, let us consider controller (4.4). If the map g in controller (4.4) satisfies the following conditions:

(1) g is continuous in $B_a(0) \cup \Omega'$ and continuously differentiable in $B_a^0(0) \cup \Omega'^0$, where $\Omega' = \{x \in X : b \leq \|x\| \leq r\}$ and Ω'^0 is the interior of Ω' with $0 < a < b < r$,
(2) $x^* = 0$ is a fixed point of g and there exists a point $\xi \in \Omega'^0$, such that $g(\xi) = 0$, and
(3) $Dg(x)$ is an invertible linear operator for each $x \in B_a^0(0) \cup \Omega'^0$ and there exists a positive constant N, such that

$$\|g(x) - g(y)\| \geq N\|x - y\|, \quad \forall x, y \in B_a(0) \text{ and } \forall\, x,\ y \in \Omega',$$

then, for each constant μ satisfying

$$\mu > \mu_0 := \max\left\{\frac{La + r}{Na}, \frac{Lr}{N(\|\xi\| - b)}, \frac{Lr}{N(r - \|\xi\|)}\right\},$$

and for any neighbourhood U of $x^*=0$, there exists a positive integer $n>2$ and a Cantor set $\Lambda\subset U$, such that $F_\mu^n:\Lambda\to\Lambda$ is topologically conjugated to the symbolic dynamical system $\sigma:\sum_2^+\to\sum_2^+$, where $F_\mu(x)=f(x)+\mu g(x)$. Consequently, as stated in Theorem 2.4, F_μ is chaotic on S in the sense of Devaney and $F\mu$ as well as Li–Yorke.

Here, g can also be easily designed and satisfies assumptions (i)–(iii). For example, g can be taken as one of the following four simple functions:

$$g_2(x)=\begin{cases}\pm x, & \text{if } \|x\|\le a\\ \text{abitrary} & a<\|x\|<b\\ \pm(x-\xi), & \text{if } b\le\|x\|\le r,\end{cases},$$

where $0<a<b<r$ and $\xi\in X$ is a fixed point satisfying $b<\|\xi\|<r$.

(b) *Chaotification in some special Banach spaces*

Let

$$R^k=\{x=\{x_j\}_{j=1}^k : x_j\in R \text{ for } 1\le j\le k\},$$

with $1\le k\le\infty$ and

$$Y_k=\{x\in R^k : \|x\|_k<\infty\},$$

with the norm

$$\|x\|_k=\sup\{|x_j| : 1\le j\le k\}.$$

It can be easily verified that $(Y_k, \|\cdot\|_k)$ is a Banach space. Clearly, in the special case of $k<\infty$, Y_k is the classical k-dimensional real space R^k and its norm $\|\cdot\|_k$ is the sup-norm, while in the special case of $k=\infty$, $Y_k=l^\infty$ and the norm $\|\cdot\|_k$ is the usual norm of l^∞.

In general, of course, chaotification of system (4.1) in $(Y_k, \|\cdot\|_k)$ can be achieved by using the controller (4.3) or (4.4) with the map g satisfying assumptions (i)–(iii) or (1)–(3) in §4*a*, respectively. Here, the chaotification of system (4.1) is considered in the space $(Y_k, \|\cdot\|_k)$ with a special feedback controller u_n given in the form of equation (4.3) or (4.4). In addition, the Chen–Lai anti-control algorithm with mod-operation in a finite-dimensional real space, proposed by Chen & Lai (1998), is extended to Y_∞.

For convenience, denote

$$I^k:=\{x=\{x_j\}_{j=1}^k : x_j\in I \text{ for } 1\le j\le k\},$$

where I is a bounded subset of R. Clearly, $I^k\subset Y_k$.

