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Pinning Control and Controllability of Complex Dynamical Networks

Guanrong Chen
Department of Electronic Engineering, City University of Hong Kong, Hong Kong, China

Abstract: In this article, the notion of pinning control for directed networks of dynamical systems is introduced, where the nodes could be either single-input single-output (SISO) or multi-input multi-output (MIMO) dynamical systems, and could be non-identical and nonlinear in general but will be specified to be identical linear time-invariant (LTI) systems here in the study of network controllability. Both state and structural controllability problems will be discussed, illustrating how the network topology, node-system dynamics, external control inputs and inner dynamical interactions altogether affect the controllability of a general complex network of LTI systems, with necessary and sufficient conditions presented for both SISO and MIMO settings. To that end, the controllability of a special temporally switching directed network of linear time-varying (LTV) node systems will be addressed, leaving some more general networks and challenging issues to the end for research outlook.

Keywords: Complex network, pinning control, controllability, linear time-invariant (LTI) system, temporally switching network, graph theory.

1 Introduction

Recently, the interplay between network science and control theory has seen rapid growth within several interdisciplinary research fields, mainly in engineering, physics, mathematics, computer science, biology and social sciences. On the one hand, network science has been extensively investigated in the past two decades, strongly stimulated by the exploration and advance of small-world networks (Watts-Strogatz[1]) and scale-free networks (Barabási-Albert[2]), which are considered as new developments to follow the classical notion of random graphs (Erdös-Rényi[3]). They are merged together as a new and fast-evolving research paradigm in the modern computation-based and data-driven engineering and technology, which have provoked a great deal of interest and great effort in studying network theory and its applications today. On the other hand, the classical control theory as a powerful tool has been indispensable to the research and development of network science and systems engineering. The classical concept of system (and network) controllability is key to both systems engineering and network science, which determines whether or not a system (or a network) is controllable and, if not, under what conditions it can be so[4].

In the big-data era and Omni-networking world today, the classical systems control theory addresses more and more large-scale networks (the Internet, wireless communication networks, transportation networks, power grids, and sensor networks alike). In the past, control theory was typically concerned about control problems and methods for a single albeit higher-dimensional dynamical system, but rarely focused on directed and inter-connected networks of many of such systems, noticeably it did not emphasize on the internal topological connectivity and directionality of the interconnected systems in interest. Today′s tremendous high-tech demands require various forms of control over complex dynamical networks such as the Internet, wireless communication networks, global transportation systems, smart grids, and gene regulation networks, using advanced facilities and devices such as supercomputers in cloud computing environments, big-data sources, GPS services, etc., so that control theory becomes more and more important and useful. In this regard, the mathematical graph theory[5] is a particularly useful tool for studying the controllability, and other relevant issues like observability, synchronizability, stability and stabilization, for different types of complex dynamical networks.

The recent fast-evolving development of network science and engineering has created a corpus of new opportunities as well as challenges to classical control systems theory, for complex dynamical networks are typically large-sized with huge numbers of nodes and edges, which are intrinsically higher-dimensional and inter-connected in a complicated manner with such structures as random-graph, small-world or scale-free topologies. Moreover, they usually involve nonlinearities, possess layered or switching structures with time-varying parameters and even evolve in multiple spatiotemporal scales.

For most large-scale and complex-structured dynamical networks, in order to achieve certain goals, practically one can only control just a few of their nodes via external inputs.

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Noticeably, in many cases, by controlling a few properly-selected nodes, one can completely control the whole network towards a desired goal. The key here is to understand, for the given network, how many and which nodes should be selected to control in order to be successful. Motivated by this application-oriented consideration, the innovative notion of "pinning control" was technically initiated in [6, 7] as an effective control strategy that answers the fundamental questions of how many and which nodes to control (to pin) for a given complex dynamical network[6−8].

The idea of pinning control can be best illustrated by some examples from biology. One is about the worm C. elegans, which has a simple neural network in its body with statistically a few hundred neurons and a few thousand synapse connections. For this worm, the question of stimulating (controlling) how many neurons one can expect to provoke its whole body has a rather surprising answer, only 49 on average, less than 17 % of the total[9]. Another example is about the beehives which migrate together to forage[10]: "Relatively few informed individuals within fish schools are known to be able to influence the foraging behaviour of the group and the ability of a school to navigate towards a target. Similarly, very few individuals (approximately 5 %) within honeybee swarms can guide the group to a new nest site."

From the control-theoretic point of view, these approximately 17 % of neurons and 5 % of honeybees can be viewed as controlled individuals, through which the entire network is manageable for completing a certain task. This control strategy is obviously very efficient and economical. Inspired by observations like these, a sensible question is: To effectively achieve some objective on a given and fixed network of dynamical systems, how many node systems one needs to control and at which nodes to apply the controllers? To answer this kind of questions, the theory and techniques of pinning control were emerged, aiming to develop an effective control approach that can "pull one hair to move the whole body".

Clearly, any answer to the above question depends upon the topology (e.g., regular, random, small-world, or scale-free structure) of the given network and its node dynamics (e.g., linear, nonlinear, impulsive, or hybrid systems), as well as the network evolution (switching, temporal, time-delayed) manner. Therefore, in pinning controllers design and implementation, one has to take into account both the node dynamics and the network topology. It is also evident that to answer such questions, the conventional control theory is essential but insufficient. In retrospect, this kind of questions would not be asked by or did not even exist in the classical control theory, which deals with large-scale systems without specifically utilizing their intrinsic topologies and the interactive dynamics among internal subsystems. The above observations thus motivate the current research efforts on pinning control, aiming to extend the classical control theory from complex systems to complex networks of such complex systems.

