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CHAOS, BIFURCATIONS, AND THEIR CONTROL

1. NONLINEAR DYNAMICS

Unlike linear systems, many nonlinear dynamical systems do not show orderly, regular, and long-term predictable responses to simple inputs. Instead, they display complex, random-like, seemingly irregular, yet well-defined output behaviors. Such dynamical phenomenon is known as *chaos* today.

The term *chaos*, originated from the Greek word $\chi\alpha o\zeta$, was designated "the primeval emptiness of the universe before things came into being of the abyss of Tartarus, the underworld. … In the later cosmologies, Chaos generally designated the original state of things, however conceived. The modern meaning of the word is derived from Ovid, who saw Chaos as the original disordered and formless mass, from which the maker of the Cosmos produced the ordered universe" (1). There also is an interpretation of chaos in ancient Chinese literature, which refers to the spirit existing in the center of the universe (2). In modern scientific terminology, chaos has a fairly precise but rather complicated definition by means of the dynamics of a generally nonlinear system. For example, in theoretical physics, "chaos is a type of moderated randomness that, unlike true randomness, contains complex patterns that are mostly unknown" (3).

Bifurcation, as a twin of chaos, is another prominent phenomenon of nonlinear dynamical systems: Quantitative change of system parameters leads to qualitative change of system properties such as the number and the stability of system equilibria. Typical bifurcations include transcritical, saddle-node, pitchfork, hysteresis, and Hopf bifurcations. In particular, period-doubling bifurcation is a route to chaos. To introduce the concepts of chaos and bifurcations as well as their control (4,5,6,7,8,9), some preliminaries on nonlinear dynamical systems are in order.

1.1. Nonlinear Dynamical Systems

A *nonlinear system* refers to a set of nonlinear equations, which can be algebraic, difference, differential, integral, functional, or abstract operator equations, even a certain combination of these. A nonlinear system is used to describe a physical device or process that otherwise cannot be well defined by a set of linear equations of any kind. *Dynamical system* is used as a synonym of a mathematical or physical system, in which the output behavior evolves with time and/or with other varying system parameters (10).

In general, a continuous-time dynamical system is described by a differential equation,

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, t; \mathbf{p}\right) \qquad t \in [t_0, \infty) \tag{1}$$

where $\mathbf{x} = \mathbf{x}(t)$ is the *state* of the system, $\mathbf{p}$ is a vector of variable *system parameters*, and f and g are continuous or smooth (differentiable) nonlinear functions of comparable dimensions, which have an explicit formulation for a specified physical system in interest.

In the discrete-time setting, a nonlinear dynamical system is described by either a difference equation,

$$\mathbf{x}_{k+1} = \mathbf{f}_k\left(\mathbf{x}_k; \mathbf{p}\right), \qquad k = 0, 1, \ldots \tag{2}$$

or a map,

$$F: \quad \mathbf{x}_k \rightarrow g_k\left(\mathbf{x}_k; \mathbf{p}\right), \qquad k = 0, 1, \ldots \tag{3}$$

where notation is similarly defined. Repeatedly iterating the discrete map F backward yields

$$\mathbf{x}_k = F\left(\mathbf{x}_{k-1}\right) = F\left(F(x_{k-2})\right) = \cdots = F^k\left(\mathbf{x}_0\right)$$

where the map can also be replaced by a function $\mathbf{f}$, if the system is given via a difference equation, leading to

$$\mathbf{x}_k = \underbrace{\mathbf{f} \circ \cdots \circ \mathbf{f}}_{k \text{ times}} \left(\mathbf{x}_0\right) = \mathbf{f}^k\left(\mathbf{x}_0\right)$$

where "$\circ$" denotes composition operation of functions or maps.

Dynamical system (eq. 1) is said to be *nonautonomous* when the time variable t appears separately in the system function $\mathbf{f}$ (e.g. a system with an external time-varying force input); otherwise, it is said to be *autonomous* and is expressed as

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}; \mathbf{p}\right), \qquad t \in (t_0, \infty) \tag{4}$$

1.2. Classification of Equilibria

For illustration, consider a general two-dimensional autonomous system:

$$\begin{cases} \dot{x} = f(x, y) \\ \\ \dot{y} = g(x, y) \end{cases} \tag{5}$$

with given initial conditions (x_0, y_0), where f and g are two smooth nonlinear functions that together describe the *vector field* of the system.

The path traveled by a solution of system (eq. 5), starting from the initial state (x_0, y_0), is a *solution trajectory*, or *orbit*, of the system, and is sometimes denoted $\varphi_t(x_0, y_0)$. For autonomous systems, two different orbits will never cross each other (i.e., never intersect) on the x–y plane. This x–y coordinate plane is called the (*generalized*) *phase plane* (*phase space*, in the higher dimensional case). The orbit family of a general autonomous system, corresponding to all possible initial conditions, is called *solution flow* in the phase space.

Equilibria, or *fixed points*, of system (eq. 5), if they exist, are the solutions of two homogeneous equations:

$$f(x, y) = 0 \qquad \text{and} \qquad g(x, y) = 0$$

An equilibrium is denoted $(\bar{x}, \bar{y})$. It is *stable* if all the nearby orbits of the system, starting from any initial conditions, approach it; it is *unstable*, if the nearby orbits are moving away from it. Equilibria can be classified, according to their stabilities, as stable or unstable *node* or *focus*, and *saddle point* or *center*, as summarized in Figure 1. The type of equilibria is determined by the eigenvalues, $\lambda_{1,2}$,

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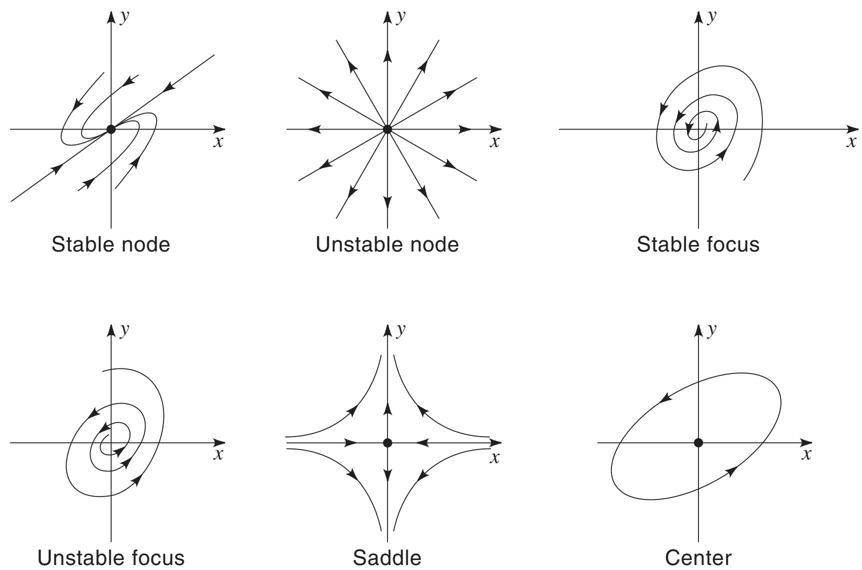

Figure 1. Classification of two-dimensional equilibria: stabilities are determined by Jacobian eigenvalues.

of the system Jacobian, $J := \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$, with $f_x := \partial f/\partial x$, $f_y := \partial f/\partial y$, and so on, all evaluated at $(\bar{x}, \bar{y})$. If the two Jacobian eigenvalues have real parts $\Re\{\lambda_{1,2}\} \neq 0$, the equilibrium $(\bar{x}, \bar{y})$, at which the linearization was taken, is said to be *hyperbolic*.

Theorem 1 (Grobman–Hartman) *If $(\bar{x}, \bar{y})$ is a hyperbolic equilibrium of the nonlinear dynamical system (eq. 5), then the dynamical behavior of the nonlinear system is qualitatively the same as (topologically equivalent to) that of its linearized system, $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} x \\ y \end{bmatrix}$, in a neighborhood of the equilibrium $(\bar{x}, \bar{y})$.*

This theorem guarantees that for the hyperbolic case, one can study the linearized system instead of the original nonlinear system, with regard to the *local* dynamical behavior of the system within a (small) neighborhood of the equilibrium $(\bar{x}, \bar{y})$. In general, "dynamical behavior" refers to such nonlinear phenomena as stabilities and bifurcations, chaos and attractors, equilibria and limit cycles, and so on. In other words, there exist some *homeomorphic maps* that transform the orbits of the nonlinear system into orbits of its linearized system in a (small) neighborhood of the equilibrium. Here, a homeomorphic map (or a *homeomorphism*) is a continuous map whose inverse exists and is also continuous. However, in the nonhyperbolic case, the situation is much more complicated, where such local topological equivalence does not hold in general.

1.3. Limit Sets and Attractors

The most basic problem in studying the general nonlinear dynamical system (eq. 1) is to understand and/or to analyze the system solutions. The asymptotic behavior of a system solution, as $t \to \infty$, is called the *steady state* of the solution, while the solution trajectory between its initial state and the steady state is the *transient state*.

For a given dynamical system, a point $\mathbf{x}_\omega$ in the state space is an *ω-limit point* of the system state orbit $\mathbf{x}(t)$ if, for every open neighborhood U of x_ω, the trajectory of $\mathbf{x}(t)$ will enter U at a (large enough) value of t. Consequently, $\mathbf{x}(t)$ will repeatedly enter U infinitely many times, as $t \to \infty$. The set of all such ω-limit points of $\mathbf{x}(t)$ is called the *ω-limit set* of $\mathbf{x}(t)$, and is denoted Ω_x. An ω-limit set of $\mathbf{x}(t)$ is *attracting*, if there exists an open neighborhood V of Ω_x such that whenever system orbit enters V, it will approach Ω_x as $t \to \infty$. The *basin of attraction* of an attracting point is the union of all such open neighborhoods. An ω-limit set is *repelling*, if the system orbits always move away from it.

For a given map F and a given initial state $\mathbf{x}_0$, an ω-limit set is obtained from the orbit $\left\{ F^k(x_0) \right\}$ as $k \to \infty$. This ω-limit set Ω_x is an invariant set of the map, in the sense that $F(\Omega_x) \subseteq \Omega_x$. Thus, ω-limit sets include equilibria and periodic orbits.

An *attractor* is an ω-limit set having the property that all orbits nearby have it as their ω-limit sets. Thus, a collection of isolated attracting points is not an attractor.

1.4. Periodic Orbits and Limit Cycles

A solution orbit $\mathbf{x}(t)$ of the nonlinear dynamical system (eq. 1) is a *periodic solution* if it satisfies $\mathbf{x}(t + t_p) = \mathbf{x}(t)$ for some constant $t_p > 0$. The minimum value of such t_p is called the (*fundamental*) *period* of the periodic solution, while the solution is said to be t_p periodic.

A *limit cycle* of a dynamical system is a periodic solution of the system that corresponds to a closed orbit on the phase plane and possesses certain attracting (or repelling) properties. Figure 2 shows some typical limit cycles: (a) an

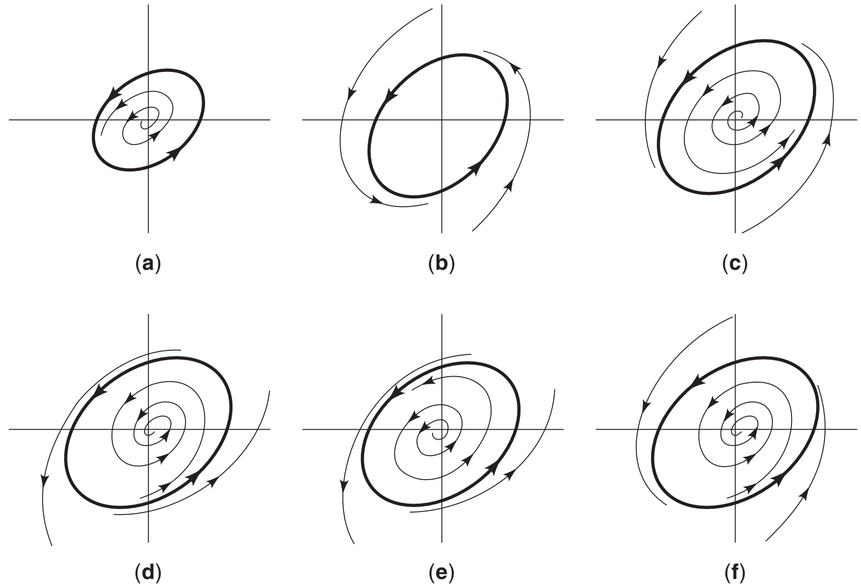

Figure 2. Periodic orbits and limit cycles.

inner limit cycle, (b) an *outer limit cycle*, (c) a *stable limit cycle*, (d) an *unstable limit cycle*, and (e and f), *saddle limit cycles*.

1.5. Poincaré Maps

Assume that the general n-dimensional nonlinear autonomous system (eq. 4) has a t_p periodic orbit, Γ, and let $\mathbf{x}^*$ be a point on the periodic orbit and Σ be an $(n-1)$-dimensional hyperplane transversal to Γ at $\mathbf{x}^*$, as shown in Figure 3. Since Γ is t_p periodic, the orbit starting from $\mathbf{x}^*$ will return to $\mathbf{x}^*$ in time t_p. Any orbit starting from a point $\mathbf{x}$ in a small neighborhood U of $\mathbf{x}^*$ on Σ will return and hit Γ at a point, denoted $P(\mathbf{x})$, in the vicinity V of $\mathbf{x}^*$. Therefore, a map $P : U \to V$ can be uniquely defined by Σ, along with the solution flow of the autonomous system. This map is called the *Poincaré map* associated with the system and the *cross section* Σ. For different choices of the cross section Σ, Poincaré maps are similarly defined.

Note that a Poincaré map is only locally defined and is a *diffeomorphism*, that is, a differentiable map that has an inverse and the inverse is also differentiable. If a cross section is suitably chosen, the orbit will repeatedly return and pass through the section. The Poincaré map together with the first return orbit is particularly important, which is called the *first return Poincaré map*. Poincaré maps can also be defined for non-autonomous systems in a similar way where, however, each return map depends on the initial time in a non-uniform fashion.

1.6. Homoclinic and Heteroclinic Orbits

Let $\mathbf{x}^*$ be a hyperbolic equilibrium of a diffeomorphism $P : R^n \to R^n$, which can be of either unstable, center, or saddle type, $\varphi_t(\mathbf{x})$ be a solution orbit passing through $\mathbf{x}^*$, and Ω_{x^*} be the ω-limit set of $\varphi_t(\mathbf{x})$. The *stable manifold* of Ω_{x^*}, denoted M_s, is the set of such points x^* that satisfy $\varphi_t(\mathbf{x}^*) \to \Omega_{x^*}$ as $t \to \infty$; the *unstable manifold* of Ω_{x^*}, M_u, is the set of such points x* that satisfy $\varphi_t(x^*) \to \Omega_{x^*}$ as $t \to -\infty$.

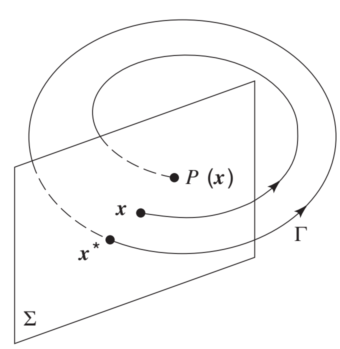

Figure 3. Illustration of the Poincaré map and cross section.

Suppose that $\Sigma_s(\mathbf{x}^*)$ and $\Sigma_u(\mathbf{x}^*)$ are cross sections of the stable and unstable manifolds of $\varphi_t(\mathbf{x})$, respectively, which intersect at $\mathbf{x}^*$. This intersection always includes one constant orbit, $\varphi_t(\mathbf{x}) = \mathbf{x}^*$. A nonconstant orbit lying in the intersection is called a *homoclinic orbit*, as illustrated in Figure 4a. For two equilibria, $\bar{\mathbf{x}}_1 \neq \bar{\mathbf{x}}_2$, of either unstable, center, or saddle type, an orbit lying in $\Sigma_s(\bar{\mathbf{x}}_1) \cap \Sigma_u(\bar{\mathbf{x}}_2)$ or in $\Sigma_u(\bar{\mathbf{x}}_1) \cap \Sigma_s(\bar{\mathbf{x}}_2)$, is called a *heteroclinic orbit*. A heteroclinic orbit is depicted in Figure 4b, which tends to one equilibrium as $t \to \infty$ but converges to another equilibrium as $t \to -\infty$.

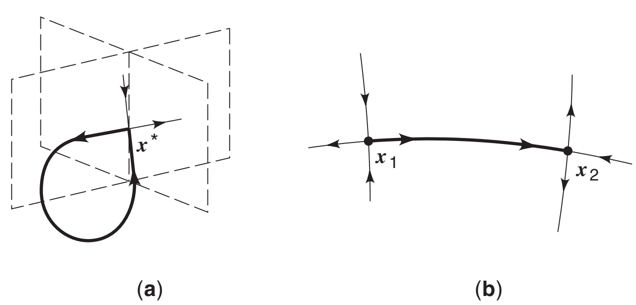

Figure 4. Illustration of homoclinic and heteroclinic orbits

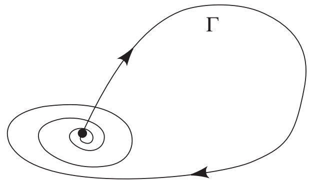

Figure 5. Illustration of a Šil'inkov-type homoclinic orbit.