Introduce the following function in Y_k for each $k\ge 1$:

$$\mathrm{Saw}_r(x)=\{\mathrm{saw}_r(x_j)\}_{j=1}^k, \tag{4.6}$$

where saw_r is the classical sawtooth function, i.e.

$$\mathrm{saw}_r(x)=(-1)^m(x-2mr), (2m-1)r\le x<(2m+1)r, m\in Z,$$

where Z is the integer set. Clearly, $\mathrm{Saw}_r(x)$ is the sawtooth function $\mathrm{saw}_r(x)$ in R, when $k=1$, and $\mathrm{Saw}_r(x)$ is the sawtooth function $\mathrm{saw}_r(x)$ in R^k, when $k<\infty$, defined by eqns (9) and (12) by Wang & Chen (2000*a*). So, Saw_r here can be regarded as a generalization of the classical sawtooth function.

First, let us consider the controlled system (4.2) with (4.4). Let us assume that f satisfies

$$\|f(x)-f(y)\| \le L\|x-y\|, \quad \forall x, y \in B_r(0), \tag{4.7}$$

for some positive constants r and L. If the map g given in equation (4.4) is taken as

$$g(x) = \mathrm{Saw}_{r/3}(x),$$

then, for each constant μ satisfying

$$\mu > \mu_0 := \max\{1 + L, 5 + 6(L + \|f(0)\|r^{-1})\},$$

the controlled system (4.2) with (4.4) is chaotic on a Cantor set in the sense of both Devaney and Li–Yorke.

Next, let us consider the controlled system (4.2) with (4.3). Under the assumption (4.7), if $f(0)=0$ and the map g given in (4.4) is taken as

$$g(x) = \epsilon\, \mathrm{Saw}_r(x),$$

where $\epsilon \neq 0$ is a fixed real constant, then, for each constant μ satisfying

$$\mu > \mu_0 := 5|\epsilon|^{-1}(1 + L),$$

the controlled system (4.2) with (4.3) is chaotic on a Cantor set in the sense of both Devaney and Li–Yorke. It is noted that the controller $g(\mu x)=\epsilon\, \mathrm{Saw}_r(\mu x)$ can be arbitrarily small in norm, since the constant ϵ can be taken as arbitrarily small in absolute value.

Several remarks on the conditions about f and g are in order. First, it is not required that f has a fixed point in the domain of interest, when the controller (4.4) is used. Second, if f is continuous in $B_r(0)$, continuously differentiable in $B_r^0(0)$, and satisfies equation (4.5), then equation (4.7) holds. Finally, it is noted that the sawtooth function $\mathrm{saw}_r(x)$ is unimodal on R. In fact, any function, as long as it has similar geometric properties to the sawtooth function, may be used to generate a function in Y_k in the same way as in equation (4.6) and the generated function can be regarded as an anti-controller, such that the controlled system is chaotic (Chen 2003). For example, the classical sine function $\sin x$ has similar geometric properties to the sawtooth function. Similar to $\mathrm{Saw}_r(x)$ in equation (4.6), the following function is generated by $\sin x$:

$$\mathrm{Sin}(x) = \{\sin x_j\}_{j=1}^{k}, \quad x = \{x_j\}_{j=1}^{k} \in Y_k.$$

If $g(x)=\epsilon\, \mathrm{Sin}\,(r^{-1}x)$ in controller (4.3) and $g(x)=\mathrm{Sin}\,(4r^{-1}x)$ in controller (4.4), then the controlled systems (4.2) with (4.3) and (4.2) with (4.4) are chaotic in the sense of Devaney and Li–Yorke, respectively, for all sufficiently large values of μ.

5. Concluding remarks

The emerging field of chaos control and anti-control (chaotification) is very stimulating and promising. This new direction of research has gradually become quite intensive and is expected to have far-reaching impacts with enormous opportunities in academic, medical, industrial and commercial applications. New theories for dynamics analysis, new methodologies for controls and new design for circuit implementation altogether are calling for new efforts and endeavours from the communities of nonlinear dynamics, circuits and systems, biological and social sciences, applied mathematics and especially control systems. Control theorists and engineers should not miss this unique opportunity, especially those who have expertise in both nonlinear dynamics and nonlinear control systems.

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