2 Pinning control of directed networks

Consider a directed network of N identical nodes:

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} c_{ij}\alpha_{ij}Hx_j,\ i = 1, 2, \cdots, N \qquad (1)$$

where $f : \mathbf{R}^n \to \mathbf{R}^n$ is a nonlinear function in general, satisfying well-posed conditions, $c_{ij} \neq 0$ are coupling strengths (weights), H is a constant inner coupling matrix, and $\Lambda = [\alpha_{ij}]$ is the outer coupling matrix in which $\alpha_{ij} = 1$ if there exists a connection from node j pointing to node i but $\alpha_{ij} = 0$ otherwise. For network (1), the Laplacian matrix is defined by $L = D - \Lambda$, with $D = \mathrm{diag}\{d_1, \cdots, d_N\}$, where d_i is the in-degree of node i, namely, the total number of its receiving connections, $i = 1, 2, \cdots, N$.

Throughout this article, for the given network (1), always apply linear state feedback controllers $u_i = -Bx_i$ to its right-hand side and always use linear state observations $y_i = Cx_i$ for coupling, where B and C are constant control and observation matrices, respectively, which together lead to a controlled network of the form

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} c_{ij}\alpha_{ij}Hy_j - \delta_i Bx_i,\ i = 1, 2, \cdots, N \qquad (2)$$

where $\delta_i = 1$ if such a feedback controller is applied to node i but $\delta_i = 0$ otherwise. Let $\Delta = \mathrm{diag}\{\delta_1, \cdots, \delta_N\}$ in the following.

Now, the pinning control problem can be precisely stated as follows: For the controlled network (2) to achieve some goal of control (e.g., synchronization or stabilization), determine, if feasible also subject to some optimization requirement (e.g., optimal energy or minimum time):

1) for how many i, one should have $\delta_i = 1$, $i = 1, 2, \cdots, N$?

2) for which i, one should have $\delta_i = 1$, $i = 1, 2, \cdots, N$?

Ever since 2002[6], this pinning control strategy has been extensively investigated, regarding for example network synchronization[8, 11, 12], stabilization[7, 13], and some applications[14], among other related topics[8].

In particular, synchronization of the complex dynamical network (1) means that

$||x_i - x_j|| \to 0$ as $t \to \infty$ for all $i, j = 1, 2, \cdots, N$.

But if this is not possible to achieve by the network itself then external control input is needed. Therefore, pinning synchronization of the network (2) means to determine how many and which nodes to control (to pin). This pinning synchronization problem has been thoroughly investigated in the current literature (see, for example, [8, 11, 12] and references therein), and so it will not be further discussed herein.

The present short article aims to briefly review the recent progress in the study of the fundamental concept of controllability for the controlled complex dynamical networks (2), leaving alone many related important topics on different kinds of control performances and their potential engineering applications.

3 Controllability of directed networks

The concept of (complete) state controllability was first introduced by Kalman in the 1960 s, for a linear time-invariant (LTI) dynamical system of the form

$$\dot{x} = Ax + Bu \tag{3}$$

where x is the system state vector and u is the control input vector, with $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times m}$ $(1 \le m \le n)$ being the constant system matrix and control matrix, respectively. This LTI dynamical system is usually denoted by the matrix pair (A, B) for simplicity. It is (completely) state controllable on the time interval $[t_0, \infty)$ if for any initial state $x_0 \in \mathbf{R}^n$ at time $t_0 \ge 0$, there exists an input $u(t) \in \mathbf{R}^r$ defined on $[t_0, \infty)$ such that $x(t_f) = 0$ at a finite time instant $t_f > 0$[4]. This is simply referred to as controllability hereinafter unless otherwise indicated. Kalman also provided a necessary and sufficient condition for the controllability of the system (A, B): The system controllability matrix

$$W = [B\ AB\ A^2B\ \cdots A^{n-1}B] \tag{4}$$

has a full row-rank[4]. Thereafter, to similarly investigate the controllability of LTI dynamical systems under the framework of directed networks, the concept of structural controllability was introduced and studied in the 1970 s-1980 s[15, 16]. The first significant contribution is attributed to Lin[15], where an elegant necessary and sufficient condition was given in geometric terms. This concept emphasizes on the structure of the system in interest, where the system and control matrices A and B are both parameterized, except all the fixed zero entries. Such a system, also denoted by the matrix pair (A, B), is structurally controllable if there is a set of nonzero parameter values of A and B, called a nonzero realization, which together satisfy condition (4). Later, in order to ensure that a structurally controllable system is always state controllable for all its nonzero realizations, the concept of strong structural controllability was introduced[17].

More precisely, an LTI system (A, B) is said to be:

1) completely state controllable (or state controllable, or simply, controllable), if its controllability matrix (4) has a full row-rank;

2) structurally controllable, if there exists a nonzero realization of (A, B) such that this system is controllable in the sense of 1);

3) strongly structurally controllable if, for any nonzero realization of (A, B), this system is always controllable in the sense of 1).

In parallel, the concepts of observability and structural observability for an LTI system $\dot{x} = Ax+Bu$ along with output $y = Cx, \quad C \in \mathbf{R}^{p\times m}$ $(1 \le p \le m)$, were introduced. Briefly, observability means that any input-output pair of an LTI system, now represented by (A, B, C), can uniquely determine the initial state on a finite time interval for some $t_f < \infty$[4]. The concept of structural observability is defined in a way similar to the structural controllability[16]. Moreover, the system (A, B, C) is (completely) state observable if and only if the system observability matrix

$$V = \left[C^{\mathrm{T}}, A^{\mathrm{T}}C^{\mathrm{T}}, \cdots, (A^{n-1})^{\mathrm{T}}C^{\mathrm{T}}\right]^{\mathrm{T}} \tag{5}$$

has a full column-rank. It can be easily verified that the two concepts of controllability and observability are dual in the sense that the system (A, B, C) is controllable if and only if the system $(A^{\mathrm{T}}, C^{\mathrm{T}}, B^{\mathrm{T}})$ is observable[4]. For this reason, the observability will not be discussed in any detail below.