Note that if a stable and an unstable manifold intersect at a point, $\mathbf{x}_0 \neq \mathbf{x}^*$, then they will do so at infinitely many points, denoted $\{\mathbf{x}_k\}_{k=-\infty}^{\infty}$, counted in both forward and backward directions, which contains $\mathbf{x}_0$. This sequence, $\{\mathbf{x}_k\}$, is a *homoclinic orbit* in which each $\mathbf{x}_k$ is called a *homoclinic point*. This special structure is called a *homoclinic structure*, in which the two manifolds usually do not intersect transversally. Hence, this structure is unstable in the sense that the connection can be destroyed by a very small perturbation. If they intersect transversally, however, the transversal homoclinic point will imply infinitely many other homoclinic points. This eventually leads to a picture of stretching and folding of the two manifolds. Such complex stretching and folding of manifolds are key to chaos. This generally implies the existence of a complicated *Smale horseshoe map*, which is supported by the following mathematical theory.

Theorem 2 (Smale–Birkhoff) *Let $P : R^n \to R^n$ be a diffeomorphism with a hyperbolic equilibrium $\mathbf{x}^*$. If the cross sections of the stable and unstable manifolds, $\Sigma_s(\mathbf{x}^*)$ and $\Sigma_u(\mathbf{x}^*)$, intersect transversally at a point other than $\mathbf{x}^*$, then P has a horseshoe map embedded within it.*

For three-dimensional autonomous systems, the case of an equilibrium with one real eigenvalue λ and two complex conjugate eigenvalues $\alpha \pm j\,\beta$ is especially interesting. For example, the case with $\lambda > 0$ and $\alpha < 0$ gives a *Šil'nikov-type* of homoclinic orbit, as illustrated in Figure 5.

Theorem 3 (Sil'nikov) *Let φ_t be the solution flow of a three-dimensional autonomous system that has a Šil'nikov-type homoclinic orbit Γ. If $|\alpha| < |\lambda|$, then φ_t can be extremely slightly perturbed to $\widetilde{\varphi}_t$, such that $\widetilde{\varphi}_t$ has a homoclinic orbit $\widetilde{\Gamma}$, near Γ, of the same type, and the Poincaré map defined by a cross section, transversal to $\widetilde{\Gamma}$, has a countable set of Smale horseshoes.*

1.7. Stabilities of Systems and Orbits

Stability theory plays a central role in both dynamical systems and automatic control. Conceptually, there are different types of stabilities, among which Lyapunov stabilities and the orbital stability are essential for chaos and bifurcations control.

Lyapunov Stabilities. In the following discussion of Lyapunov stabilities for the general nonautonomous system (eq. 1), the parameters are not indicated explicitly for simplicity. Thus, consider the general nonautonomous nonlinear system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)\,, \qquad \mathbf{x}(t_0) = x_0 \tag{6}$$

and, by changing variables if necessary, assume that the origin, $\mathbf{x} = 0$, is an equilibrium of the system. Lyapunov stability theory concerns with various types of stabilities of the zero equilibrium of system (eq. 6).

Stability in the Sense of Lyapunov. The equilibrium $\bar{\mathbf{x}} = 0$ of system (eq. 6) is said to be *stable in the sense of Lyapunov* if, for any $\varepsilon > 0$ and any initial time $t_0 \geq 0$, there exists a constant, $\delta = \delta(\varepsilon, t_0) > 0$, such that

$$||\mathbf{x}(t_0)|| < \delta \implies ||x(t)|| < \varepsilon\,, \qquad \forall\, t \geq t_0 \tag{7}$$

Here and throughout, $||\cdot||$ denotes the standard Euclidean norm of a vector.

It should be emphasized that the constant δ in the above generally depends on both ε and t_0. It is particularly important to point out that, unlike autonomous systems, one cannot simply assume the initial time $t_0 = 0$ for a nonautonomous system in a general situation. The stability is said to be *uniform* with respect to the initial time, if this constant, $\delta = \delta(\varepsilon)$, is indeed independent of t_0 over the entire time interval $(0, \infty)$.

Asymptotic Stability. In both theoretical studies and practical applications, the concept of asymptotic stability is of most importance.

The equilibrium $\bar{\mathbf{x}} = 0$ of system (eq. 6) is said to be *asymptotically stable*, if there exists a constant, $\delta = \delta(t_0) > 0$, such that

$$||\mathbf{x}(t_0)|| < \delta \implies ||\mathbf{x}(t)|| \to 0 \qquad \text{as } t \to \infty \tag{8}$$

This asymptotical stability is said to be *uniform*, if the existing constant δ is independent of t_0 over $[0, \infty)$, and is said to be *global*, if the convergence $(||\mathbf{x}|| \to 0)$ is independent of the initial point $x(t_0)$ over the entire domain on which the system is defined (i.e., when $\delta = \infty$).

Orbital Stability. The orbital stability differs from the Lyapunov stabilities in that it concerns with the structural stability of a system orbit under perturbation.

Let $\varphi_t(\mathbf{x}_0)$ be a t_p periodic solution of the autonomous system:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x})\,, \qquad \mathbf{x}(t_0) = x_0 \tag{9}$$

and let Γ be the closed orbit of $\varphi_t(\mathbf{x}_0)$ in the phase space, namely,

$$\Gamma = \left\{ y \mid y = \varphi_t(\mathbf{x}_0)\,,\ \ 0 \leq t < t_p \right\}$$

The solution trajectory $\varphi_t(\mathbf{x}_0)$ is said to be *orbitally stable* if, for any $\varepsilon > 0$, there exits a $\delta = \delta(\varepsilon) > 0$ such that for any $\mathbf{x}_0$ satisfying

$$d(\mathbf{x}_0, \Gamma) := \inf_{y \in \Gamma} ||\mathbf{x}_0 - y|| < \delta$$

the solution $\varphi_t(\mathbf{x}_0)$ of the autonomous system satisfies

$$d(\varphi_t(\mathbf{x}_0), \Gamma) < \varepsilon\,, \qquad \forall\, t \geq t_0$$

Lyapunov Stability Theorems. Two cornerstones in the Lyapunov stability theory for dynamical systems are the *Lyapunov first* (or *indirect*) *method* and the *Lyapunov second* (or *direct*) *method*.

The Lyapunov first method, known also as the *Jacobian* or local *linearization method*, is applicable to autonomous systems. This method is based on the fact that the stability of an autonomous system, in a neighborhood of an equilibrium of the system, is essentially the same as its linearized model operating at the same point, and under certain conditions local system stability and associated dynamical behavior are qualitatively the same as (topologically equivalent to) that of its linearized model (in some sense, similar to the Grobman–Hartman theorem). The Lyapunov first method provides a theoretical justification for applying linear analysis and linear feedback controllers to nonlinear autonomous systems in the study of asymptotic stability and stabilization.

The Lyapunov second method, on the other hand, originated from the concept of energy decay (i.e., dissipation) associated with a stable mechanical or electrical system, is applicable to both autonomous and nonautonomous systems. Hence, the second method is more powerful, also more useful, for global stability analysis of dynamical systems.

For the general autonomous system (eq. 9), under the assumption that $\mathbf{f} : \mathcal{D} \to R^n$ is continuously differentiable in a neighborhood, $\mathcal{D}$, of the origin in R^n, the following theorem of stability for the Lyapunov first method is convenient to use.

Theorem 4 (Lyapunov first method) (for continuous-time autonomous systems)

In equation 9, let

$$J_0 = \left.\frac{\partial \mathbf{f}}{\partial x}\right|_{x=\bar{\mathbf{x}}=0}$$

be the Jacobian of the system at the equilibrium $\bar{\mathbf{x}} = 0$. *Then*

(i) $\bar{\mathbf{x}} = 0$ *is asymptotically stable if all the eigenvalues of* J_0 *have negative real parts;*
(ii) $\bar{\mathbf{x}} = 0$ *is unstable if one of the eigenvalues of* J_0 *have positive real part.*

Note that the region of asymptotic stability given in this theorem is local. It is important to emphasize that this theorem cannot be applied to non-autonomous systems in general, not even locally.

For a general nonautonomous system (eq. 6), the following criterion can be used.

Theorem 5 (Lyapunov second method) (for continuous-time nonautonomous systems)

Let $\bar{\mathbf{x}} = 0$ *be an equilibrium of the nonautonomous system* (6). *Let*

$$\mathcal{K} = \left\{ g(t) :\ g(t_0) = 0,\ \ g(t) \text{ is continuous and nondecreasing on } [t_0, \infty) \right\}$$

The zero equilibrium of the system is globally (over the domain $\mathcal{D} \subseteq R^n$ *containing the origin), uniformly (with respect to the initial time), and asymptotically stable, if there exists a scalar-valued function* $V(\mathbf{x}, t)$ *defined on* $\mathcal{D} \times [t_0, \infty)$ *with three functions* $\alpha(\cdot), \beta(\cdot), \gamma(\cdot) \in \mathcal{K}$, *such that*

(i) $V(0, t_0) = 0$;
(ii) $V(\mathbf{x}, t) > 0$, *for all* $\mathbf{x} \neq 0$ *in* $\mathcal{D}$ *and all* $t \geq t_0$;
(iii) $\alpha(||\mathbf{x}||) \leq V(\mathbf{x}, t) \leq \beta(||\mathbf{x}||)$, *for all* $t \geq t_0$;
(iv) $\dot{V}(\mathbf{x}, t) \leq -\gamma(||\mathbf{x}||) < 0$, *for all* $t \geq t_0$.

In this theorem, the uniform stability is usually necessary since the solution of a nonautonomous system may depend on the initial time, often sensitively. As a special case for autonomous systems, the above theorem reduces to the following version.

Theorem 6 (Lyapunov second method) (for continuous-time autonomous systems)

Let $\bar{\mathbf{x}} = 0$ *be an equilibrium for the autonomous system (eq. 9). This zero equilibrium is globally (over the domain* $\mathcal{D} \subseteq R^n$ *containing the origin) and asymptotically stable, if there exists a scalar-valued function* $V(\mathbf{x})$ *defined on* $\mathcal{D}$ *such that*

(i) $V(0) = 0$;
(ii) $V(\mathbf{x}) > 0$, *for all* $\mathbf{x} \neq 0$ *in* $\mathcal{D}$;
(iv) $\dot{V}(\mathbf{x}) < 0$, *for all* $\mathbf{x} \neq 0$ *in* $\mathcal{D}$.

In the above two theorems, the function V is called a *Lyapunov function*, which is generally not unique for a given system.

Similar stability theorems can be established for discrete-time systems (via properly replacing derivatives with differences).

To this end, it is important to remark that the Lyapunov theorems only offer *sufficient* conditions for determining the asymptotic stability. Yet, the power of the Lyapunov second method lies in its generality: It works for all kinds of dynamical systems (linear and nonlinear, continuous-time and discrete-time, autonomous and nonautonomous, time-delayed, functional, etc.), and it does not require any knowledge of the solution formula of the underlying system. In a particular application, the key is to construct a working Lyapunov function for the system, which can be technically difficult if the system is higher dimensional and complicated.

2. CHAOS

Nonlinear systems have various complex behaviors that would never be anticipated in the (finite-dimensional) linear systems. Chaos is a typical one of this kind.

In the development of chaos theory, the first evidence of physical chaos is Edward Lorenz's discovery in 1963 (11). The first underlying mechanism within chaos was observed by Mitchell Feigenbaum, who in 1976 found that "when an ordered system begins to break down into chaos, a consistent pattern of rate doubling occurs" (3).

2.1. What Is Chaos

There is no unified, universally accepted, rigorous definition of *chaos* in the current scientific literature.

The term chaos was first introduced into mathematics by Li and Yorke (12). Since then, there have been several different but closely related proposals for defining chaos, among which Devaney's definition is perhaps the most popular one (13). It states that a map $F : S \to S$, where S is a set, is said to be *chaotic*, if

(1) F is *transitive* on S: For any pair of nonempty open sets U and V in S, there is an integer $k > 0$ such that $F^k(U) \cap V \neq \phi$.
(2) F has *sensitive dependence on initial conditions*: There is a real number $\delta > 0$, depending only on F and S, such that in every nonempty open subset of S there is a pair of points that, through iterations under F, are eventually separated by a distance of at least δ.
(3) The periodic points of F are *dense* in S.

Another definition requires the set S be compact, but drops condition (3) from the above. There even is a view that only the transitivity property is essential in this definition, while another view is that the sensitivity to initial conditions is so.

Although a precise and rigorous mathematical definition of chaos does not seem to be available any time soon, some fundamental features of chaos are well understood, which can be used to signify or identify chaos in most cases, especially from an application perspective.

2.2. Features of Chaos

A hallmark of chaos is its fundamental property of extreme sensitivity to initial conditions. Other features of chaos include the embedding of a dense set of unstable periodic orbits in its strange attractor, positive leading (maximal) Lyapunov exponent, finite Kolmogorov–Sinai entropy or positive topological entropy, continuous power spectrum, positive algorithmic complexity, ergodicity and mixing (Arnold's cat map), Smale horseshoe map, a statistical-oriented definition of Šhil'nikov, and some unusual (strange) limiting properties (5).

Extreme Sensitivity to Initial Conditions. The first signature of chaos is its *extreme sensitivity* to initial conditions, associated with its bounded (or compact) region of orbital patterns. It implies that two sets of slightly different initial conditions can lead to two dramatically different asymptotic states of the system orbit after some time. This is the so-called butterfly effect, which is a metaphor saying that a single flap of a butterfly's wings in China today *may* alter the initial conditions of the global weather dynamical system, thereby leading to a significantly different weather pattern in Argentina at a future time. In other words, for a dynamical system to be chaotic, it must have a (large) set of such "unstable" initial conditions that cause orbital divergence within a bounded region in the phase space of the system.

Positive Leading Lyapunov Exponents. The sensitive dependence on initial conditions for chaotic systems possesses exponential growth rate in general. This exponential growth is related to the existence of at least one positive Lyapunov exponent, the leading (largest) one. Among all main characteristics of chaos, positive leading Lyapunov exponent is perhaps the most convenient one to verify in engineering applications.

To introduce this concept, consider an n-dimensional discrete-time dynamical system described by a smooth map $\mathbf{f}$, or the Poincaré map of a continuous-time system. The ith Lyapunov exponent of the orbit $\{\mathbf{x}_k\}_{k=0}^{\infty}$, generated by the iterations of the map starting from any given initial state $\mathbf{x}_0$, is defined to be

$$\lambda_i(\mathbf{x}_0) = \lim_{k\to\infty} \frac{1}{k} \ln \left| \mu_i\Big(J_k(\mathbf{x}_k) \cdots J_0(\mathbf{x}_0) \Big) \right|, \qquad i = 1, \cdots, n \quad (10)$$

where $J_i = \mathbf{f}'(\mathbf{x}_i)$ is the Jacobian and $\mu_i(\cdot)$ denotes the ith eigenvalue of a matrix (numbered in the decreasing order of magnitudes).

Lyapunov exponents are generalization of eigenvalues of linear systems, which provide a measure for the mean convergence or divergence rate of neighboring orbits of a dynamical system. For an n-dimensional continuous-time system, depending on the direction (but not the position) of the initial state vector, its n Lyapunov exponents, $\lambda_1 \geq \cdots \geq \lambda_n$, describe different types of attractors. For example, for nonchaotic attractors (e.g., limit sets),

$$\begin{aligned}
&\lambda_i < 0,\ i = 1, \ldots, n \quad \Longrightarrow \text{ stable equilibrium}\\
&\lambda_1 = 0, \lambda_i < 0,\ i = 2, \ldots, n \quad \Longrightarrow \text{ stable limit cycle}\\
&\lambda_1 = \lambda_2 = 0, \lambda_i < 0,\ i = 3, \ldots, n \quad \Longrightarrow \text{ stable two} - \text{torus}\\
&\lambda_1 = \cdots = \lambda_m = 0, \lambda_i < 0,\ i = m+1, \ldots, n \quad \Longrightarrow \text{ stable } m - \text{torus}
\end{aligned}$$

Here, a two-torus is a bagel-shaped surface in the three-dimensional space, and an m-torus is its geometrical generalization in the $(m+1)$-dimensional space.

Note that for a three-dimensional continuous-time dynamical system, the only possibility for chaos to exist is that the three-system Lyapunov exponents satisfy

$$(+, 0, -) := (\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0) \quad \text{and} \quad \lambda_3 < -\lambda_1$$

Intuitively, this means that the system orbit in the phase space expands in one direction but shrinks in another direction, thereby yielding complex (stretching and folding) dynamical dynamics within a bounded region. Discrete-time case is different, however. A prominent example is the one-dimensional logistic map, discussed in more detail below, which is chaotic but has (the only) one positive Lyapunov exponent. For four-dimensional continuous-time systems, there are only three possibilities for chaos to emerge:

(1) $(+, 0, -, -)$: $\lambda_1 > 0, \lambda_2 = 0, \lambda_4 \leq \lambda_3 < 0$; leading to chaos.
(2) $(+, +, 0, -)$: $\lambda_1 \geq \lambda_2 > 0, \lambda_3 = 0, \lambda_4 < 0$; leading to "hyperchaos."
(3) $(+, 0, 0, -)$: $\lambda_1 > 0, \lambda_2 = \lambda_3 = 0, \lambda_4 < 0$; leading to a "chaotic two-torus."

Simple Zero of the Melnikov Function. The Melnikov theory for chaotic dynamics deals with the saddle points of

Poincaré maps of continuous solution flows in the phase space. The Melnikov function provides a measure of the distance between the stable and the unstable manifolds near a saddle point.

To introduce the Melnikov function, consider a nonlinear oscillator described by a Hamiltonian system:

$$\begin{cases} \dot{p} = -\frac{\partial H}{\partial q} + \varepsilon f_1 \\ \dot{q} = \frac{\partial H}{\partial p} + \varepsilon f_2 \end{cases}$$

where $\mathbf{f} := [f_1(p,q,t),\ f_2(p,q,t)]^\top$ has state variables $(p(t), q(t))$, $\varepsilon > 0$ is small, and $H = H(p,q) = E_K + E_P$ is the Hamilton function for the undamped, unforced (when $\varepsilon = 0$) oscillator, in which E_K and E_P are the kinetic and potential energies of the system, respectively.

Suppose that the unperturbed (unforced and undamped) oscillator has a saddle-node equilibrium (e.g., the undamped pendulum), and that $\mathbf{f}$ is t_p periodic with phase frequency $\omega > 0$. When the forced motion is described in the three-dimensional phase space $(p, q, \omega t)$, the Melnikov function is defined by

$$F(d^*) = \int_{-\infty}^{\infty} \left[\nabla H(\bar{p}, \bar{q})\right] \mathbf{f}_* \, dt \tag{11}$$

where $(\bar{p}, \bar{q})$ is the solution of the unperturbed homoclinic orbit starting from the saddle point of the original Hamiltonian system, $\mathbf{f}_* = \mathbf{f}(\bar{p}, \bar{q}, \omega t + d^*)$, and $\nabla H = [\partial H/\partial p,\ \partial H/\partial q]$. The variable d^* gives a measure of the distance between the stable and unstable manifolds near the saddle-node equilibrium.

The Melnikov theory states that chaos is possible if the two manifolds intersect, which corresponds to that the Melnikov function has a simple zero: $F(d^*) = 0$ at a single point, d^*.

Strange Attractors. Attractors are typical in nonlinear systems. The most interesting attractors, very closely related to chaos, are strange attractors. A *strange attractor* is a bounded attractor, which exhibits sensitive dependence on initial conditions but cannot be decomposed into two invariant subsets contained in disjoint open sets. Most chaotic systems have strange attractors; however, not all strange attractors are associated with chaos.

Generally speaking, a strange attractor is not any of the stable equilibria or limit cycles, but rather consists of some limit sets associated with some kind of Cantor sets. In other words, it has a "strange" and complicated structure that may possess a noninteger dimension (fractals), and have some special properties of a Cantor set. For instance, a chaotic orbit usually appears to be "strange" in that the orbit moves toward a certain point (or limit set) for some time but then moves away from it for some other time, although the orbit repeats this process infinitely and many time it never settles anywhere. Figure 6 shows a typical Chua attractor that has such strange behavior.

Fractals. An important concept that is related to Lyapunov exponent is the Hausdorff dimension. Let S be a set in R^n and C be a covering of S by countably many balls

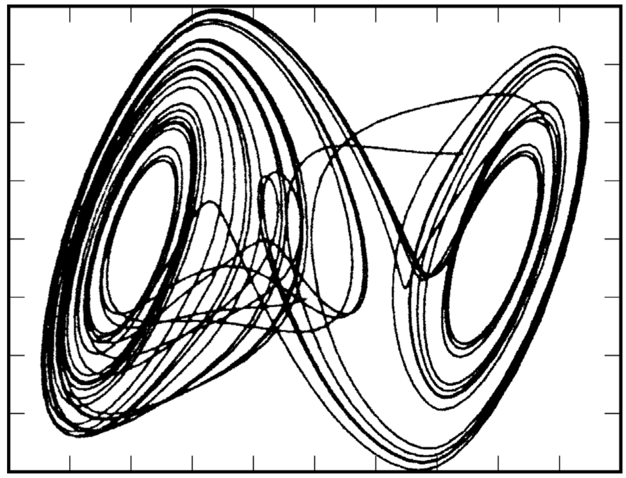

Figure 6. A typical example of strange attractor: the double scroll of Chua's circuit response.

of radii $d_1, d_2, \ldots$, satisfying $0 < d_k < \varepsilon$ for all k. For a constant $\rho > 0$, consider $\sum_{k=1}^{\infty} d_k^\rho$ for different coverings, and let $\inf_C \sum d_k^\rho$ be the smallest value of the sum over all such coverings. In the limit $\varepsilon \to 0$, this value will diverge if $\rho < h$ but tends to zero if $\rho > h$ for some constant h (need not be an integer). This value h is called the Hausdorff dimension of the set S. If the above limit exists for $\rho = h$, then the Hausdorff measure of the set S is defined to be

$$\mu_h(S) := \lim_{\varepsilon \to 0} \inf_C \sum_{k=1}^{\infty} d_k^\rho$$

There is an interesting conjecture that the Lyapunov exponents $\{\lambda_k\}$ (indicating the dynamics) and the Hausdorff dimension h (indicating the geometry) of a strange attractor have the relation

$$h = k + \frac{1}{\left|\lambda_{k+1}\right|} \sum_{i=1}^{k} \lambda_i$$

where k is the largest integer that satisfies $\sum_{i=1}^{k} \lambda_i > 0$. This formula has been mathematically proved for large families of three-dimensional continuous-time autonomous systems and of two-dimensional discrete-time systems.

A notion that is closely related to Hausdorff dimension is fractal, which was first conned and defined by Mandelbrot in the 1970s, to be a set with Hausdorff dimension strictly greater than its topological dimension, where the latter is always an integer. Roughly, a *fractal* is a set that has a fractional Hausdorff dimension and possesses certain self-similarities. An illustration of the concept of self-similarity and fractal is given in Figure 13.

There is a strong connection between fractal and chaos. Chaotic orbits often possess fractal structures in the phase space. For conservative systems, the Kolmogorov–Arnold–Moser (KAM) theorem implies that the boundary between the region of regular motion and that of chaos is fractal. However, some chaotic systems have nonfractal limit sets, and some fractal structures are not chaotic.

Finite Kolmogorov–Sinai Entropy. Another important feature of chaos and strange attractors is quantified by the Kolmogorov–Sinai (KS) entropy, a concept based on Shannon's information theory.

The classical statistical entropy is defined by

$$E = -c \sum_{k} P_k \ln(P_k)$$

where c is a constant and P_k is the probability of the system state being at the stage k of the process. According to Shannon's information theory, this entropy is a measure of the amount of information needed to determine the state of the system. This idea can be used to define a measure for the intensity of a set of system states, which gives the mean loss of information at the state of the system when it evolves with time. To do so, let $x(t)$ be a system orbit and partition its m-dimensional phase space into cells of small volumes, ε^m. Let P_{k_i} be the probability of the sampling orbital state, $x(k)$, with sampling time $t_s > 0$, that belongs to the k_ith cell at instant $k = it_s$, $i = 0, 1, \ldots, n$. Similarly, $P_{k_0 \cdots k_i}$ denotes the probability of the orbital state moving from the k_0th cell through the k_ith cell. Then, Shannon defined the information index to be

$$\mathcal{I}_n := - \sum_{k_0, \cdots, k_n} P_{k_0 \ldots k_n} \ln(P_{k_0 \cdots k_n})$$

which is proportional to the amount of the information needed to determine the orbit, if the probabilities are known. Consequently, $\mathcal{I}_{n+1} - \mathcal{I}_n$ gives additional information for predicting the state $x(k_{n+1})$, if the states $x(k_0), \ldots, x(k_n)$ are known. This difference is also the information lost during the process. The KS entropy is then defined by

$$\begin{aligned} E_{\mathrm{KS}} &:= \lim_{t_s \to 0} \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{nt_s} \sum_{i=0}^{n-1} (\mathcal{I}_{i+1} - \mathcal{I}_i) \\ &= -\lim_{t_s \to 0} \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{nt_s} \sum_{k_0, \ldots, k_n} P_{k_0 \cdots k_n} \ln(P_{k_0 \cdots k_n}) \quad (12) \end{aligned}$$

This entropy, E_{KS}, quantifies the degree of *disorder*: (i) $E_{\mathrm{KS}} = 0$ indicates regular attractors, such as stable equilibria, limit cycles, and tori; (ii) $E_{\mathrm{KS}} = \infty$ implies totally random dynamics, which have no correlations in the phase space; (iii) $0 < E_{\mathrm{KS}} < \infty$ signifies strange attractors and chaos.

It is interesting to note that there is a connection between the Lyapunov exponents and the KS entropy:

$$E_{\mathrm{KS}} \le \sum_{i} \lambda_i^+$$

where λ_i^+ are positive Lyapunov exponents of the same system.

2.3. Chaos in Control Systems

Chaos is ubiquitous, indeed. Chaotic behaviors have been found in many typical mathematical maps such as the logistic map, Arnold's circle map, Hénon map, Lozi map, Ikeda map, and Bernoulli shift; in various physical systems, including the Duffing oscillator, van der Pol oscillator, forced pendula, hopping robot, brushless DC motor, rotor with varying mass, Lorenz model, and Rössler system, electrical and electronics systems (e.g., Chua's circuit and electric power systems), digital filters, celestial mechanics (the three-body problem), fluid dynamics, lasers, plasmas, solid states, quantum mechanics, nonlinear optics, chemical reactions, neural networks, fuzzy systems, economic and financial systems, biological systems (heart, brain, and population models), and various Hamiltonian systems (5).

Chaos also exists in many engineering processes and, perhaps unexpectedly, in some continuous-time and discrete-time feedback control systems. For instance, in the continuous-time case, chaos has been found in very simple dynamical systems such as a first-order autonomous feedback system with a time-delay feedback channel, surge tank dynamics under a simple liquid level control system with time-delayed feedback, and several other types of time-delayed feedback control systems. It also exists in automatic gain control loops, which are very popular in industrial applications such as in most receivers of communication systems. Most fascinating of all, very simple pendula can display complex dynamical phenomena; in particular, pendula subject to linear feedback controls can exhibit even richer bifurcations and chaotic behaviors. As an example, pendulum controlled by a proportional-derivative controller can behave chaotically when the tracking signal is periodic, with energy dissipation, even for the case of small controller gains. In addition, chaos has been found in many engineering applications such as in the designs of control circuits for switched mode power conversion equipment, high-performance digital robot controllers, second-order systems containing a relay with hysteresis, and in various biochemical control systems.

Chaos occurs also frequently in discrete-time feedback control systems due to sampling, quantization, and roundoff effects. Discrete-time linear control systems with deadzone nonlinearity have global bifurcations, unstable periodic orbits, scenarios leading to chaotic attractors, crisis of chaotic attractors reducing to periodic orbits, and so on. Chaos also exists in digitally controlled systems, feedback-type of digital filtering systems (either with or without control), and even the linear Kalman filter when numerical issues are involved.

Many adaptive systems are inherently nonlinear, thus bifurcations and chaos in such systems are often inevitable. The instances of chaos in adaptive control systems usually come from several possible sources: the nonlinearities of the plant and the estimation scheme, external excitation or disturbances, the adaptation mechanism, and so on. Chaos can occur in typical model-referenced adaptive control (MRAC) and self-tuning adaptive control (STAC) systems, as well as some other classes of adaptive feedback control systems of arbitrary order that contain unmodeled dynamics and disturbances. In such adaptive control systems, typical failure modes include convergence to undesirable local minima and nonlinear self-oscillation, such as bursting, limit cycling, and chaos. In indirect adaptive control of linear discrete-time plants, strange system behaviors can arise due to unmodeled dynamics (or disturbances), bad combinations of parameter estimation and/or control laws, and lack of persistency of excitation. As an example, chaos is found in setpoint tracking control of a linear discrete-time system of unknown order, where the adaptive control scheme is either to estimate the order of the plant or to directly track the reference.

Chaos also emerges from various types of neural networks. Similar to biological neural networks, most artificial neural networks can display complex dynamics, including bifurcations, strange attractors, and chaos. Even a very simple recurrent two-neuron model with only one self-interaction can produce chaos. A simple three-neuron recurrent neural network can also create period-doubling bifurcations leading to chaos. A four-neuron network and multi-neuron networks, of course, have more chances to produce complex dynamical patterns such as bifurcations and chaos. A typical example in point is the cellular neural networks, which have very rich complex dynamical behaviors.

Chaos has also been experienced in some fuzzy control systems. The fact that fuzzy logic can produce complex dynamics is more or less intuitive, inspired by the nonlinear nature of the fuzzy systems. This has been justified, not only experimentally but also both mathematically and logically. Chaos has been observed, for example, from a coupled simple fuzzy control system, among others. The change in the shapes of the fuzzy membership functions can significantly alter the dynamical behavior of a fuzzy control system, potentially leading to the occurrence of chaos.

Many specific examples of chaos in control systems can be given. Therefore, controlling chaos is not only interesting as a subject for scientific research but also very much relevant to the objectives of traditional control and systems engineering. Simply put, it is not an issue that can be treated with ignorance or neglect.

3. BIFURCATIONS

Associated with chaos is bifurcation, another typical phenomenon of nonlinear dynamical systems that quantifies the change of system properties (such as the number and the stabilities of the system equilibria) due to the variation of system parameters. Chaos and bifurcations have a very strong connection; oftentimes they coexist in a complex dynamical system.

3.1. Basic Types of Bifurcations

To illustrate various bifurcation phenomena, it is convenient to consider a two-dimensional parameterized nonlinear dynamical system:

$$\begin{cases} \dot{x} = f(x, y; p) \\ \dot{y} = g(x, y; p) \end{cases} \tag{13}$$

where p is a real and variable system parameter.

Let $(\bar{x}, \bar{y}) = \left(\bar{x}(t; p_0), \bar{y}(t; p_0)\right)$ be an equilibrium of the system when $p = p_0$, at which $f(\bar{x}, \bar{y}; p_0) = 0$ and $g(\bar{x}, \bar{y}; p_0) = 0$. If the equilibrium is stable (respectively, unstable) for $p > p_0$ but unstable (respectively, stable) for $p < p_0$, then p_0 is a *bifurcation value* of p, and $(0, 0; p_0)$ is a *bifurcation point* in the *parameter space*, (x, y, p). A few examples are given below to distinguish several typical bifurcations.

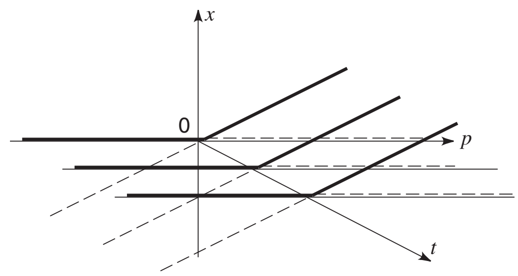

Figure 7. The transcritical bifurcation.

Transcritical Bifurcation. The one-dimensional system

$$\dot{x} = f(x; p) = p\,x - x^2$$

has two equilibria: $\bar{x}_1 = 0$ and $\bar{x}_2 = p$. When p is varied, there emerge two equilibrium curves as shown in Figure 7. Since the Jacobian for this one-dimensional system is simply $J = p$, it is clear that for $p < p_0 = 0$, the equilibrium $\bar{x}_1 = 0$ is stable, but for $p > p_0 = 0$ it changes to be unstable. Thus, $(\bar{x}_1, p_0) = (0, 0)$ is a bifurcation point. In the figure, the solid curves indicate stable equilibria and the dashed curves, the unstable ones. Likewise, $(\bar{x}_2, p_0)$ is another bifurcation point. This type of bifurcation is called the *transcritical bifurcation*.

Saddle-Node Bifurcation. The one-dimensional system

$$\dot{x} = f(x; p) = p - x^2$$

has an equilibrium $\bar{x}_1 = 0$ at $p_0 = 0$, and an equilibrium curve $\bar{x}^2 = p$ at $p \geq 0$, where $\bar{x}_2 = \sqrt{p}$ is stable and $\bar{x}_3 = -\sqrt{p}$ is unstable for $p > p_0 = 0$. This bifurcation, as shown in Figure 8, is called the *saddle-node bifurcation*.

Pitchfork Bifurcation. The one-dimensional system

$$\dot{x} = f(x; p) = p\,x - x^3$$

has an equilibrium $\bar{x}_1 = 0$ at $p_0 = 0$, and an equilibrium curve $\bar{x}^2 = p$ at $p \geq 0$. Here, $\bar{x}_1 = 0$ is unstable for $p > p_0 = 0$ and stable for $p < p_0 = 0$, and the entire equilibrium curve $\bar{x}^2 = p$ is stable for all $p > 0$ at which it is defined. This situation, as depicted in Figure 9, is called the *pitchfork bifurcation*.

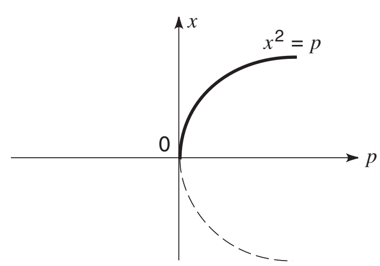

Figure 8. The saddle-node bifurcation.