Today, spurred by the rapid and promising development of network science and engineering, the aforementioned concepts and notions have been revisited and reused from a network-theoretic perspective, which brought up many new interesting problems as well as technical challenges to both network science and control systems communities.

To further discuss the controllability issue for directed networks, the nonlinear node-system function in network (1) and (2) is now specified to be linear, in the form of $f(x) = Ax$, which could be the linearized model of a general nonlinear system.

Recently, there has been great progress in the study of linear network controllability[18]. For instance, it was found[19] that, regarding control cost, good pinned-nodes typically are not big hub-nodes but some small nodes of lower degrees. In [20], it is shown that scale-free networks are easier to control than those networks with weak degree-degree correlations. In [21], the issue of controlling edge dynamics is discussed, while in [22], an analytical framework is developed for identifying critical intermittent or redundant nodes, revealing two distinct control modes for complex systems: centralized versus distributed controls. In [23], a control capacity measure is formulated for quantifying the likelihood that a node should be pinned, demonstrating that the possibility of being pinned decreases with the in-degree but is independent of the out-degree of the node. In [24], an exact controllability framework is introduced for identifying the minimum set of input-control nodes for a general network with an arbitrary link-weight distribution. In [25], it is shown that the density of nodes with in-degree and out-degree equal to one and two determines the number of external control inputs needed to control the network, with an algorithm developed for network controllability improvement. In [26], target control of networks is studied, with a "k-walk" theory developed for directed tree networks, which shows that one node can control a set of target nodes if the path length to each target node is unique. In [27], a mathematical and computational approach is proposed for controlling higher-dimensional nonlinear networks under some general constraints on admissible interventions. In [28], it shows that the variability of control energy and observational uncertainty for different directions in the state space depend strongly on the number of external control inputs. In [29], effects of both structure and dynamics on the network controllability are studied. In [30], several metrics based on the network controllability and observability Gramian matrices are suggested for efficient global optimization or its greedy heuristic approximation. The abovementioned publications are by no means exhaustive, and some other studies may be found from the references given in the aforementioned research papers, as well as some related investigations[31].

As can be seen from the above research developments, the network controllability is a focal subject of common interest in recent years, and there are many relevant fundamental theoretic and applied research problems waiting for further exploration under the general framework of complex dynamical networks.

3.1 SISO networks

In this subsection, consider the directed network (2) of single-input single-output (SISO) LTI dynamical systems, where all node systems are identical and one-dimensional, in the form of

$$\dot{x}_i = ax_i \tag{6}$$

namely, with $f(x_i) = ax_i$ in (2), and the node output

$$y_i = cx_i \tag{7}$$

where both a and c are scalers, hence both inputs and outputs of the nodes are one-dimensional, $u_i, y_i \in \mathbf{R}$, $i = 1, 2, \cdots, N$. The next subsection will discuss the multi-input multi-output (MIMO) setting, which includes also MISO and SIMO configurations.

In the SISO setting, the controllability of the controlled network (2) towards synchronizing to a reference state can be studied via an augmented system approach[32]. The idea is to include a virtual node, related to control input, into the network, so as to augment the network coupling matrix from dimension N to $N+1$, and then apply the master stability function method[33] to perform analysis. A further development of this approach can be found in [34].

The controllability problem for asymmetrical weighted scale-free networks was discussed in [35], where a threshold for pinning control was formulated: when the ratio of the pinned to non-pinned nodes increases to be over this threshold, the network controllability will be achieved and the control performance will also be improved.

The pinning control problem for non-diagonalized directed networks was studied in [36], using the spectral gap as the controllability index. It was found that the controllability is closely related to both the node dynamical functions and the control gains, revealing the key issue and also the essential difficulty in controllability analysis of directed networks.

For the above SISO networks with LTI node systems, a special formulation and a general setting are respectively discussed in more detail below.

In general, the SISO setting with network (2) and node systems (6) and (7) can be put into the form of (3). In this case, a complete solution to the pinning control problem for a general formulation of network structural controllability, which is not restricted to any particular task (e.g., synchronization or stabilization), can be obtained via a graph-theoretical approach. More precisely, the minimum number of external control inputs needed to ensure the given directed network be structurally controllable can be determined by a maximum matching on the controlled network (2). Here, a subset E^* of directed edges on a directed network is called a matching if every pair of edges in E^* do not have common starting node nor common ending node. A node is called a matched node if it is the ending node of an edge in E^*. Otherwise, it is an unmatched node. In a directed network, a matching with the maximum number of matched nodes is called a maximum matching, which is not unique in general. Furthermore, a maximum matching is a perfect matching if all of its nodes are matched nodes[37].

The so-called "minimum input theorem"[38] shows that, for the SISO directed network (2) of size N with identical SISO node-systems (6) and (7), the pinning control problem is completely solved: For the network to be structurally controllable, the minimum number of external control inputs is $N_D = \max\{\ N - |E^*|,\ 1\ \}$, where $|E^*|$ is the number of edges in E^*. Specifically, if a network has a perfect matching, then $N_D = 1$ and the external control input can be pinned at any node of the network. Otherwise, $N_D = N - |E^*|$, which is the number of unmatched nodes in a maximum matching of the network, and in this case, the external control inputs should be pinned at the unmatched nodes therein.