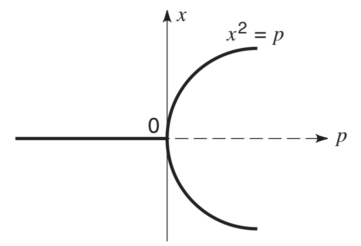

Figure 9. The pitchfork bifurcation.

Note, however, that not all nonlinear parameterized dynamical systems have bifurcations. A simple example is

$$\dot{x} = f(x; p) = p - x^3$$

that has an entire stable equilibrium curve $\bar{x} = p^{1/3}$, which does not have any bifurcation.

Hysteresis Bifurcation. The dynamical system

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = p + x_2 - x_2^3 \end{cases}$$

has equilibria

$$\bar{x}_1 = 0 \qquad \text{and} \qquad p - \bar{x}_2 + \bar{x}_2^3 = 0$$

According to different values of p, there are either one or three equilibrium solutions, where the second equation yields one bifurcation point at $p_0 = \pm 2\sqrt{3}/9$, but three equilibria for $|p_0| < 2\sqrt{3}/9$.

The stabilities of the equilibrium solutions are shown in Figure 10. This type of bifurcation is called the *hysteresis bifurcation*.

Hopf Bifurcation and Hopf Theorems. In addition to the bifurcations described above, called *static bifurcations*, the parameterized dynamical system (eq. 13) can have another type of bifurcation, the *Hopf bifurcation* (or *dynamical bifurcation*).

Hopf bifurcation corresponds to the situation where, as the parameter p is varied to pass the critical value p_0, the system Jacobian has one pair of complex conjugate eigenvalues moving from the left-half plane to the right, crossing the imaginary axis, while all the other eigenvalues remain to be stable. At the moment of the crossing,

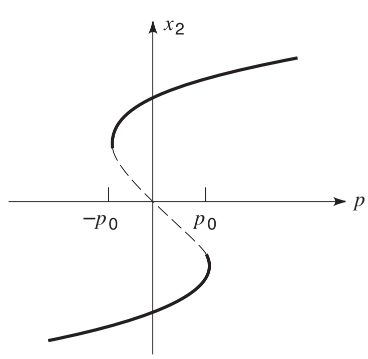

Figure 10. The hysteresis bifurcation.

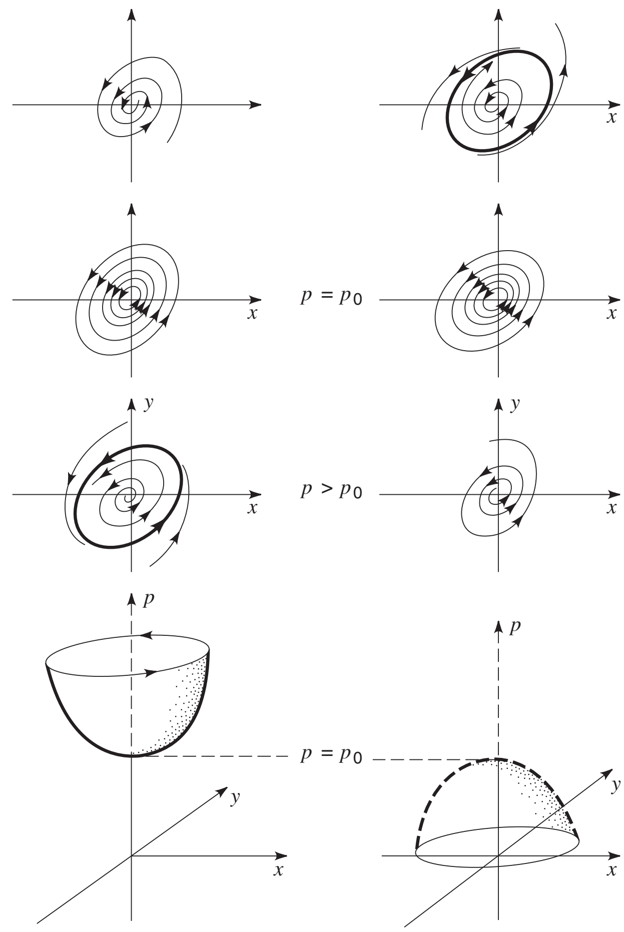

Figure 11. Two types of Hopf bifurcations illustrated in the phase plane.

the real parts of the eigenvalue pair are zero, and the stability of the existing equilibrium changes to opposite, as shown in Figure 11. In the meantime, a limit cycle will emerge. As indicated in the figure, Hopf bifurcation can be classified as *supercritical* (respectively, *subcritical*), if the equilibrium is changed from stable to unstable (respectively, from unstable to stable). The same terminology of supercritical and subcritical bifurcations applies also to other non-Hopf types of bifurcations.

Hopf Bifurcation Theorems. Consider a general nonlinear parameterized autonomous system,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; p), \qquad \mathbf{x}(t_0) = x_0 \tag{14}$$

where $\mathbf{x} \in R^n$, p is a real variable parameter, and $\mathbf{f}$ is differentiable.

The most fundamental result on the Hopf bifurcation of this system is the following theorem, which is stated here only for the special two-dimensional setting.

Theorem 7 (Poincaré–Andronov–Hopf) *Suppose that the two-dimensional system (eq. 14) has a zero equilibrium, $\bar{\mathbf{x}} = 0$, and assume that its associate Jacobian $\bar{A} = \frac{\partial f}{\partial x}\Big|_{x=\bar{\mathbf{x}}=0}$ has a pair of purely imaginary eigenvalues, $\lambda(p)$ and $\lambda^*(p)$.*

If

$$\left.\frac{\mathrm{d}\Re\{\lambda(\mathrm{p})\}}{\mathrm{d}\,\mathrm{p}}\right|_{p=p_0} > 0$$

for some p_0, where $\Re\{\cdot\}$ denotes the real part of the complex eigenvalues, then

(i) *$p = p_0$ is a bifurcation point of the system;*
(ii) *for close enough values of $p < p_0$, the equilibrium $\bar{\mathbf{x}} = 0$ is asymptotically stable;*
(iii) *for close enough values of $p > p_0$, the equilibrium $\bar{\mathbf{x}} = 0$ is unstable;*
(iv) *for close enough values of $p \neq p_0$, the equilibrium $\bar{\mathbf{x}} = 0$ is surrounded by a limit cycle of magnitude $O(\sqrt{|p|})$.*

Graphical Hopf Bifurcation Theorem. The Hopf bifurcation can also be analyzed in the frequency domain setting (14). In this approach, the nonlinear parameterized autonomous system (eq. 14) is first rewritten in the following Lur'e form:

$$\begin{cases} \dot{\mathbf{x}} = A(p)\,\mathbf{x} + B(p)\,u \\ \mathbf{y} = -C(p)\,\mathbf{x} \\ \mathbf{u} = g(y;p) \end{cases} \tag{15}$$

where the matrix $A(p)$ is chosen to be invertible for all values of p, and $\mathbf{g} \in C^4$ depends on the chosen matrices A, B, and C. Assume that this system has an equilibrium solution, $\bar{\mathbf{y}}$, satisfying

$$\bar{\mathbf{y}}(t;p) = -H(0;p)\,g(\bar{\mathbf{y}}(t;p);p)$$

where

$$H(0;p) = -C(p)\,A^{-1}(p)\,B(p)$$

Let $J(p) = \partial\mathbf{g}/\partial\mathbf{y}\Big|_{\mathbf{y}=\bar{\mathbf{y}}}$ and let $\hat{\lambda} = \hat{\lambda}(j\,\omega;p)$ be the eigenvalue of the matrix $[G(j\,\omega;p)J(p)]$ that satisfies

$$\hat{\lambda}(j\,\omega_0;p_0) = -1 + j\,0\,, \qquad j = \sqrt{-1}$$

Then, fix $p = \tilde{p}$ and let ω vary. In so doing, a trajectory of the function $\hat{\lambda}(\omega;\tilde{p})$, the "eigenlocus," can be obtained. This locus traces out from the frequency $\omega_0 \neq 0$. In much the same way, a real zero eigenvalue (a condition for the *static bifurcation*) is replaced by a characteristic gain locus that crosses the point $(-1 + j\,0)$ at frequency $\omega_0 = 0$.

For illustration, consider a single-input single-output (SISO) system. In this case, the matrix $[H(j\,\omega;p)J(p)]$ is merely a scalar, and

$$y(t) \approx \bar{y} + \Re\left\{\sum_{k=0}^{n} y_k\,\mathrm{e}^{j\,k\omega t}\right\}$$

where $\bar{y}$ is the equilibrium solution and the complex coefficients $\{y_k\}$ are determined as follows. For the approximation with $n = 2$, first define an auxiliary vector:

$$\xi_1(\tilde{\omega}) = \frac{-\mathbf{l}^{\top}\left[H(j\,\tilde{\omega};\tilde{p})\right]\mathbf{h}_1}{\mathbf{l}^{\top}\mathbf{r}} \tag{16}$$

where $\tilde{p}$ is the fixed value of the parameter p, $\mathbf{l}^{\top}$ and $\mathbf{r}$ are the left and right eigenvectors of $[H(j\,\tilde{\omega};\tilde{p})J(\tilde{p})]$, respectively, associated with the eigenvalue $\hat{\lambda}(j\,\tilde{\omega};\tilde{p})$, and

$$\mathbf{h}_1 = \left[D_2\left(\mathbf{z}_{02} \otimes \mathbf{r} + \frac{1}{2}\mathbf{r}^* \otimes \mathbf{z}_{22}\right) + \frac{1}{8}D_3\mathbf{r} \otimes \mathbf{r} \otimes \mathbf{r}^*\right]$$

where * denotes the complex conjugate, $\tilde{\omega}$ is the frequency of the intersection between the $\hat{\lambda}$ locus and the negative real axis that is closest to the point $(-1 + j\,0)$, $\otimes$ is the tensor product operator, and

$$\begin{aligned}
D_2 &= \left.\frac{\partial^2\mathbf{g}(y;\tilde{p})}{\partial y^2}\right|_{y=\bar{y}} \\
D_3 &= \left.\frac{\partial^3\mathbf{g}(y;\tilde{p})}{\partial y^3}\right|_{y=\bar{y}} \\
\mathbf{z}_{02} &= -\frac{1}{4}\left[1 + H(0;\tilde{p})J(\tilde{p})\right]^{-1} G(0;\tilde{p})D_2\mathbf{r} \otimes \mathbf{r}^* \\
\mathbf{z}_{22} &= -\frac{1}{4}\left[1 + H(2j\,\tilde{\omega};\tilde{p})J(\tilde{p})\right]^{-1} H(2j\,\tilde{\omega};\tilde{p})D_2\mathbf{r} \otimes \mathbf{r} \\
y_0 &= \mathbf{z}_{02}\,|\tilde{p} - p_0| \\
y_1 &= \mathbf{r}\,|\tilde{p} - p_0|^{1/2} \\
y_2 &= \mathbf{z}_{22}\,|\tilde{p} - p_0|
\end{aligned}$$

The graphical Hopf bifurcation theorem (for SISO systems) formulated in the frequency domain, based on the generalized Nyquist criterion, is stated as follows.

Theorem 8 (Graphical Hopf Bifurcation Theorem)

Suppose that, whenever ω is varied, the vector $\xi_1(\tilde{\omega}) \neq 0$. Assume also that the half-line, starting from $-1 + j\,0$ and pointing to the direction parallel to that of $\xi_1(\tilde{\omega})$, first intersects the locus of the eigenvalue $\hat{\lambda}(j\,\omega;\tilde{p})$ at the point

$$\hat{P} = \hat{\lambda}(\hat{\omega};\tilde{p}) = -1 + \xi_1(\tilde{\omega})\,\theta^2$$

at which $\omega = \hat{\omega}$ and the constant $\theta = \theta(\hat{\omega}) \geq 0$, as shown in Figure 12. Suppose, furthermore, that the above intersection is transversal, namely,

$$\det\begin{bmatrix} \Re\{\xi_1(j\,\hat{\omega})\} & \Im\{\xi_1(j\,\hat{\omega})\} \\ \Re\left\{\left.\frac{d}{d\omega}\hat{\lambda}(\omega;\tilde{p})\right|_{\omega=\hat{\omega}}\right\} & \Im\left\{\left.\frac{d}{d\omega}\hat{\lambda}(\omega;\tilde{p})\right|_{\omega=\hat{\omega}}\right\} \end{bmatrix} \neq 0$$

Then

(i) *The nonlinear system (eq. 15) has a periodic solution (output) $y(t) = y(t;\bar{y})$. Consequently, there exists a unique limit cycle for the nonlinear equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, in a ball of radius $O(1)$ centered at the equilibrium $\bar{\mathbf{x}}$.*
(ii) *If the total number of anticlockwise encirclements of the point $p_1 = \hat{P} + \varepsilon\xi_1(\tilde{\omega})$, for a small enough $\varepsilon > 0$, is equal to the number of poles of $[G(s;p)J(p)]$ that have positive real parts, then the limit cycle is stable.*

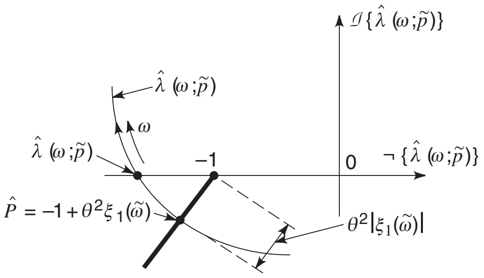

Figure 12. The frequency domain version of the Hopf bifurcation theorem.

3.2. Period-Doubling Bifurcations to Chaos

There are several routes to chaos from a regular state of a nonlinear system, provided that the system is chaotic in nature.

One route is that after three Hopf bifurcations a regular motion can become highly unstable, leading to a strange attractor and then to chaos. Actually, even pitchfork and saddle-node bifurcations can be routes to chaos under certain circumstances. For motion on a normalized two-torus, if the ratio of the two fundamental frequencies $\omega_1/\omega_2 = p/q$ is rational, then the orbit returns to the same point after a q-cycle; but if the ratio is irrational, then this (*quasiperiodic*) orbit never returns to the starting point. Quasiperiodic motion on a two-torus provides another common route to chaos.

Period-doubling bifurcation is perhaps the most typical route that leads system dynamics to chaos. Consider, as an example, the logistic map

$$x_{k+1} = p\,x_k\left(1 - x_k\right) \tag{17}$$

where $p > 0$ is a variable parameter. With $0 < p < 1$, the origin $x = 0$ is stable, so the orbit approaches it as $k \to \infty$. However, for $1 < p < 3$, all points converge to another equilibrium, denoted $\bar{x}$.

The evolution of the system dynamics, as p is gradually increased from 3.0 to 4.0 by small steps, is mostly interesting, which is depicted in Figure 13. The figure shows that at $p = 3$, a (stable) period-two orbit is bifurcated out of $\bar{x}$, which becomes unstable at that moment, and, in addition to 0, there emerge two (stable) equilibria:

$$\bar{x}^{1,2} = \left(1 + p \pm \sqrt{p^2 - 2p - 3}\right)\Big/(2p)$$

When p continues to increase to the value of $1 + \sqrt{6} = 3.544090\ldots$, each of these two points bifurcates to other two, as can be seen from the figure. As p moves consequently through the values $3.568759\ldots$, $3.569891\ldots$, $\ldots$, an infinite sequence of bifurcations is generated by such *period-doubling*, which eventually leads to chaos:

$$\text{period } 1 \to \text{period } 2 \to \text{period } 4 \to \cdots \text{ period } 2^k \to \cdots \to \text{chaos}$$

It is also interesting to note that in certain regions (e.g., the three windows magnified in the figure) of the logistic map, it appears *self-similarity* of the bifurcation diagram of the map, which is a typical *fractal structure*.

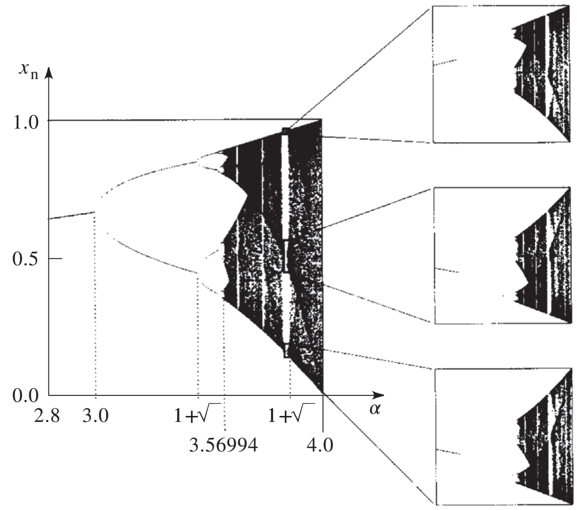

Figure 13. Period-doubling of the logistic system with self-similarity.

Figure 14 shows the Lyapunov exponent λ versus the parameter p, over the interval of $[2.5, 4]$. This figure corresponds to the period-doubling diagram shown in Figure 13.

The most significant discovery about the phenomenon of period-doubling bifurcation route to chaos is Feigenbaum's observation in 1978: The convergence of the period-doubling bifurcating parameters has a geometric rate, $p_\infty - p_k \propto \delta^{-k}$, where

$$\delta_k := \frac{p_{k+1} - p_k}{p_{k+2} - p_{k+1}} \to \delta = 4.6692\ldots \quad (k \to \infty)$$

which is known as a *universal number* for a large class of chaotic dynamical systems.

3.3. Bifurcations in Control Systems

Not only chaos but also bifurcations can exist in feedback and adaptive control systems. Generally speaking, local instability and complex dynamical behavior can result from feedback and adaptive mechanisms when ad-

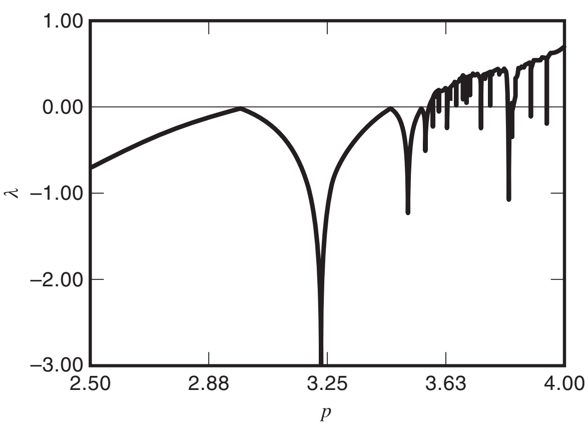

Figure 14. Lyapunov exponent versus parameter p for the logistic map.

equate process information is not available for feedback transmission or parameter estimation. In this situation, one or more poles of the linearized closed-loop transfer function may move to cross over the stability boundary, thereby causing signal divergence as the control process continues. However, this sometimes may not lead to global unboundedness, but rather to self-excited oscillations or self-stabilization, leading to very complex dynamical phenomena.

Real examples of bifurcations in feedback control systems include the automatic gain control loop system, which has bifurcations transmitting to Smale horseshoe chaos and the common route of period-doubling bifurcations to chaos. Surprising enough, in some situations even a single pendulum controlled by a linear proportional-derivative controller can display rich bifurcations in addition to chaos.

Adaptive control systems are more likely to produce bifurcations than a simple feedback control system, due to the changes of stabilities in adaptation. The complex dynamics emerging from an adaptive control system is often caused by estimation instabilities. Moreover, certain prototypes of MRAC systems can experience various bifurcations.

Bifurcation theory has been employed for analyzing complex dynamical systems. For instance, in an MRAC system, a few pathways leading to estimator instability have been identified via bifurcation analysis:

(i) A sign change in the adaptation law, leading to a reversal of the gradient direction as well as an infinite linear drift.
(ii) The instability caused by high control gains, leading to global divergence through period-doubling bifurcations.
(iii) A Hopf bifurcation-type of instability, leading to parameter drift and bursting in a bounded regime through a sequence of global bifurcations.

Both instabilities of types (i) and (ii) can be avoided by gain-tuning or simple algorithmic modifications. The third instability, however, is generally due to the unmodeled dynamics and a poor signal-to-noise ratio, and so cannot be avoided by simple tuning methods. This instability is closely related to the presence of a degenerate set and a period-two attractor.

Similarly, in the discrete-time case, a simple adaptive control system can have rich bifurcation phenomena such as period-doubling bifurcation (due to high adaptive control gains) and Hopf and global bifurcations (due to insufficient excitations).

Like the omnipresent chaos, bifurcations exist in many physical systems (5). For instance, power systems generally have various bifurcation dynamics. When the consumers' demands for power reach peaks, the stability of an electric power network may move to its margin, leading to serious oscillations and stability bifurcations, which may quickly result in voltage collapse. As another example, a typical double pendulum can display bifurcations as well as chaotic motions. Some rotational mechanical systems also have similar behavior. Even a common road vehicle driven by a pilot with driver steering control can have Hopf bifurcation when its stability is lost, which may also develop chaos and even hyperchaos. A hopping robot, or a simple two-degree-of-freedom flexible robot arm, can respond to strange vibrations undergoing period-doubling bifurcations, which eventually lead to chaos. An aircraft stalls for flight below a critical speed or over a critical angle-of-attack can respod to various bifurcations. Dynamics of a ship can exhibit stability bifurcation according to wave frequencies that are close to the natural frequency of the ship, which creates oscillations and chaotic motions leading to ship capsize. Simple nonlinear circuits are rich sources of different types of bifurcations as well as chaos. Other systems that have bifurcation properties include cellular neural networks, lasers, aeroengine compressors, weather systems, and biological population dynamics, to name but a few.

4. CONTROLLING CHAOS

Understanding chaos has long been the main focus of research in the field of nonlinear science. The idea that chaos can in fact be controlled is perhaps counterintuitive. Indeed, the extreme sensitivity of a chaotic system to initial conditions once led to the impression and argument that chaotic motion is in general neither predictable nor controllable.

However, recent research effort has shown that not only (short term) prediction but also (long term) control of chaos are possible. It is now well known that most conventional control methods and many special techniques can be used for controlling chaos (5,15,16). In this pursuit, no matter the purpose is to reduce "bad" chaos or to introduce "good" ones, numerous control strategies have been proposed, developed, tested, and applied to many case studies. Numerical and experimental simulations have demonstrated that chaotic physical systems respond quite well to these controls. In about the same time, applications are proposed in such diverse fields as biology, medicine, physiology, chemical engineering, laser physics, electric power systems, fluid mechanics, aerodynamics, circuits and electronic devices, signal processing and communications, and so on. The fact that researchers from vast scientific and engineering backgrounds are joining together and aiming at one central theme – bringing order to chaos – indicates that the study of nonlinear dynamics and their control has progressed into a new era. Much has been accomplished in the past decade, and yet much more remains a challenge for the near future.

Similar to conventional systems control, the concept of "controlling chaos" is first to mean suppressing chaos in the sense of stabilizing chaotic system responses, oftentimes unstable periodic outputs. However, controlling chaos has also encompassed many nontraditional tasks, particularly those in creating or enhancing chaos when it is useful. The process of chaos control is now understood as a transition between chaos and order and, sometimes, the transition from chaos to chaos, depending on the application at hand. In fact, the notion of chaos control is neither exclusive of nor conflicting with the purposes of conventional control systems theory. Rather, it targets at better managing the dynamics of a nonlinear system on a wider scale, with the hope that more benefits may thus be derived from the special features of chaos.

4.1. Why Chaos Control

There are many practical reasons for controlling or ordering chaos. First of all, "chaotic" (messy, irregular, or disordered) system response with little useful information content is unlikely to be desirable. Second, chaos can lead systems to harmful or even catastrophic situations. In these troublesome cases, chaos should be reduced as much as possible, or be totally suppressed. Traditional engineering design always tries to reduce irregular behaviors of a system and, therefore, completely eliminates chaos. Such "over-design" is needed in the aforementioned situations. However, this is usually accomplished at the price of loosing great opportunities and benefits in achieving high performance near the stability boundaries or at the expense of radically modifying the original system dynamics.

Conversely, recent research has shown that chaos can actually be useful under certain circumstances, and there is growing interest in utilizing the very nature of chaos (5). For example, it was observed (17) that a chaotic attractor typically has embedded within it a dense set of unstable periodic orbits. Thus, if any of these periodic orbits can be stabilized, it may be desirable to select one that gives rise to a certain maximal system performance. In other words, when the design of a dynamical system is intended for multiple usages, purposely building chaotic dynamics into the system may allow for the desired flexibilities. A control design of this kind is certainly nonconventional.

Fluid mixing is a good example in which chaos is not only useful but actually necessary (18). Chaos is desirable in many applications of liquid mixing, where two fluids are to be thoroughly mixed while the required energy is minimized. For this purpose, it turns out to be much easier if the dynamics of the particle motion of the two fluids are strongly chaotic, since it is difficult to obtain rigorous mixing properties otherwise, due to the possibility of invariant two-tori in the flow. This has been one of the main subjects in fluid mixing, known as "chaotic advection." Chaotic mixing is also important in applications involving heating, such as plasma heating for a nuclear fusion reactor. In such plasma heating, heat waves are injected into the reactor, for which the best result is obtained when the heat convection inside the reactor is chaotic.

Within the context of biological systems, the controlled biological chaos seems to be important with the way a human brain executes its tasks. For years, scientists have been trying to unravel how our brains endow us with inference, thought, perception, reasoning and, most fascinating of all, emotion such as happiness and sadness. There were experimental suggestions that human brain can process massive information in almost no time, in which chaotic dynamics could be a fundamental reason: "The controlled chaos of the brain is more than an accidental by-product of the brain complexity, including its myriad connections," but rather "it may be the chief property that makes the brain different from an artificial-intelligence machine" (19). The idea of *anticontrol of chaos* has been proposed for solving the problem of driving the system responses of a human brain model away from the stable direction, and hence away from the stable (saddle-type) equilibrium. As a result, the periodic behavior of neuronal population bursting can be prevented (20). Control tasks of this type are also nontraditional.

Other potential applications of chaos control in biological systems have reached out from the brain to elsewhere, particularly to the human heart. In physiology, healthy dynamics has been regarded as regular and predictable, whereas disease, such as fatal arrhythmias, aging, and drug toxicity, are commonly assumed to produce disorder and even chaos. However, recent laboratory studies have seemingly demonstrated that the complex variability of healthy dynamics in a variety of physiological systems has features reminiscent of deterministic chaos, and a wide class of disease processes (including drug toxicities and aging) may actually decrease (yet not completely eliminate) the amount of chaos or complexity in physiological systems, known as "decomplexification." Thus, in contrast to the common belief that healthy heartbeats are completely regular, a normal heart rate may fluctuate in a highly erratic fashion, even at rest, and may actually be chaotic (21). It has also been observed that, in the heart, the amount of intracellular Ca is closely regulated by coupled processes that cyclically increase or decrease this amount, in a way similar to a system of coupled oscillators. This cyclical fluctuation in the amount of intracellular Ca is a cause of after-depolarizations, which triggers activities in the heart — the so-called arrhythmogenic mechanism. Medical evidence reveals that controlling (while not completely eliminating) the chaotic arrhythmia can be a new, safe, and promising approach to regulating heartbeats (22,23).

The sensitivity of chaotic systems to small perturbations can be used to direct system trajectories to a desired target quickly with very low (or minimum) control energy. As an example, NASA scientists used small amounts of residual hydrazine fuel to send the spacecraft ISEE-3/IEC more than 50 million miles across the solar system, achieving the first scientific cometary encounter. This control action utilized the sensitivity to small perturbations of the three-body problem of celestial mechanics, which would not be possible in a nonchaotic system since it normally requires a huge control effort (24).

4.2. Chaos Control: An Example

To appreciate the challenge of chaos control, consider the one-dimensional logistic map (eq. 17) with the period-doubling bifurcations route to chaos as shown in Figure 13.

Chaos control problems in this situation include, but not limited to, the following:

> Is it possible (and, if so, how) to design a simple controller, u_k, for the given system, in the form of
>
> $$x_{k+1} = p\,x_k\left(1 - x_k\right) + u_k$$
>
> such that

(i) the limiting chaotic behavior of the period-doubling bifurcations process is suppressed?
(ii) the first bifurcation is delayed to take place, or some bifurcations are changed either in form or in stability?
(iii) when the parameter p is currently not in the bifurcating range, the asymptotic behavior of the system becomes chaotic?

Many of such nonconventional control problems emerging from chaotic dynamical systems have posed a real challenge to both nonlinear dynamics analysts and control engineers — they have become, in effect, motivation and stimuli for the current endeavor devoted to the new research direction in control systems: controlling bifurcations and chaos.

4.3. Some Distinctive Features of Chaos Control

At this point, it is illuminating to highlight some distinctive features of chaos control theory and methodology, in contrast to other conventional approaches regarding such issues as objectives, perspectives, problem formulations, and performance measures.

1. The targets in chaos control are usually unstable periodic orbits (including equilibria and limit cycles), perhaps of higher periods. The controller is designed to stabilize some of these unstable orbits or to drive the trajectories of the controlled system to switch from one orbit to another. This inter-orbit switching can be either chaos → order, chaos → chaos, order → chaos, or order → order, depending on the application in interest. Conventional control, on the other hand, does not normally investigate such inter-orbit switching problems of a dynamical system, especially not those problems that involve guiding a system trajectory to an unstable or chaotic state by any means.
2. A chaotic system typically has embedded within it a dense set of unstable orbits, and is extremely sensitive to tiny perturbations to its initial conditions and system parameters. Such a special property, useful for chaos control, is not available in nonchaotic systems and is not utilized in any forms by conventional control.
3. Most conventional control schemes work within the *state space* framework. In chaos control, however, one more often deals with *parameter space* and *phase space*. Poincaré maps, delay-coordinates embedding, parametric variation, entropy reduction, bifurcation monitoring, and so on, are some typical but nonconventional tools for design and analysis.
4. In conventional control, a target for tracking is usually a constant vector in the state space, with very few exceptions such as some model-referenced control schemes. This target is generally not a state of the given (uncontrolled) system (otherwise, perhaps no control is needed or can be easily achieved). Also, the terminal time for the control is usually finite (e.g., the fundamental concept of "controllability" is typically defined using a finite, and often fixed, terminal time, at least for linear systems and affine-nonlinear systems). However, in chaos control, a target for tracking is not limited to constant vectors in the state space but often is an unstable periodic orbit of the given system. In addition, the terminal time for chaos control is usually infinite to be meaningful and practical, because many nonlinear dynamical behaviors such as steady states, limit cycles, attractors, and chaos are asymptotic properties.
5. Depending on different situations or purposes, the performance measure in chaos control can be different from those for conventional controls. Chaos control generally uses criteria like Lyapunov exponents, Kolmogorov–Sinai entropy, power spectra, ergodicity, bifurcation changes, and so on, whereas conventional controls normally emphasize on robustness of the system stability or control performance, optimality of control energy or time, ability of disturbances rejection, and so on.
6. Chaos control includes a unique task — anticontrol, required by some unusual applications such as those in biomedical engineering mentioned above. This anticontrol tries to create, maintain, or enhance chaos for improving system performance. Bifurcation control is another example of this kind, where a bifurcation point is expected to be delayed in case it cannot be avoided or stabilized. This delay can significantly extend the operating time (or system parameter range) for a time-critical process such as chemical reaction, voltage collapse of electric power systems, and compressor of stall of gas turbine jet engines. These are in direct contrast to traditional control tasks such as the typical problem of stabilizing an equilibrium position of a nonlinear system.
7. Due to the inherent association of chaos and bifurcations with various related issues, the scope of chaos control and the variety of problems that chaos control deals with are quite diverse, including creation and/or management of self-similarity and symmetry, pattern formation, amplitudes of limit cycles and size of attractor basins, birth and change of bifurcations and limit cycles, and so on, in addition to some conventional tasks such as target tracking and system regulation.

It is also worth mentioning an additional distinctive feature of a controlled chaotic system that differs from an uncontrolled chaotic system. When the controlled chaotic system is non-autonomous, it cannot be reformulated as an autonomous system by defining the control input as a new state variable, since the control input is physically not a system state variable and, moreover, it has to be determined via design for performance specifications. Hence, a controlled chaotic system is intrinsically much more difficult to design than it appears; for instance, many invariant properties of autonomous systems are no longer valid. This observation raises the question of extending some existing theories and techniques from autonomous system dynamics to nonautonomous controlled dynamical systems, including such complex phenomena as degenerate bifurcations and hyperchaos in the system dynamics when a controller is involved. Unless suppressing complex dynamics in a process is the only purpose for control, understanding and utilizing the rich dynamics of a controlled chaotic system are very important for design and applications.

4.4. Representative Approaches to Chaos Control

There are various conventional and nonconventional control methods available for bifurcations and chaos control (5,15,16). To introduce a few representative control techniques, two major categories of methodologies are briefly described.

Parametric Variation Control. This approach to controlling a chaotic dynamical system, proposed by Ott *et al.* (17), known as the OGY method, is to stabilize one of its unstable periodic orbits embedded in an existing chaotic attractor, via *small* time-dependent perturbations of a variable system parameter. This methodology utilizes the special feature of chaos that a chaotic attractor typically has embedded within it a dense set of unstable periodic orbits.

To introduce this control strategy, consider a general continuous-time parameterized nonlinear autonomous system:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), p) \tag{18}$$

where, for illustration, $\mathbf{x} = [x \;\; y \;\; z]^\top$ denotes the state vector and p is a system parameter accessible for adjustment. Assume that when $p = p^*$ the system is chaotic, and it is desired to control the system orbit, $\mathbf{x}(t)$, to reach a saddle-type equilibrium (or periodic orbit), Γ, which otherwise would not be able to arrive by the system orbits due to its unstable nature.

Suppose that within a small neighborhood of p^*, that is,

$$p^* - \Delta p_{\max} < p < p^* + \Delta p_{\max} \tag{19}$$

where $\Delta p_{\max} > 0$ is the maximum allowable perturbation, both the chaotic attractor and the target orbit Γ do not disappear (i.e., within this small neighborhood of p^*, there are no bifurcation points of the periodic orbit Γ). By the structural stability of chaotic systems, this can be guaranteed. Then, let P be the underlying Poincaré map and Σ be a surface of cross section of Γ. For simplicity, assume that this two-dimensional hyperplane is orthogonal to the third axis, and thus is given by

$$\Sigma = \left\{ \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix}^\top \in R^3 : \quad \gamma = z_0 \text{ (a constant)} \right\}$$

Moreover, let ξ be the coordinates of the surface of cross section, that is, a vector satisfying

$$\xi_{k+1} = P(\xi_k, p_k)$$

where

$$p_k = p^* + \Delta p_k, \qquad |\Delta p_k| \le \Delta p_{\max}$$

At each iteration, $p = p_k$ is chosen to be a constant.