In the general SISO setting, namely for a directed network of identical node systems with higher-dimensional states, described by (2) with $f(x) = Ax$ and $n > 1$, a necessary and sufficient condition was derived in [38]. To introduce this result, some more notations are needed. Let

$$U = \{\ i|\delta_i = 1,\ i = 1, 2, \cdots, N\}$$

and define

$$\Lambda(s) = \left\{ \left[\left[\alpha_1^{\rm T}, \cdots, \alpha_N^{\rm T}\right] \ \middle|\ \begin{array}{l} \alpha_i \in \Lambda_1(s) \text{ for } i \notin U \\ \alpha_i \in \Lambda_2(s) \text{ for } i \in U \end{array} \right] \right\}$$

where $s \in \sigma(A)$, which is the spectrum of matrix A, and

$$\Lambda_1(s) = \{\ \xi \in {\rm C}^{1\times n} | \xi(sI - A) = 0\ \}$$

$$\Lambda_2(s) = \{\ \xi \in \Lambda_1(s),\ \xi B = 0\ \}.$$

The following result was established in [39].

Theorem 1. The controlled network (2) with identical SISO node systems (A, B), with a state dimension $n \geq 1$, is structurally controllable if and only if

1) (A, H) is controllable;

2) (A, C) is observable;

3) $\beta L \neq 0$ for $\beta \in \Lambda(s), \beta \neq 0,\ \forall\ s \in \sigma(A)$;

4) ${\rm rank}(I - \gamma L, \eta\Delta) = N$, where $\gamma = C(sI - A)^{-1}H$, $\eta = C(sI - A)^{-1}B,\ \forall\ s \in \sigma(A)$.

3.2 MIMO networks

In retrospect, in the study of structural controllability of systems, the setting of SISO systems was extended to multi-input single-output (MISO) systems in [40]. To that end, there was a long period of silence, during which the subject of structural controllability was not investigated much more and further, until recently it was being revisited under a more general network framework, e.g., in [41, 42]. To prevent a network from becoming not state controllable when it is structurally controllable, the problem of strongly structural controllability was investigated in [43−45], where parts of the issues were put forward even to MIMO settings.

Now, consider a general dynamical network (2) with identical MIMO LTI node systems described by (3) and

$$y_i = Cx_i, \; i = 1, 2, \cdots, N \tag{8}$$

with $C \in \mathbf{R}^{m\times n}$, $x_i \in \mathbf{R}^n$, $i = 1, 2, \cdots, N$, $\quad 1 \le m \le n$.

When the directed network of MIMO node systems can be decomposed into leaders and followers, the network controllability was studied in [46], and by a few other papers subject to various constraints. In [47], a general linear network of leader-followers was studied, which is described by the tensor-product form of

$$\dot{X}(t) = [(D - L) \otimes H]X(t) + (\Delta \otimes B)U(t) \tag{9}$$

where $X(t)$ is the state vector composing of all node state vectors, $U(t)$ is the pinning control input vector composing of all control input vectors, Δ is a $0{-}1$ diagonal matrix determining how many and which nodes to pin, B is the control gain matrix, D is a diagonal constant matrix describing intrinsic dynamics, H is the inner coupling matrix, and L is the Laplacian matrix of the network.

The following result was established in [47].

Theorem 2. The directed leader-followers network (9) is state controllable if and only if

1) system (H, B) is controllable;

2) there exists no left-eigenvector of matrix $[L - D]$ with the first k entries being zeros, where k is the number of pinning-controlled nodes.

As consequences, Theorem 2 implies several specific results, such as the following:

1) A directed path is controllable if the beginning node is selected to be the only leader;

2) A directed cycle with a single leader is always controllable;

3) A complete digraph with a single leader is not controllable;

4) A star digraph (with more than 2 nodes) is not controllable if the center node is the only leader.

Now, consider a general MIMO formulation, namely network (2) with identical node systems described by (3) and (8). Early work along this line includes [48], where all node systems are subjected to external control inputs which, however, is not quite the same as the pinning control discussed herein.

Somewhat surprisingly, it is not straightforward to extend the analysis and results from SISO to the MIMO setting, as shown by the following two counterintuitive examples[49].

Example 1. Consider a directed network (2) of two identical LTI node systems with $x_i = [x_i^1 \; x_i^2]^{\mathrm{T}}$, $i = 1, 2$, and

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [0 \; 1],$$

$$\Lambda = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

as shown in Fig. 1 (a). It can be easily verified that the node system is both controllable and observable, but the network with $\Delta = \mathrm{diag}\{1, 0\}$ is not controllable.

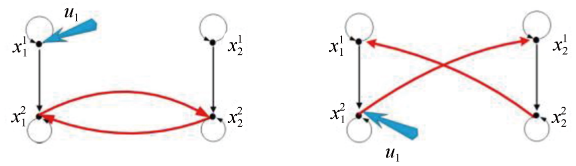
Example 2. Consider a directed network (2) of two identical LTI node systems with $x_i = [x_i^1 \; x_i^2]^{\mathrm{T}}$, $i = 1, 2$, and

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [0 \; 1],$$

$$\Lambda = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

as shown in Fig. 1 (b). Here, conversely, the node system is observable but not controllable, and yet the network with $\Delta = \mathrm{diag}\{1, 0\}$ is controllable.

These two examples demonstrate the complexity and imply the difficulty in studying the controllability of a general directed network of MIMO LTI node systems, even if all node systems are identical.