Many distinct unstable periodic orbits within the chaotic attractor can be determined by the Poincaré map. Suppose that an unstable period-one orbit ξ_f^* has been selected, which maximizes certain desired system performance with respect to the dynamical behavior of the system. This target orbit satisfies

$$\xi_f^* = P(\xi_f^*, p^*)$$

The iterations of the map near the desired orbit are then observed, and the local properties of this chosen periodic orbit are obtained. To do so, the map is first locally linearized, yielding a linear approximation of P near ξ_f^* and p^*, as

$$\xi_{k+1} \approx \xi_f^* + L_k(\xi_k - \xi_f^*) + \mathbf{v}_k(p_k - p^*) \tag{20}$$

or

$$\Delta\xi_{k+1} \approx L_k \Delta\xi_k + \mathbf{v}_k \Delta p_k \tag{21}$$

where

$$\Delta\xi_k = \xi_k - \xi_f^*, \qquad \Delta p_k = p_k - p^*$$
$$L_k = \partial P(\xi_f^*, p^*) \big/ \partial \xi_k, \qquad \mathbf{v}_k = \partial P(\xi_f^*, p^*) \big/ \partial p_k$$

The stable and unstable eigenvalues, $\lambda_{s,k}$ and $\lambda_{u,k}$ satisfying $|\lambda_{s,k}| < 1 < |\lambda_{u,k}|$, can be calculated from the Jacobian L_k. Let M_s and M_u be the stable and unstable manifolds, whose directions are specified by the eigenvectors $e_{s,k}$ and $e_{u,k}$ that are associated with $\lambda_{s,k}$ and $\lambda_{u,k}$, respectively. If $g_{s,k}$ and $g_{u,k}$ are the basis vectors defined by

$$\mathbf{g}_{s,k}^\top \mathbf{e}_{s,k} = \mathbf{g}_{u,k}^\top \mathbf{e}_{u,k} = 1$$
$$\mathbf{g}_{s,k}^\top \mathbf{e}_{u,k} = \mathbf{g}_{u,k}^\top \mathbf{e}_{s,k} = 0$$

then the Jacobain L_k can be expressed as

$$L_k = \lambda_{u,k} \mathbf{e}_{u,k} \mathbf{g}_{u,k}^\top + \lambda_{s,k} \mathbf{e}_{s,k} \mathbf{g}_{s,k}^\top \tag{22}$$

To start the parametric variation control scheme, one may open a window covering the target equilibrium, and wait until the system orbit travels into the window (i.e., till ξ_k falls close enough to ξ_f^*). Due to the ergodicity of chaos, this is always possible. To that end, the nominal value of the parameter p_k is adjusted by a small amount, Δp_k, using a control formula given below. In so doing, both the location of the orbit and its stable manifold are changed, such that the next iteration, represented by ξ_{k+1} in the surface of cross section, is forced toward the local stable manifold of the original equilibrium point. Since the system has been linearized, this control action is usually unable to bring the moving orbit to the target at one iteration step. As a matter of fact, the controlled orbit will leave the small neighborhood of the equilibrium, and continue to wander chaotically as if there was no control on it at all. However, since the chaotic attractor has a dense set of unstable periodic orbits embedded within it, sooner or later the orbit returns to the window again, but would be closer to the target due to the control effect. Then, the next cycle of iteration is applied, with an even smaller control action, to nudge the orbit to move toward the target.

For the case of a saddle-node equilibrium target, this control procedure is illustrated by Figure 15.

Now, suppose that ξ_k has approached sufficiently close to ξ_f, so that (eq. 20) holds. For the next iteration, ξ_{k+1}, to fall onto the local stable manifold of ξ_f^*, the parameter

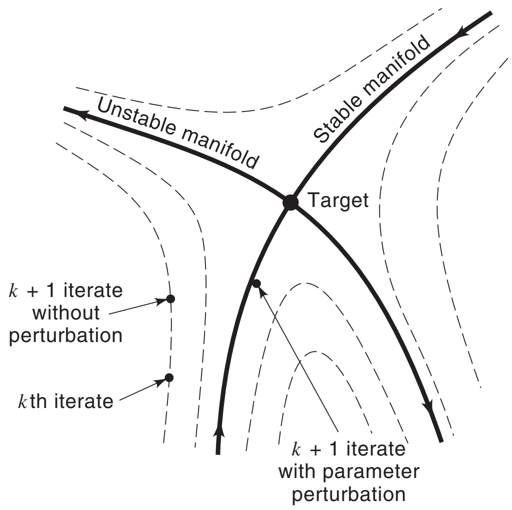

Figure 15. Schematic diagram for the parametric variation control method.

$p_k = p^* + \Delta p_k$ has to be chosen such that

$$\mathbf{g}_{\mathrm{u},k}^{\top} \, \Delta\xi_{k+1} = \mathbf{g}_{\mathrm{u},k}^{\top} \, (\xi_{k+1} - \xi_f^*) = 0$$

This simply means that the direction of the next iteration is perpendicular to the direction of the current local unstable manifold. For this purpose, taking the inner product of equation 21 with $g_{u,k}$ and using equation 22 yields

$$\Delta p_k = -\lambda_{\mathrm{u},k} \frac{\mathbf{g}_{\mathrm{u},k}^{\top} \Delta\xi_k}{\mathbf{g}_{\mathrm{u},k}^{\top} \mathbf{v}_k} \tag{23}$$

where it is assumed that $g_{\mathrm{u},k}^{\top} v_k \neq 0$. This is the control formula for determining the variations of the adjustable system parameter p at each step, $k = 1, 2, \ldots$. The controlled orbit thus is expected to approach ξ_f^* at a geometrical rate, λ_s.

Note that this calculated Δp_k is used to adjust the parameter p only if $\Delta p_k \leq \Delta p_{\max}$. When $\Delta p_k > \Delta p_{\max}$, however, one should set $\Delta p_k = 0$. Also, when ξ_{k+1} falls on a local stable manifold of ξ_f^*, one set $\Delta p_k = 0$, because the stable manifold would lead the orbit directly to the target.

Note also that the above derivation is based on the assumption that the Poincaré map, P, always possesses a stable and an unstable direction (saddle-type orbits). This may not be the case in many systems, particularly those with higher periodic orbits. Moreover, it is necessary that the number of accessible parameters for control is at least equal to the number of unstable eigenvalues of the periodic orbit to be stabilized. In particular, when some of such key system parameters are unaccessible, the algorithm is not applicable or has to be modified. Also, if a system has multi-attractors, the system orbit may never return to the opened window but, instead, move to another nontarget (attracting) limit set. In addition, the technique is successful only if the control is applied after the system orbit moves into the small window covering the target orbit, over which the local linear approximation is still valid. In this case, the waiting time can be quite long for some chaotic systems. To improve this situation, a method called *targeting* may help persuade the system dynamics to quickly approach the region of control. While this algorithm is effective, it generally requires good knowledge of the equations governing the system, so that computing Δp_k by (eq. 23) is possible. In the case where only time series data from the system are available, the *delay-coordinate* technique may be used to construct a faithful dynamical model for control (25).

Engineering Feedback Controls. From a control theoretic point of view, if only suppression of chaos is concerned, chaos control may be considered as a particular nonlinear control problem, and so may not be much harder than conventional nonlinear systems control. This was not clear a few years ago, however, since even if chaos could be controlled was questionable in the old days.

A distinctive characteristic of control engineering is that it always employs some kind of feedback mechanism. In fact, feedback is pervasive in modern control theories and technologies. For instance, the parametric variation control method discussed above is a special type of feedback control method by its very nature. In engineering control systems, conventional feedback controllers are used to be designed for non-chaotic systems. In particular, linear feedback controllers are typically designed for linear systems. It has been widely experienced that with careful designs of various conventional controllers, controlling chaotic systems using feedback strategies is not only possible but indeed quite successful. One basic reason for the success is that chaotic systems, although nonlinear and sensitive to initial conditions with complex dynamical behaviors, belong to deterministic systems without stochastic components or random parameters.

Some Features of Feedback Control. Feedback is one of the most fundamental principles prevalent in the world. The idea of using feedback, originated from Isaac Newton and Gottfried Leibniz some 300 years ago, has been applied in various forms to natural science and modern technology.

One basic feature of conventional feedback control is, while achieving target tracking, that it can guarantee the stability of the controlled system even if the original uncontrolled system is unstable. This implies its intrinsic robustness against external disturbances or internal variations to a certain extent, which is desirable and often necessary for the well performance of a control system. The intuition of feedback control always consuming strong control energy perhaps lead to a false impression that feedback mechanism may not be suitable for chaos control due to the extreme sensitive nature of chaos. However, feedback control under certain optimality criteria, such as a minimum control energy requirement, can provide the best possible performance including the lowest control energy consumption. This is not only supported by theory but also confirmed by experiments with comparison.

Another advantage of using feedback control is that it normally does not change the structure and parameters of the given system and, whenever the feedback is disconnected, the given system retains the original form and dynamics without modification. In most engineering applications, the system parameters are not accessible or not allowed for direct tuning and modifying. In such cases, parameter variation method cannot be used, but feedback control provides a practical approach.

An additional advantage of feedback control is its automatic fashion in processing control tasks without further human interference after being designed and implemented. As long as a feedback controller is correctly designed to satisfy the stability criteria and performance specifications, it works off hand. This is important for automation, reducing the dependence on human operator's skills and avoiding human errors in monitoring control.

A shortcoming of feedback control methods employing tracking errors is the explicit or implicit use of reference signals. This has never been a problem in conventional feedback control of non-chaotic systems where the reference signals are always some designated, well-behaved ones. However, in chaos control, typically a reference signal is an unstable equilibrium or unstable limit cycle, which is difficult if not impossible to be implemented physically as a reference input. This critical issue has stimulated some new research efforts, for instance, to use an auxiliary reference as control input in a self-tuning feedback manner.

Engineering feedback control approaches have seen an alluring future in more advanced theories and applications in controlling complex dynamics. Utilization of feedback is among the most inspiring concepts that engineering mind has ever contributed to modern sciences and advanced technologies.

A Typical Feedback Control Problem. Only the simple tracking control problem of suppressing chaos is discussed here for illustration.

A general feedback approach to controlling a dynamical system, not necessarily chaotic nor even nonlinear, can be illustrated by starting from the following general form of an n-dimensional control system:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{24}$$

where $\mathbf{x}(t)$ is the system state, $\mathbf{u}$ is the controller, $\mathbf{x}_0$ is a given initial state, and $\mathbf{f}$ is a piecewise continuous or smooth nonlinear function.

Given a reference signal, $\mathbf{r}(t)$, which can be either a constant (set point) or a function (time-varying trajectory), the automatic feedback control problem is to design a controller in the following *state-feedback* form:

$$\mathbf{u}(t) = \mathbf{g}(\mathbf{x}, t) \tag{25}$$

where $\mathbf{g}$ is generally a piecewise continuous nonlinear function, such that the feedback-controlled system

$$\dot{\mathbf{x}}(t) = \mathbf{f}\Big(\mathbf{x}, \mathbf{g}(\mathbf{x}, t), t\Big) \tag{26}$$

can achieve the goal of tracking:

$$\lim_{t\to\infty} ||\, \mathbf{x}(t) - \mathbf{r}(t)\, || = 0 \tag{27}$$

For discrete-time systems, the problem and notation are similar. More precisely, for a system

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k) \tag{28}$$

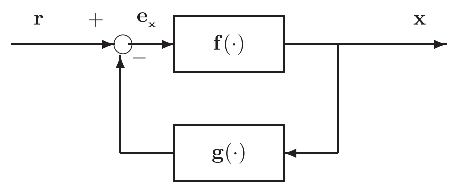

Figure 16. Configuration of a general feedback control system.

with a given target trajectory $\{\mathbf{r}_k\}$ and initial state $\mathbf{x}_0$, find a (nonlinear) controller

$$\mathbf{u}_k = \mathbf{g}_k(\mathbf{x}_k) \tag{29}$$

to achieve the tracking-control goal:

$$\lim_{k\to\infty} ||\, \mathbf{x}_k - r_k\, || = 0 \tag{30}$$

A closed-loop continuous-time feedback control system has a configuration as shown in Figure 16, where $\mathbf{e}_\mathbf{x} := \mathbf{r} - \mathbf{x}$ is the tracking error, $\mathbf{f}$ is the given system, and $\mathbf{g}$ is the feedback controller to be designed, in which $\mathbf{f}$ and $\mathbf{g}$ can be either linear or nonlinear. In particular, for a linear system in the state-space form in connection with a linear state-feedback controller, namely,

$$\dot{\mathbf{x}} = A\,\mathbf{x} + B\,\mathbf{u} = A\,\mathbf{x} + BK_c\Big(\mathbf{r} - \mathbf{x}\Big),$$

where K_c is a constant control gain matrix to be determined, the closed-loop configuration is shown in Figure 17.

A Control Engineer's Perspective. In controllers design, particularly in finding a nonlinear controller for a system, it is important to emphasize that the designed controller should be (much) simpler than the given system to make sense of the world.

For instance, suppose that one wants to find a nonlinear controller, $u(t)$, in the continuous-time setting, to guide the state vector $\mathbf{x}(t) = \Big[x_1(t), \ldots, x_n(t)\Big]^\top$ of a given nonlinear control system:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \quad\vdots \\ \dot{x}_n(t) = f\Big(x_1(t), \ldots, x_n(t)\Big) + u(t) \end{cases}$$

to a target state, y, namely,

$$\mathbf{x}(t) \to \mathbf{y} \qquad \text{as} \qquad t \to \infty\,.$$

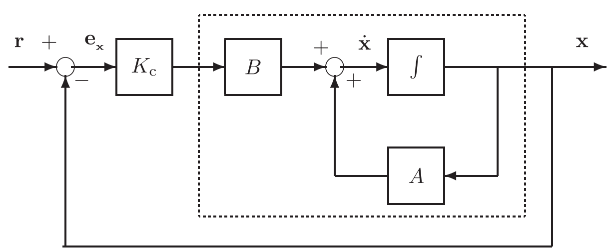

Figure 17. Configuration of a state feedback control system.

It is then mathematically straightforward to use the "controller"

$$u(t) = -f\Big(x_1(t), \dots, x_n(t)\Big) + k_c\Big(x_n(t) - y_n\Big)$$

with an arbitrary constant $k_c < 0$. This controller leads to

$$\dot{x}_n(t) = k_c\Big(x_n(t) - y_n\Big)$$

which yields $e_n(t) := x_n(t) - y_n \to 0$ as $t \to \infty$. Overall, it results in a completely controllable linear system, so that $\mathbf{x}(t) \to \mathbf{y}$ as $t \to \infty$.

Another example is that for the control system

$$\dot{\mathbf{x}}(t) = \mathbf{f}\Big(\mathbf{x}(t), t\Big) + \mathbf{u}(t),$$

use

$$\mathbf{u}(t) = -\mathbf{f}(\mathbf{x}(t), t) + \dot{\mathbf{y}}(t) + K\Big(\mathbf{x}(t) - \mathbf{y}(t)\Big)$$

with a stable constant gain matrix K, to drive its trajectory to the target $\mathbf{y}(t)$ directly.

This kind of "design," however, is unimplementable, because the controller is even more complicated than the given system. Physically, it needs to cancel the nonlinear elements in the given system, which means to remove part of the given machine, and then put a new part into the given system, which will change the given system to another one.

In the discrete-time setting, for a given nonlinear system, $\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k) + \mathbf{u}_k$, one may also find a similar nonlinear feedback controller, or even simpler, use $\mathbf{u} = -\mathbf{f}_k(\mathbf{x}_k) + \mathbf{g}_k(\mathbf{x}_k)$ to achieve any desired dynamics satisfying $\mathbf{x}_{k+1} = \mathbf{g}_k(\mathbf{x}_k)$ in just one step!

Simple "mathematical tricks" like these are certainly not of any engineering design, nor any valuable methodology, for any real-world application other than illusive computer simulations.

Therefore, in designing a feedback controller, it is very important to come out with a simplest possible working controller: If a linear controller can be designed to do the job, use a linear controller; otherwise, try the simplest possible nonlinear controllers (starting, for example, from piecewise linear or quadratic controllers). Whether or not can one find a simple, physically meaningful, easily implementable, low-cost, and effective controller for a designated control task can be quite technical, which relies on the designer's theoretical background and practical experience.

A General Approach to Feedback Control of Chaos. To outline the basic idea of a general feedback approach to chaos suppression and tracking control, consider system (eq. 24) that is now assumed to be chaotic and possess an unstable periodic orbit (or equilibrium), $\bar{\mathbf{x}}$, of period $t_p > 0$, namely, $\bar{\mathbf{x}}(t + t_p) = \bar{\mathbf{x}}(t)$, $t_0 \le t < \infty$. The task is to design a feedback controller in the form of equation 25, such that the tracking control goal (eq. 26), with $\mathbf{r} = \bar{\mathbf{x}}$ therein, is achieved.

Since the target periodic orbit $\bar{\mathbf{x}}$ is itself a solution of the original system, it satisfies

$$\dot{\bar{\mathbf{x}}} = \mathbf{f}(\bar{\mathbf{x}}, t) \tag{31}$$

Subtracting equation 31 from equation 24 yields the error dynamics:

$$\dot{\mathbf{e}}_\mathbf{x} = \mathbf{f}_\mathbf{e}(\mathbf{e}_\mathbf{x}, mat\bar{h}bf\,x, t) \tag{32}$$

where

$$\mathbf{e}_\mathbf{x}(t) = \mathbf{x}(t) - \bar{\mathbf{x}}(t), \qquad \mathbf{f}_\mathbf{e}(\mathbf{e}_\mathbf{x}, \bar{\mathbf{x}}, t) = \mathbf{f}\Big(\mathbf{x}, \mathbf{g}(\mathbf{x}, \bar{\mathbf{x}}(t), t)\Big) - \mathbf{f}(\bar{\mathbf{x}}, t)$$

Here, it is important to note that in order to perform correct stability analysis later on, in the error dynamical system (eq. 32) the function $\mathbf{f}_\mathbf{e}$ must not explicitly contain $\mathbf{x}$; if so, $\mathbf{x}$ should be replaced by $\mathbf{e}_\mathbf{x} + \bar{\mathbf{x}}$. This is because system (eq. 32) should only contain the dynamics of $\mathbf{e}_\mathbf{x}$ but not $\mathbf{x}$, while the system may contain $\bar{\mathbf{x}}$ that merely is a specified time function but not a system variable.

Thus, the design problem becomes to determine the controller, $\mathbf{u}(t)$, such that

$$\lim_{t\to\infty} ||\, \mathbf{e}_\mathbf{x}(t)\, || = 0 \tag{33}$$

which implies that the goal of tracking control described by equation 27 is achieved.

It is clear from equations 32 and 33 that if zero is an equilibrium of the error dynamical system (eq. 32), then the original control problem has been converted to the asymptotic stability problem for this equilibrium. As a result, Lyapunov stability methods and theorems can be directly applied or modified to obtain rigorous mathematical techniques for the controllers design, as discussed in more detail in the following section.

Chaos Control via Lyapunov Methods. The key in applying the Lyapunov second method to a nonlinear dynamical system is to construct a Lyapunov function that describes some kind of energy governing the system motion. If this function is constructed appropriately, so that it decays monotonically to zero as time evolves, then the system motion, which falls on the surface of this decaying function, will be asymptotically stabilized to zero. A controller, then, may be designed to force this Lyapunov function of the system, stable or not originally, to decay to zero. As a result, the stability of tracking error equilibrium, hence the original goal of trajectory tracking, is achieved.

For a chaos control problem with a target trajectory $\bar{\mathbf{x}}$, typically an unstable periodic solution of the given system, a design can be carried out by determining the controller $u(t)$ via the Lyapunov second method such that the zero equilibrium of the error dynamics, $\mathbf{e}_\mathbf{x} = 0$, is asymptotically stable. In this approach, since a linear feedback controller alone is usually not sufficient for the control of a nonlinear system, particularly a chaotic one, it is desirable to find some criteria for the design of simple nonlinear feedback controllers.

In so doing, consider the feedback controller candidate of the form

$$\mathbf{u}(t) = K_c(\mathbf{x} - \bar{\mathbf{x}}) + \mathbf{g}(\mathbf{x} - \bar{\mathbf{x}}, \mathbf{k}_c, t) \tag{34}$$

where K_c is a constant matrix, which can be zero, and $\mathbf{g}$ is a simple nonlinear function with constant parameters $\mathbf{k}_c$, satisfying $\mathbf{g}(0, k_c, t) = 0$ for all $t \ge t_0$. Both K_c and $\mathbf{k}_c$ are determined in the design. Adding this controller to the given system gives

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{u} = \mathbf{f}(\mathbf{x}, t) + K_c(\mathbf{x} - \bar{\mathbf{x}}) + \mathbf{g}(\mathbf{x} - \bar{\mathbf{x}}, \mathbf{k}_c, t) \qquad (35)$$

The controller is required to drive the orbit of the controlled system (eq. 35) to approach the target trajectory $\bar{x}$.

The error dynamics (eq. 32) now takes the following form:

$$\dot{\mathbf{e}}_x = \mathbf{f}_\mathbf{e}(\mathbf{e}_x, t) + K_c \mathbf{e}_x + \mathbf{g}(e_x, \mathbf{k}_c, t) \qquad (36)$$

where

$$\mathbf{e}_\mathbf{x} = \mathbf{x} - \bar{\mathbf{x}}, \qquad \mathbf{f}_\mathbf{e}(\mathbf{e}_\mathbf{x}, t) = \mathbf{f}(e_\mathbf{x} + \bar{\mathbf{x}}, t) - \mathbf{f}(\bar{\mathbf{x}}, t)$$

It is clear that $\mathbf{f}_\mathbf{e}(0, t) = 0$ for all $t \in [t_0, \infty)$, namely, $\mathbf{e}_x = 0$ is an equilibrium of the tracking-error dynamical system (eq. 36).

Next, Taylor expand the right-hand side of the controlled system (eq. 36) at $\mathbf{e}_\mathbf{x} = 0$ (i.e., at $\mathbf{x} = \bar{\mathbf{x}}$) and remember that the nonlinear controller will be designed to satisfy $\mathbf{g}(0, \mathbf{k}_c, t) = 0$. Then, the error dynamics are reduced to

$$\dot{\mathbf{e}}_\mathbf{x} = A(\bar{\mathbf{x}}, t)\mathbf{e}_\mathbf{x} + \mathbf{h}(\mathbf{e}_\mathbf{x}, K_c, \mathbf{k}_c, t) \qquad (37)$$

where

$$A(\bar{\mathbf{x}}, t) = \left[\frac{\partial \mathbf{f}_\mathbf{e}(e_\mathbf{x}, t)}{\partial \mathbf{e}_\mathbf{x}}\right]_{\mathbf{e}_\mathbf{x}=0}$$

and $\mathbf{h}(\mathbf{e}_\mathbf{x}, K_c, \mathbf{k}_c, t)$ contains the rest of the Taylor expansion.

The design is then to determine both the constant control gains K_c and $\mathbf{k}_c$ as well as the nonlinear function $\mathbf{g}(\cdot, \cdot, t)$, based on the linearized model (eq. 37), such that $\mathbf{e}_\mathbf{x} \to 0$ as $t \to \infty$. When this controller is applied to the original system, the goal of both chaos suppression and target tracking will be achieved.

For illustration, two controllability conditions established based on the boundedness of the chaotic attractors as well as the Lyapunov first and second methods, respectively, are summarized below (5).

In system (eq. 37), suppose that $\mathbf{h}(0, K_c, t) = 0$ and $A(\bar{\mathbf{x}}, t) = A$ is a constant matrix with eigenvalues having negative real parts, and let P be the positive definite and symmetric matrix solution of the Lyapunov equation:

$$PA + A^\top P = -I$$

where I is the identity matrix. If K_c is designed to satisfy

$$||\,\mathbf{h}(\mathbf{e}_\mathbf{x}, K_c, \mathbf{k}_c, t)\,|| \le c\,||\,\mathbf{e}_\mathbf{x}\,||$$

for a constant $c < \frac{1}{2}\lambda_{\max}(P)$ for $t_0 \le t < \infty$, where $\lambda_{\max}(P)$ is the maximum eigenvalue of P, then the controller $\mathbf{u}(t)$, defined in equation 34, will drive the orbits $\mathbf{x}$ of the controlled system (eq. 35) to the target trajectory, $\bar{\mathbf{x}}$, as $t \to \infty$.

For system (eq. 37), since $\bar{\mathbf{x}}$ is t_p periodic, associated with the matrix $A(\bar{\mathbf{x}}, t)$, there always exist a t_p periodic nonsingular matrix $M(\bar{x}, t)$ and a constant matrix Q, such that the fundamental matrix (consisting of n independent solution vectors) has the expression

$$\Phi(\bar{\mathbf{x}}, t) = M(\bar{\mathbf{x}}, t)\, e^{tQ}$$

The eigenvalues of the constant matrix $e^{t_p Q}$ are called the *Floquet multipliers* of the system matrix $A(\bar{\mathbf{x}}, t)$.

In system (eq. 37), assume that $\mathbf{h}(0, K_c, \mathbf{k}_c, t) = 0$ and $\mathbf{h}(\mathbf{e}_\mathbf{x}, K_c, \mathbf{k}_c, t)$ and $\partial\mathbf{h}(\mathbf{e}_\mathbf{x}, K_c, \mathbf{k}_c, t)/\partial\mathbf{e}_\mathbf{x}$ both are continuous in a bounded neighborhood of the origin in R^n. Assume also that

$$\lim_{||\mathbf{e}_\mathbf{x}|| \to 0} \frac{||\,\mathbf{h}(e_\mathbf{x}, K_c, \mathbf{k}_c, t)\,||}{||\,e_\mathbf{x}\,||} = 0$$

uniformly with respect to $t \in [t_0, \infty)$. If the nonlinear controller (eq. 34) is designed such that all Floquet multipliers $\{\lambda_i\}$ of the system matrix $A(\bar{\mathbf{x}}, t)$ satisfy

$$\left|\lambda_i(t)\right| < 1, \qquad i = 1, \ldots, n, \qquad \forall\, t \in [t_0, \infty)$$

then the controller will drive the chaotic orbit $\mathbf{x}$ of the controlled system (eq. 35) to the target trajectory, $\bar{\mathbf{x}}$, as $t \to \infty$.

Various Feedback Methods for Chaos Control. In addition to the general nonlinear feedback control approach described above, adaptive and intelligent controls are two large classes of engineering feedback control methods that have been shown to be effective for chaos control. Other successful feedback control methods include optimal control, sliding-mode, and robust controls (e.g., H^∞ control), digital controls, occasionally proportional and time-delayed feedback controls, and so on. Linear feedback controls are also useful, but generally for simple chaotic systems. Various variants of classical control methods that have demonstrated great potentials for controlling chaos include distortion control, dissipative energy method, absorber as a controller, external weak periodic forcing, Kolmogorov–Sinai entropy reduction, stochastic controls, chaos filtering, and so on (5).

Finally, it should be noted that there are indeed many valuable ideas and methodologies that, by their nature and forms, cannot be well classified into any of the aforementioned categories, not to mention that many novel approaches are still emerging, improving, and developing (5).

5. CONTROLLING BIFURCATIONS

Ordering chaos via bifurcations control has never been a subject in conventional control studies. This seems to be a unique approach valid only for those nonlinear dynamical systems that possess some special characteristics with a route to chaos from bifurcations.

5.1. Why Bifurcation Control

Bifurcation and chaos are often twins and, in particular, period-doubling bifurcation is a route to chaos. Hence, by monitoring and managing bifurcations, one can expect achieving certain type of control over chaotic dynamics.

Even bifurcation control itself is very important. In some physical systems such as a stressed system, delaying bifurcations offers an opportunity to obtain stable operating conditions for the machine beyond the margin of operability in a normal situation. Also, relocating and ensuring stability of bifurcated limit cycles can be applied to some conventional control problems, such as thermal convection, to obtain better results. Other examples include stabilization of some critical situations for tethered satellites, magnetic bearing systems, voltage dynamics of electric power systems, and compressor stall in gas turbine jet engines (5).

Bifurcation control essentially means to design a controller for a system to obtain some desired behaviors such as stabilization of bifurcated dynamics, to modify some properties of bifurcations, or to tame chaos via controlling bifurcations. Typical examples include delaying the onset of an inherent bifurcation, relocating an existing bifurcation, changing the shape or type of a bifurcation, introducing a bifurcation at a desired parameter value, stabilizing (at least locally) a bifurcated periodic orbit, optimizing the performance near a bifurcation point for a system, or a combination of some of these. Such tasks have practical values and great potentials in many nontraditional real-world applications.

5.2. Bifurcation Control via Feedback

Bifurcations can be controlled by different methods, among which the feedback strategy is especially effective.

For illustration, consider a general discrete-time parametric nonlinear system:

$$\mathbf{x}_{k+1} = \mathbf{f}\Big(\mathbf{x}_k; p\Big), \qquad k = 0, 1, \ldots \tag{38}$$

where $\mathbf{f}$ is assumed to be sufficiently smooth with respect to both the state $\mathbf{x}_k \in R^n$ and the parameter $p \in R$, and has an equilibrium at $(\bar{\mathbf{x}}, \bar{p}) = (0, 0)$. In addition, assume that the Jacobian of the linearized system of equation 38, evaluated at the equilibrium that is the continuous extension of the origin, has an eigenvalue $\lambda_1(p)$ satisfying $\lambda_1(0) = -1$ and $\lambda'(0) \neq 0$, while all remaining eigenvalues have magnitude strictly less than 1. Under these conditions, the nonlinear function has a Taylor expansion:

$$\mathbf{f}\Big(\mathbf{x}; p\Big) = J(p)\,\mathbf{x} + Q(\mathbf{x}, x; p) + C(\mathbf{x}, \mathbf{x}, \mathbf{x}; p) + \cdots$$

where $J(p)$ is the parametric Jacobian, and Q and C are quadratic and cubic terms in symmetric bilinear and trilinear forms, respectively.

This system has the following property (8): A period-doubling orbit can bifurcate from the origin of system (eq. 38) at $p = 0$; the period-doubling bifurcation is supercritical and stable if $\beta < 0$ but is subcritical and unstable if $\beta > 0$, where

$$\beta = 2\,\mathbf{l}^\top \Big[\, C_0(\mathbf{r}, \mathbf{r}, \mathbf{r}; p) - 2Q_0\Big(\mathbf{r}, J_0^- Q_0(\mathbf{r}, \mathbf{r}; p)\Big) \Big]$$

in which $\mathbf{l}^\top$ is the left eigenvector and $\mathbf{r}$ is the right eigenvector of $J(0)$, respectively, both associated with the eigenvalue -1, and

$$\begin{aligned}
Q_0 &= J(0)Q(\mathbf{x}, \mathbf{x}; p) + Q\Big(J(0)\mathbf{x}, J(0)\mathbf{x}; p\Big) \\
C_0 &= J(0)C(\mathbf{x}, \mathbf{x}, \mathbf{x}; p) \\
&\quad + 2Q\Big(J(0)\mathbf{x}, Q(\mathbf{x}, \mathbf{x}; p)\Big) \\
&\quad + C\Big(J(0)\mathbf{x}, J(0)\mathbf{x}, Q(\mathbf{x}, \mathbf{x}; p); p\Big) \\
J_0^- &= \Big[J^\top(0)J(0) + \mathbf{l}\mathbf{l}^\top\Big]^{-1} J^\top(0)
\end{aligned}$$

Now, consider system (eq. 38) with a control input:

$$\mathbf{x}_{k+1} = \mathbf{f}\Big(\mathbf{x}_k; p, u_k\Big), \quad k = 0, 1, \ldots$$

which is assumed to satisfy the same assumptions when $\mathbf{u}_k = 0$. If the critical eigenvalue -1 is controllable for the associated linearized system, then there is a feedback controller, $\mathbf{u}_k(\mathbf{x}_k)$, containing only third-order terms in the components of $\mathbf{x}_k$, such that the controlled system has a locally stable bifurcated period-two orbit for p near zero. Also, this feedback stabilizes the origin for $p = 0$. If, however, -1 is uncontrollable for the associated linearized system, then generically there is a feedback controller, $\mathbf{u}_k(\mathbf{x}_k)$, containing only second-order terms in the components of $\mathbf{x}_k$, such that the controlled system has a locally stable bifurcated period-two orbit for p near 0. Moreover, this feedback controller stabilizes the origin for $p = 0$ (8).

5.3. Bifurcation Control via Harmonic Balance

For continuous-time systems, limit cycles in general cannot be expressed in analytic forms, and so limit cycles corresponding to period-two orbits in the period-doubling bifurcation diagram have to be approximated in analysis and applications. In this case, the harmonic balance approximation technique (14) can be applied, which is also useful for controlling bifurcations such as delaying or stabilizing the onset of period-doubling bifurcations (26).

Consider a feedback control system in the Lur'e form described by

$$\mathbf{f} * \Big(\mathbf{g}\circ\mathbf{y} + K_c\, y\Big) + \mathbf{y} = 0$$

where $*$ and $\circ$ represent the convolution and composition operations, respectively, as shown in Figure 18.

First, suppose that a system $S = S(\mathbf{f}, \mathbf{g})$ is given as shown in the figure without the feedback controller, K_c. Assume also that two system parameter values, p_h and p_c, are specified, which define a Hopf bifurcation and a supercritical predicted period-doubling bifurcation, respectively. Moreover, assume that the system has a family of predicted first-order limit cycles, which are stable within the range of $p_h < p < p_c$.

Under this framework, the objective now is to design a feedback controller, K_c, added to the system as shown in Figure 18, such that the controlled system, $S^* = S^*\Big(\mathbf{f}, \mathbf{g}, K_c\Big)$, has the following properties:

(i) S^* has a Hopf bifurcation at $p_h^* = p_h$.

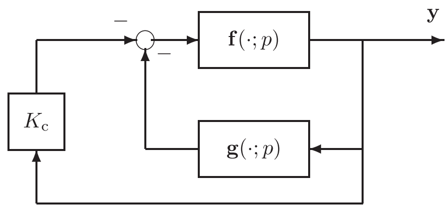

Figure 18. A feedback system in the Lur'e form.