(a) Example 1 (b) Example 2

Fig. 1 Two counterintuitive examples

Nevertheless, a necessary and sufficient controllability condition was established in [49], for the directed network (2) of identical MIMO LTI node systems (3) and (8), which is now formulated by incorporating both the control and the observation equations together:

$$\dot{x}_i = Ax_i + \sum_{j=1}^{N} c_{ij}\alpha_{ij}HCx_j - \sum_{k=1}^{r} \delta_{ik}Bx_i, \; i = 1, 2, \cdots, N \tag{10}$$

where $1 \le r \le N$ and other notations are the same as before (e.g., $\Lambda = [\alpha_{ij}]_{N\times N}$, $\Delta = [\delta_{ik}]_{n\times r}$, etc.).

The following result was established in [49].

Theorem 3. The directed network (10) is state controllable if and only if the following system of two matrix equations has a unique zero solution $X = 0$:

$$\Delta^{\mathrm{T}} XB = 0 \ \text{ and } \ \Lambda^{\mathrm{T}} XHC = X(\lambda I - A) \quad \text{for all} \quad \lambda \in \mathbf{C}. \tag{11}$$

When Theorem 3 is specified to several typical types of digraphs, such as directed cycles and trees, including stars and chains, more precise conditions can be obtained[49]. Moreover, Theorem 1 above can also be derived from Theorem 3, but the former is easier to use (e.g., condition 3) are automatically satisfied for cycles and condition 4) holds automatically for chains)[39, 50].

3.3 Temporally switching networks

In the real world, most complex dynamical networks are evolving in time, such as mobile communication networks where contact patterns among individuals such as emailing and telephoning are typically temporal. Therefore, attention was attracted to the controllability and observability of time-varying networks recently.

The first attempt was to consider complex networks with switching topologies[51−53], but more realistically one should consider truly temporal networks[54, 55], which differs from the conventional switching networks in that when the network switches back to an earlier topology, it is not the same as before due to time causality. Regarding the notion of temporal structural controllability, some analysis was performed based on temporal characteristics[56]. The notion of strong structural controllability was further investigated[57]. The concept of control centrality was introduced to individual node systems in the form of a temporal tree[58]. Although temporal networks are by nature time-varying, while the controllability of linear time-varying (LTV) systems is not a new topic[59], due to the irreversibility of time, the deterministic chronological order in a temporally network distinguishes itself from general LTV systems[60].

It is important to note that a temporal network can have topological evolution in a continuous and even smooth fashion, but can also in a discontinuous and even switching manner, where the latter is obviously much more difficult to deal with.

To introduce the recent progress in the study of state and structural controllability of temporally switching networks, some new concepts and notations are in order.

A temporally switching network G is characterized by a set of nodes, $N(G) = \{1, 2, \cdots, n\}$, and a set of static connection topologies in a specified order, $G_1, G_2, \cdots, G_m$, where G_k (and G_m) exists only on the time interval $[t_{k-1}, t_k)$, $k = 1, 2, \cdots, m-1$ (and $[t_{m-1}, t_m]$), and each G_k has a coupling matrix $A(G_k) = [a_{ij}(k)]_{n\times n}$ with

$$a_{ij}(k) \begin{cases} \neq 0, & \text{if} \quad \mathrm{edge}(i, j, [t_{k-1}, t_k)) \neq \phi \\ = 0, & \text{otherwise} \end{cases}$$

denoting a directed edge from node j to node i on the time intervals $[t_{k-1}, t_k)$, $k = 1, 2, \cdots, m-1$ (and $[t_{m-1}, t_m]$).

In the graphic representation of a temporally switching network, as shown in Fig. 2, a temporal walk is a sequence of altering nodes and edges in a certain order and manner, e.g., $a_{21}(4)a_{13}(3)a_{32}(2)a_{21}(1)$ is a temporal walk $1 \to 2 \to 3 \to 1 \to 2$ on $[t_0, t_4)$, where $a_{21}(1)$ means that a walk starts from node 1 to node 2 on $[t_0, t_1)$, then walk to node 3 on $[t_1, t_2)$, which is described by $a_{32}(2)$, etc. Clearly, $a_{21}(4)$ is different from $a_{21}(1)$ in the above temporal walk because of the time stamps in $a_{21}(k)$ shown by $k = 1, 4$, respectively. Hence, a network node is temporally accessible if and only if there exists a walk starting from some external input node and ending at this node; otherwise, this node is non-accessible.

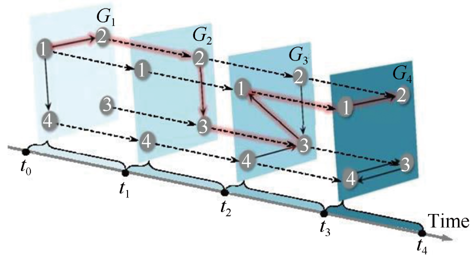


Fig. 2 Illustration of a temporally switching network

Connected nodes in a temporally switching network with fixed external inputs can be interpreted as an overall linear temporally switching system having the state matrix $A(t)$ with entrances being piecewise constant functions over the entire time interval $t \in [t_0, t_1) \cup [t_1, t_2) \cup \cdots \cup [t_{m-1}, t_m]$, where the fixed input matrix B shows how many and where the control-input nodes are located. Therefore, one can literally consider a linear temporally switching system in the general compact form of

$$\dot{x}(t) = A(t)x(t) + Bu(t), \quad x(t_0) = x_0 \tag{12}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^r$ is the control input vector, $B \in \mathbf{R}^{n\times r}$ is the constant input matrix, and the entries of the adjacency matrix $A(t) : \mathbf{R} \to \mathbf{R}^{n\times n}$ are piecewise constant functions of $t \in [t_0, \infty)$. The concept of (complete) state controllability for temporally switching systems is similar to the classical one. To precisely describe it, some more concepts are needed[60].

The linear temporally switching system (11) is (completely) state controllable on the time interval $[t_0, t_m]$ if, for any initial state $x_0 \in \mathbf{R}^n$ at $t_0 \geq 0$, there exists an input $u(t) \in \mathbf{R}^r$ defined on $[t_0, t_1]$ such that $x(t_m) = 0$, $t \in [t_0, t_1) \cup \cdots \cup [t_{m-1}, t_m]$, for some $t_m < \infty$.

Two linear temporally switching systems, $(A_1(t), B)$ and $(A_2(t), B)$, have the same structure if and only if they have the same number of static connection topologies, $G_1^1, G_2^1, \cdots, G_m^1$ and $G_1^2, G_2^2, \cdots, G_m^2$, and moreover they have the same (fixed) zero and (parametric) nonzero patterns in their corresponding adjacency matrices $A_1^1, A_2^1, \cdots, A_m^1$ and $A_1^2, A_2^2, \cdots, A_m^2$.

A temporally switching network G with fixed external inputs is structurally controllable if and only if there exists a state controllable linear temporally switching system (11) with the same structure as $(A(t), B)$, namely, if and only if for each admissible realization of the independent nonzero parameters on the time intervals $[t_{k-1}, t_k)$, $i = 1, 2, \cdots, m-1$, and $[t_{m-1}, t_m]$, the corresponding system (A_k, B) is state controllable, $k = 1, 2, \cdots, m-1$.

The following result was established in [60].

Theorem 4. The linear temporally switching system (11) is structurally controllable on the time interval $[t_0, t_m]$ if and only if the network Gramian matrix

$$\Big[\mathrm{e}^{(t_m - t_{m-1})A_m} \cdots \mathrm{e}^{(t_2 - t_1)A_2} W_1, \cdots, \\ \mathrm{e}^{(t_m - t_{m-1})A_m} W_{m-1}, W_m\Big] \tag{13}$$

has a full row-rank, where $W_i = \left[B, A_i B, \cdots, A_i^{n-1} B\right]$ is the state controllability matrix (4) of the system on the i-th time interval $[t_i, t_{i+1})$, $i = 1, 2, \cdots, m-1$, and $[t_{m-1}, t_m]$.

In [60], a sufficient condition on the strong structural controllability of the linear temporally switching system (12) was also established based on a new concept of n-walk, which is a generalization of the concept of cactus from classical graph theory to temporal networks.

4 Research outlook

Controllability is a fundamental issue to be addressed in network science and engineering before considering how to control a network of dynamical systems in applications.

Due to the well-known duality between controllability and observability, theoretically one can convert all results on controllability to observability, but the latter has some particular features and properties that have found specific applications related to network estimation, identification and prediction, therefore is still worth investigating[61, 62].

This article mainly discusses some basis of pinning control and controllability of complex dynamical networks with identical LTI node systems and fixed topologies, although the controllability of temporally switching networks has also been discussed, which is still not in a general setting with time-varying topologies. Networks with time-varying node systems or time-varying topologies deserve more attention and further investigation[63]. In addition, complex dynamical networks of non-identical node systems post a great challenge, not to mention settings with nonlinear node systems such as bilinear systems[64]. Last but not least, since both topology and dynamics contribute to control performances of complex dynamical networks[65], their integrated effects on pinning control, controllability as well as observability are calling for further efforts to study.

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References

[1] D. J. Watts, S. H. Strogatz. Collective dynamics of ‘small-world’ networks. *Nature*, vol. 393, no. 6684, pp. 440–442, 1998.

[2] A. L. Barabási, R. Albert. Emergence of scaling in random networks. *Science*, vol. 286, no. 5439, pp. 509–512, 1999.

[3] P. Erdős, A. Rényi. On the evolution of random graphs. *Publication of the Mathematical Institute of the Hungarian Academy Sciences*, vol. 5, pp. 17–60, 1960.

[4] C. K. Chui, G. R. Chen. *Linear Systems and Optimal Control*, New York, USA: Springer-Verlag, 1989.

[5] G. R. Chen, Z. S. Duan. Network synchronizability analysis: A graph-theoretic approach. *Chaos*, vol. 18, Article number 037102, 2008.

[6] X. F. Wang, G. R. Chen. Pinning control of scale-free dynamical networks. *Physica A: Statistical Mechanics and its Applications*, vol. 310, no. 3–4, pp. 521–531, 2002.

[7] X. Li, X. F. Wang, G. R. Chen. Pinning a complex dynamical network to its equilibrium. *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 51, no. 10, pp. 2074–2087, 2004.

[8] G. R. Chen, X. F. Wang, X. Li. *Introduction to Complex Networks: Models, Structures and Dynamics*, 2nd ed., Beijing, China: Higher Education Press.

[9] A. Cho. Scientific link-up yields ‘control panel’ for networks. *Science*, vol. 332, no. 6031, pp. 777, 2011.

[10] I. D. Couzin, J. Krause, N. R. Franks, S. A. Levin. Effective leadership and decision-making in animal groups on the move. *Nature*, vol. 433, no. 7025, pp. 513–516, 2005.

[11] G. R. Chen. Pinning control and synchronization on complex dynamical networks. *International Journal of Control, Automation and Systems*, vol. 12, no. 2, pp. 221–230, 2014.