(ii) S^* has a supercritical predicted period-doubling bifurcation for $p_c^* > p_c$.
(iii) S^* has a one-parameter family of stable predicted limit cycles for $p_h^* < p < p_c^*$.
(iv) S^* has the same set of equilibria as S.

Only the one-dimensional case is discussed here for illustration. First, one can design a washout filter with the transfer function $s/(s+a)$, where $a > 0$, such that it preserves the equilibria of the given nonlinear system. Then, note that any predicted first-order limit cycle can be well approximated by

$$y^{(1)}(t) = y_0 + y_1 \sin(\omega t)$$

In so doing, the controller transfer function becomes

$$K_c(s) = k_c \frac{s\left(s^2 + \omega^2(p_h)\right)}{(s+a)^3}$$

where k_c is the constant control gain and $\omega(p_h)$ is the frequency of the limit cycle emerged from the Hopf bifurcation at the point $p = p_h$. This controller also preserves the Hopf bifurcation at the same point. More importantly, since $a > 0$, the controller is stable, so by continuity in a small neighborhood of k_c, the Hopf bifurcation of S^* not only remains supercritical but also has a supercritical predicted period-doubling bifurcation (say, at $p_c(k_c)$, close to p_h) and a one-parameter family of stable predicted limit cycles for $p_h < p < p_c(k_c)$.

The design is then to determine k_c such that the predicted period-doubling bifurcation can be delayed to a desired parameter value p_c^*. For this purpose, the harmonic balance approximation method (14) is useful, which leads to a solution of $y^{(1)}$ by obtaining values of y_0, y_1, and ω (they are functions of p, depending on k_c and a, within the range $p_h < p < p_c^*$). The harmonic balance also yields conditions, in terms of k_c, a, and a new parameter, for the period-doubling prediction to occur at the point p_c^*. To this end, the controller design can be completed by choosing a suitable value for k_c to satisfy such conditions (26).

5.4. Controlling Multiple Limit Cycles

As indicated by the Hopf bifurcation theorem, limit cycles are frequently associated with bifurcations. In fact, one type of *degenerate* (or *singular*) Hopf bifurcations (when some of the conditions stated in the Hopf theorems are not satisfied) determines the birth of multiple limit cycles under system parameters variation. Hence, the appearance of multiple limit cycles can be controlled by managing the corresponding degenerate Hopf bifurcations. This task can be conveniently accomplished in the frequency domain setting.

Again, consider the feedback system (eq. 15), which can be illustrated by a variant of Figure 18. For harmonic expansion of the system output, $\mathbf{y}(t)$, the first-order formula is (14)

$$\mathbf{y}^1 = \theta\,\mathbf{r} + \theta^3 z_{13} + \theta^5 \mathbf{z}_{15} + \cdots$$

where θ is shown in Figure 12, r is defined in equation 16, and $\mathbf{z}_{13}, \ldots, z_{1,2m+1}$ are some vectors orthogonal to $\mathbf{r}$, $m = 1, 2, \ldots$, given by explicit formulas (14).

Observe that for a given value of $\widehat{\omega}$, defined in the graphical Hopf theorem, the SISO system transfer function satisfies

$$H(j\,\widehat{\omega}) = H(s) + (-\alpha + j\,\delta\omega)H'(s) + \frac{1}{2}(-\alpha + j\,\delta\omega)^2 H''(s) + \cdots \tag{39}$$

where $\delta\omega = \widehat{\omega} - \omega$, with ω being the imaginary part of the bifurcating eigenvalues, and $H'(s)$ and $H''(s)$ are the first and second derivatives of $H(s)$, defined in equation 15, respectively. On the other hand, with the higher order approximations, the following equation of harmonic balance can be derived:

$$[H(j\,\omega)J + I] \sum_{i=0}^{m} \mathbf{z}_{1,2i+1}\theta^{2i+1} = -H(j\,\omega) \sum_{i=1}^{m} \mathbf{r}_{1,2i+1}\theta^{2i+1}$$

where $\mathbf{z}_{11} = \mathbf{r}$ and $\mathbf{r}_{1,2m+1} = \mathbf{h}_m$, $m = 1, 2, \ldots$, in which $\mathbf{h}_1$ has the formula shown in equation 16, and the others also have explicit formulas (14).

In a general situation, the following equation has to be solved:

$$\begin{aligned}&[H(j\,\widehat{\omega})J + I]\left(\mathbf{r}\,\theta + z_{13}\,\theta^3 + \mathbf{z}_{15}\,\theta^5 + \cdots\right)\\ &= -H(j\,\widehat{\omega})\left[\mathbf{h}_1\,\theta^3 + \mathbf{h}_2\,\theta^5 + \cdots\right].\end{aligned} \tag{40}$$

In so doing, by substituting equation 39 into equation 40, one obtains the expansion

$$(\alpha - j\,\delta\omega) = \gamma_1\,\theta^2 + \gamma_2\,\theta^4 + \gamma_3\,\theta^6 + \gamma_4\,\theta^8 + O(\theta^9) \tag{41}$$

in which all the coefficients γ_i, $i = 1, 2, 3, 4$, can be calculated explicitly (14). Then, taking the real part of equation 41 gives

$$\alpha = -\sigma_1\,\theta^2 - \sigma_2\,\theta^4 - \sigma_3\,\theta^6 - \sigma_4\,\theta^8 - \cdots$$

where $\sigma_i = -\Re\{\gamma_i\}$ are the *curvature coefficients* of the expansion.

To this end, notice that multiple limit cycles will emerge when the curvature coefficients are varied near the value zero, after alternating the signs of the curvature coefficients in increasing (or decreasing) order. For example, to have four limit cycles in the vicinity of a type of degenerate Hopf bifurcation that has $\sigma_1 = \sigma_2 = \sigma_3 = 0$ but $\sigma_4 \neq 0$ at the criticality, the system parameters have to be varied in such a way that, for example, $\alpha > 0$, $\sigma_1 < 0$, $\sigma_2 > 0$, $\sigma_3 < 0$, and $\sigma_4 > 0$. This condition suggests a methodology for controlling the birth of multiple limit cycles associated with degenerate Hopf bifurcations.

One advantage of this methodology is that there is no need to modify the feedback control path by adding any nonlinear components, in order to drive the system orbit to a desired region: One can simply modify the system parameters, a kind of parametric variation control, according to the expressions of the curvature coefficients, so as to achieve the goal of controlling bifurcations and limit cycles.

6. ANTICONTROL OF CHAOS

Anticontrol of chaos, in contrast to the main stream of ordering or suppressing chaos, is to make a nonchaotic dynamical system chaotic, or to retain/enhance the existing chaos of a chaotic system. Anticontrol of chaos as one of the unique features of chaos control has emerged as a theoretically attractive and potentially useful new subject in systems control theory and time-critical or energy-critical high-performance applications.

6.1. Why Anticontrol of Chaos

Chaos has long been considered as a disaster phenomenon and so is very fearsome to many. However, chaos "is dynamics freed from the shackles of order and predictability." Under good conditions or suitable control, it "permits systems to randomly explore their every dynamical possibility. It is exciting variety, richness of choice, a cornucopia of opportunities" (27).

Today, chaos theory has been anticipated to be potentially useful in many novel and time- or energy-critical applications. In addition to those potential utilization of chaos that mentioned earlier in the discussion of chaos control, to name but a few more, it is worth mentioning navigation in the multibody planetary system, secure information processing via chaos synchronization, dynamic crisis management, critical decision-making in political, economical, as well as military events, and so on In particular, it has been observed that a transition of a biological system's state from being chaotic to being pathophysiologically periodic can cause the so-called dynamical disease, and so is undesirable. Examples of dynamical diseases include cell counts in hematological disorder, stimulant drug-induced abnormalities in time of the patterns of brain enzymes, receptors, and animal explorations in space, cardiac interbeat interval patterns in a variety of cardiac disorders, the resting record in a variety of signal-sensitive biological systems following desensitization, experimental epilepsy, hormone release patterns correlated with the spontaneous mutation of a neuroendocrine cell to a neoplastic tumor, prediction of immunological rejection of heart transplants, electroencephalographic behaviors of the human brain in the presence of neurodegenerative disorder, neuroendocrine, cardiac, and electroencephalographic changes with aging, and imminent ventricular fibrillation in human subjects, and so on (28). Hence, preserving chaos in these cases is important and healthy, which presents a real challenge for creative research on anticontrol of chaos (5).

6.2. Some Approaches to Anticontrolling Chaos

Anticontrol of chaos is a new research direction, which has just started to develop. Different methods for anticontrolling chaos are quite possible (5), but only two preliminary approaches are presented here for illustration.

Preserving Chaos by Small Control Perturbations. Consider an n-dimensional discrete-time nonlinear system:

$$\mathbf{x}_{k+1} = \mathbf{f}\left(\mathbf{x}_k, p, u_k\right)$$

where $\mathbf{x}_k$ is the system state, u_k is a scalar-valued control input, p is a variable parameter, and $\mathbf{f}$ is a locally invertible nonlinear map. Assume that with $u_k = 0$, the system orbit behaves chaotically at some value of p, and that when p increases and passes a critical value, p_c, inverse bifurcation emerges leading the chaotic state to becoming periodic.

Within the biological context, such a bifurcation is often undesirable: There are many cases where the loss of complexity and the emergence of periodicity are associated with pathology (dynamical disease). The question, then, is whether it is possible (and, if so, how) to keep the system state chaotic even if $p > p_c$, by using small control inputs, $\{u_k\}$.

It is known that there are at least three common bifurcations that can lead chaotic motions directly to low-periodic attracting orbits: (i) crises, (ii) saddle-node type of intermittency, and (iii) inverse period-doubling type of intermittency. Here, *crises* refer to a sudden change caused by the collision of an attractor with an unstable periodic orbit; *intermittency* is a special route to chaos, where regular orbital behavior is intermittently interrupted by a finite-duration "burst," in which the orbit behaves in a decidedly different fashion; *inverse period-doubling bifurcation* has a diagram in reverse form as that shown in Figure 13 (i.e., from chaos back to less and less bifurcating points, leading back to a periodic motion), while the parameter remains increasing.

In all these cases, one can identify a *loss region*, G, which has the property that after the orbit falls into G, it is rapidly drawn to a periodic orbit. Thus, a strategy to retain chaos for $p > p_c$ is to avoid this from happening by successively iterating G in such a way that

$$\begin{aligned} G_1 &= \mathbf{f}^{-1}\left(G, p, 0\right) \\ G_2 &= f^{-1}\left(G_1, p, 0\right) = \mathbf{f}^{-2}\left(G, p, 0\right) \\ &\vdots \\ G_m &= \mathbf{f}^{-m}\left(G, p, 0\right) \end{aligned}$$

As m increases, the width of G_m in the unstable direction(s) has a general tendency to shrink exponentially. This suggests the following control scheme (28):

> Pick a suitable value of m, denoted m_0. Assume that the orbit initially starts from outside the region $G_{m_0+1} \cup G_{m_0} \cup \cdots \cup G_1 \cup G$. If the orbit lands in G_{m_0+1} at iterate ℓ, the control u_ℓ is applied to kick the orbit out of G_{m_0} at the next iterate. Since G_{m_0} is thin, this control can be very small. After the orbit is kicked

out of G_{m_0}, it is expected to behave chaotically, until it falls again into G_{m_0+1}, and at that moment another small control is applied, and so on. This procedure can keep the motion chaotic.

Anticontrol of Chaos via State Feedback. An approach to anticontrol of discrete-time systems can be made mathematically rigorous by means of applying the engineering feedback control strategy. This anticontrol technique is first to make the Lyapunov exponents of the controlled system either strictly positive, or arbitrarily assigned (positive, zero, and negative in any desired order), and then applying the simple mod-operations, which keeps the orbits to remain bounded (5,29). This task can be accomplished for any given higher dimensional discrete-time dynamical system that could be originally nonchaotic or even asymptotically stable. The argument used is purely algebraic and the design procedure is completely schematic, in which no approximations are needed.

Specifically, consider a nonlinear dynamical system, not necessarily chaotic nor unstable to start with, in the general form of

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k) \tag{42}$$

where $\mathbf{x}_k \in R^n$, $\mathbf{x}_0$ is given, and f_k is assumed to be continuously differentiable, at least locally in the region of interest.

The anticontrol problem for this dynamical system is to design a linear state-feedback control sequence, $\mathbf{u}_k = B_k\mathbf{x}_k$, with uniformly bounded constant control gain matrices, $||B_k||_s \le \gamma_u < \infty$, where $||\cdot||_s$ is the spectral norm for a matrix, such that the output states of the controlled system,

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k) + \mathbf{u}_k$$

behaves chaotically within a bounded region. Here, chaotic behavior is in the mathematical sense of Devaney described above, namely, the controlled map (i) is transitive, (ii) has sensitive dependence on initial conditions, and (iii) has a dense set of periodic solutions.

In the controlled system

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k) + B_k\mathbf{x}_k$$

let

$$J_k(\mathbf{x}_k) = \mathbf{f}'_k(\mathbf{x}_k) + B_k$$

be the system Jacobian, and let

$$T_k(\mathbf{x}_0) := T_j(\mathbf{x}_0, \ldots, \mathbf{x}_k) = J_k(x_k)\cdots J_1(\mathbf{x}_1)J_0(\mathbf{x}_0), \quad k = 0, 1, 2, \ldots$$

Moreover, let $\mu_i^k = \mu_i(T_k^\top T_k)$ be the ith eigenvalue of the kth product matrix $\left[T_k^\top T_k\right]$, where $i = 1, \ldots, n$ and $k = 0, 1, 2, \ldots$.

The first attempt is to determine the constant control gain matrices, $\{B_k\}$, such that the Lyapunov exponents of the controlled system orbit can be arbitrarily assigned:

$$\lambda_i(\mathbf{x}_0) = \sigma_i \qquad \text{(arbitrarily given)}, \qquad i = 1, \ldots, n$$

where each value σ_i may be positive, zero, or negative. It turns out that this is possible under a natural condition that all the Jacobians $\mathbf{f}'_k(\mathbf{x}_k)$ are uniformly bounded:

$$\sup_{0\le k\le\infty} \left|\left| \mathbf{f}'_k(\mathbf{x}_k)\right|\right| \le \gamma_f < \infty \tag{43}$$

To come up with a design methodology, first observe that if $\left\{\theta_i^{(k)}\right\}_{i=1}^n$ are the singular values of the matrix $T_k(\mathbf{x}_0)$, then $\theta_i^{(k)} \ge 0$ for all $i = 1, \ldots, n$ and $k = 0, 1, \ldots$, with the relationship

$$\sigma_i = \lim_{k\to\infty} \frac{1}{k} \ln\ \theta_i^{(k)} \qquad \left(\text{for } \theta_i^{(k)} > 0\right), \quad i = 1, \ldots, n$$

Clearly, if $\theta_i^{(k)} = \mathrm{e}^{(k+1)\sigma_i}$ is used in the design, then all $\theta_i^{(k)}$ will not be zero for any finite values of σ_i, for all $i = 1, \ldots, n$ and $k = 0, 1, \ldots$. Thus, $T_k(\mathbf{x}_0)$ is always nonsingular. Consequently, a control-gain sequence $\{B_k\}$ can be designed such that the singular values of the product matrix $T_k(\mathbf{x}_0)$ are exactly equal to $\left\{e^{k\sigma_i}\right\}_{i=1}^n$: at the kth step, $k = 0, 1, 2, \ldots$, one may choose the control gain matrix to be

$$B_k = -\,\mathbf{f}'_k(\mathbf{x}_k) + \begin{bmatrix} e^{\sigma_1} & & \\ & \ddots & \\ & & e^{\sigma_n} \end{bmatrix}, \quad k = 0, 1, \ldots$$

where $\{\sigma_i\}_{i=1}^n$ are the desired Lyapunov exponents.

Actually, if the objective is only to have all the resulting Lyapunov exponents of the controlled system orbit be finite and strictly positive:

$$0 < c \le \lambda_i(\mathbf{x}_0) < \infty, \qquad i = 1, \ldots, n \tag{44}$$

then one can simply use

$$B_k = \left(\gamma_f + e^c\right) I_n, \qquad \text{for all } k = 0, 1, 2, \ldots$$

where the constants c and γ_f are given in equations 44 and 43, respectively (29).

Finally, in conjunction with the above-designed controller, that is,

$$\mathbf{u}_k = B_k\mathbf{x}_k = \left(\gamma_f + e^c\right)\mathbf{x}_k$$

anti-control is accomplished by imposing the mod-operation onto the controlled system:

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k) + \mathbf{u}_k \qquad (\text{mod } 1)$$

This results in the expected chaotic system with trajectories remaining within a bounded region in the phase space and, moreover, satisfying the aforementioned three basic properties that together define discrete chaos.

This approach yields rigorous anticontrol of chaos for any given discrete-time systems, including all higher dimensional, linear time-invariant systems, that is, with $\mathbf{f}_k(\mathbf{x}_k) = A\,\mathbf{x}_k$ in system (eq. 42), where the constant matrix A can be arbitrary (even asymptotically stable).

Although $\mathbf{u}_k = B_k\mathbf{x}_k$ is a linear state-feedback controller, the mod-operation is inherently nonsmooth. Recently, other types of (smooth) feedback controllers have been developed for rigorous anticontrol of chaos, particularly for continuous-time dynamical systems (30) and hyperchaotic systems (31,32).

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