[12] X. F. Wang, H. S. Su. Pinning control of complex networked systems: A decade after and beyond. *Annual Reviews in Control*, vol. 38, no. 1, pp. 103–111, 2014.

[13] F. F. Li. Pinning control design for the stabilization of Boolean networks. *IEEE Transactions on Neural Networks and Learning Systems*, vol. 27, no. 7, pp. 1585–1590, 2016.

[14] Y. Tang, H. J. Gao, J. Kurths, J. A. Fang. Evolutionary pinning control and its application in UAV coordination. *IEEE Transactions on Industrial Informatics*, vol. 8, no. 4, pp. 828–838, 2012.

[15] C. T. Lin. Structural controllability. *IEEE Transactions on Automatic Control*, vol. 19, no. 3, pp. 201–208, 1974.

[16] J. L. Willems. Structural controllability and observability. *Systems & Control Letters*, vol. 8, no. 1, pp. 5–12, 1986.

[17] H. Mayeda, T. Yamada. Strong structural controllability. *SIAM Journal on Control and Optimization*, vol. 17, no. 1, pp. 123–138, 1979.

[18] Y. Y. Liu, A. L. Barabási. Control principles of complex networks, [Online], Available: https://arxiv.org/abs/1508.05384, 2015.

[19] G. Yan, J. Ren, Y. C. Lai, C. H. Lai, B. W. Li. Controlling complex networks: How much energy is needed?. *Physical Review Letters*, vol. 108, no. 21, Article number 218703, 2012.

[20] W. X. Wang, X. Ni, Y. C. Lai, C. Grebogi. Optimizing controllability of complex networks by minimum structural perturbations. *Physical Review E*, vol. 85, no. 2, Article number 026115, 2012.

[21] T. Nepusz, T. Vicsek. Controlling edge dynamics in complex networks. *Nature Physics*, vol. 8, no. 7, pp. 568–573, 2012.

[22] T. Jia, Y. Y. Liu, E. Csóka, M. Pósfai, J. J. Slotine, A. L. Barabási. Emergence of bimodality in controlling complex networks. *Nature Communications*, vol. 4, Article number 2002, 2013.

[23] T. Jia, A. L. Barabási. Control capacity and a random sampling method in exploring controllability of complex networks. *Scientific Reports*, vol. 3, Article number 2354, 2013.

[24] Z. Z. Yuan, C. Zhao, Z. R. Di, W. X. Wang, Y. C. Lai. Exact controllability of complex networks. *Nature Communications*, vol. 4, Article number 2447, 2013.

[25] G. Menichetti, L. Dall′Asta, G. Bianconi. Network controllability is determined by the density of low in-degree and out-degree nodes. *Physical Review Letters*, vol. 113, no. 7, Article number 078701, 2014.

[26] J. X. Gao, Y. Y. Liu, R. M. D′Souza, A. L. Barabási. Target control of complex networks. *Nature Communications*, vol. 5, Article number 5415, 2014.

[27] A. E. Motter. Networkcontrology. *Chaos*, vol. 25, no. 9, Article number 097621, 2015.

[28] G. Yan, G. Tsekenis, B. Barzel, J. J. Slotine, Y. Y. Liu, A. L. Barabási. Spectrum of controlling and observing complex networks. *Nature Physics*, vol. 11, no. 9, pp. 779–786, 2015.

[29] A. J. Gates, L. M. Rocha. Control of complex networks requires both structure and dynamics. *Scientific Reports*, vol. 6, Article number 24456, 2016.

[30] T. H. Summers, F. L. Cortesi and J. Lygeros. On submodularity and controllability in complex dynamical networks. *IEEE Transactions on Control of Network Systems*, vol. 3, no. 1, pp. 91–101, 2016.

[31] B. Das, B. Subudhi, B. B. Pati. Cooperative formation control of autonomous underwater vehicles: An overview. *International Journal of Automation and Computing*, vol. 13, no. 3, pp. 199–225, 2016.

[32] F. Sorrentino, M. di Bernardo, F. Garofalo, G. R. Chen. Controllability of complex networks via pinning. *Physical Review E*, vol. 75, no. 4, Article number 046103, 2007.

[33] L. M. Pecora, T. L. Carroll. Master stability functions for synchronized coupled systems. *Physical Review Letters*, vol. 80, no. 10, pp. 2109–2112, 1998.

[34] M. Porfiri, M. di Bernardo. Criteria for global pinning-controllability of complex networks. *Automatica*, vol. 44, no. 12, pp. 3100–3106, 2008.

[35] Y. L. Zou, G. R. Chen. Pinning controllability of asymmetrical weighted scale-free networks. *Europhysics Letters*, vol. 84, no. 5, Article number 58005, 2008.

[36] L. Y. Xiang, F. Chen, G. R. Chen. Pinning synchronization of networked multi-agent systems: Spectral analysis. *Control Theory and Technology*, vol. 13, no. 1, pp. 45–54, 2015.

[37] L. Lováz, M. D. Plummer. *Matching Theory*, New York: Elsevier, 1986.

[38] Y. Y. Liu, J. J. Slotine, A. L. Barabási. Controllability of complex networks. *Nature*, vol. 473, no. 7346, pp. 167–173, 2011.

[39] L. Wang, X. F. Wang, G. Chen. Controllability of networked higher-dimensional systems with one-dimensional communication channels. *Philosophical Transactions of the Royal Society A*, to be published.

[40] R. Shields, J. Pearson. Structural controllability of multiinput linear systems. *IEEE Transactions on Automatic Control*, vol. 21, no. 2, pp. 203–212, 1976.

[41] J. M. Dion, C. Commaulta, J. van der Woude. Generic properties and control of linear structured systems: A survey. *Automatica*, vol. 39, no. 7, pp. 1125–1144, 2003.

[42] A. Lombardi, M. Hornquist. Controllability analysis of networks. *Physical Review E*, vol. 75, no. 5, Article number 056110, 2007.

[43] C. T. Lin. System structure and minimal structure controllability. *IEEE Transactions on Automatic Control*, vol. 22, no. 5, pp. 855–862, 1977.

[44] J. C. Jarczyk, F. Svaricek, B. Alt. Strong structural controllability of linear systems revisited. In *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference*, IEEE, Orlando, USA, pp. 1213–1218, 2011.

[45] A. Chapman. Strong structural controllability of networked dynamics. *Semi-Autonomous Networks*, A. Chapman, Ed., New York: Springer, pp. 135–150, 2015.

[46] H. G. Tanner. On the controllability of nearest neighbor interconnections. In *Proceedings of the 43rd IEEE Conference on Decision and Control*, IEEE, Nassau, Bahamas, 2004, vol. 3, pp. 2467–2472.

[47] L. Y. Xiang, J. J. H. Zhu, F. Chen, G. R. Chen. Controllability of weighted and directed networks with nonidentical node dynamics. *Mathematical Problems in Engineering*, vol. 2013, Article number 405034, 2013.

[48] T. Zhou. On the controllability and observability of networked dynamic systems. *Automatica*, vol. 52, pp. 63–75, 2015.

[49] L. Wang, G. R. Chen, X. F. Wang, W. K. S. Tang. Controllability of networked MIMO systems. *Automatica*, vol. 69, pp. 405–409, 2016.

[50] L. Wang, G. R. Chen, X. F. Wang, W. K. S. Tang. Controllability of networked MIMO systems, [Online], Available: https://arxiv.org/abs/1505.01255v3, 2015.

[51] B. Liu, T. G. Chu, L. Wang, G. M. Xie. Controllability of a leader-follower dynamic network with switching topology. *IEEE Transactions on Automatic Control*, vol. 53, no. 4, pp. 1009–1013, 2008.

[52] X. M. Liu, H. Lin, B. M. Chen. Graph-theoretic characterisations of structural controllability for multi-agent system with switching topology. *International Journal of Control*, vol. 86, no. 2, pp. 222–231, 2013.

[53] X. M. Liu, H. Lin, B. M. Chen. Structural controllability of switched linear systems. *Automatica*, vol. 49, no. 12, pp. 3531–3537, 2013.

[54] P. Holme, J. Saramäki. Temporal networks. *Physics Reports*, vol. 519, no. 3, pp. 97–125, 2012.

[55] X. Li, P. Yao, Y. J. Pan. Towards structural controllability of temporal complex networks. In *Complex Systems and Networks: Dynamics, Controls and Applications*, J. H. Lü, X. H. Yu, G. R. Chen, W. W. Yu, Eds., Berlin Heidelberg: Springer, pp. 341–371, 2015.

[56] M. Pósfai, P. Hövel. Structural controllability of temporal networks. *New Journal of Physics*, vol. 16, no. 12, Article number 123055, 2014.

[57] G. Reissig, C. Hartung, F. Svaricek. Strong structural controllability and observability of linear time-varying systems. *IEEE Transactions on Automatic Control*, vol. 59, no. 11, pp. 3087–3092, 2014.

[58] Y.J. Pan, X. Li. Structural controllability and controlling centrality of temporal networks. *PLoS One*, vol. 9, no. 4, Article number 0094998, 2014.

[59] L. M. Silverman, H. E. Meadows. Controllability and observability in time-variable linear systems. *SIAM Journal on Control*, vol. 5, no. 1, pp. 64–73, 1967.

[60] B. Y. Hou, X. Li, G. R. Chen. Structural controllability of temporally switching networks. *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 63, no. 10, pp. 1771–1781, 2016.

[61] Y. Y. Liu, J. J. Slotine, A. L. Barabási. Observability of complex systems. *Proceedings of the National Academy of Sciences of the United States of America*, vol. 110, no. 7, pp. 2460–2465, 2013.

[62] B. B. Wang, L. Gao, Y. Gao, Y. Deng, Y. Wang. Controllability and observability analysis for vertex domination centrality in directed networks. *Scientific Reports*, vol. 4, Article number 5399, 2014.

[63] A. M. Li, S. P. Cornelius, Y. Y. Liu, L. Wang, A. L. Barabási. The fundamental advantages of temporal networks, [Online], Available: https://arxiv.org/abs/1607.06168, 2016.

[64] S. Ghosh, J. Ruths. Structural control of single-input rank one bilinear systems. *Automatica*, vol. 64, pp. 8–17, 2016.

[65] A. J. Gates, L. M. Rocha. Control of complex networks requires both structure and dynamics. *Scientific Reports*, vol. 6, Article number 24456, 2016.

Guanrong Chen is a chair professor and the director of the Centre for Chaos and Complex Networks at the City University of Hong Kong, China. He was elected a Fellow of the IEEE in 1997, awarded the 2011 Euler Gold Medal from Russia, and conferred Honorary Doctor Degrees by the Saint Petersburg State University, Russia in 2011 and by the University of Normandy, France in 2014. He is a Highly Cited Researcher in Engineering (since 2009), in Physics (2014) and also in Mathematics (2015) according to Thomson Reuters. He is a member of the Academy of Europe (2014) and a Fellow of The World Academy of Sciences (2015).

His research interests include complex systems and networks with regard to their modeling, dynamics and control.

E-mail: eegchen@cityu.edu.hk (Corresponding author)

ORCID iD: 0000-0003-1381-7418