STABILITY OF NONLINEAR SYSTEMS

Guanrong Chen

City University of Hong Kong, Kowloon, Hong Kong SAR, P. R. China

1 INTRODUCTION

A *nonlinear system* refers to a set of nonlinear equations, which can be algebraic, difference, differential, integral, functional, or abstract operator equations, or a combination of some of these. A nonlinear system is used to describe a physical device or process that otherwise cannot be well defined and characterized by a set of linear equations of any kind. *Dynamical system* is used as a synonym for mathematical or physical system when the describing equation represents evolution of a solution with time, which sometimes has control inputs and variable parameters as well.

The theory of nonlinear dynamical systems, or *nonlinear control systems* if control inputs are involved, has been greatly advanced since the nineteenth century. Today, nonlinear control systems are used to describe a great variety of scientific and engineering phenomena ranging from natural, social, and physical sciences to engineering and technology. This theory has been applied to a broad spectrum of problems in physics, chemistry, mathematics, biology, medicine, economics, and various engineering disciplines.

Stability theory plays a central role in system engineering, especially in the field of control systems and automation, with regard to both dynamics and control.

Stability of a dynamical system, with or without control and disturbance inputs, is a fundamental requirement for its practical use, particularly in real-world applications. Roughly speaking, stability means that the system outputs and its internal signals are bounded within admissible limits (the so-called bounded-input bounded-output stability) or, sometimes more strictly, the system outputs tend to an equilibrium state of interest (the so-called asymptotic stability). Conceptually, there are different kinds of stabilities, among which three basic notions are the main concerns in nonlinear dynamics and control systems: the stability of a system with respect to its equilibria, the orbital stability of a system output trajectory, and the structural stability of a system itself.

The basic concept of stability emerged from the study of an equilibrium state of a mechanical system, dated back to as early as 1644 when E. Torricelli studied the equilibrium of a rigid body under the natural force of gravity. The classical stability theorem of G. Lagrange, formulated in 1788, is likely the most well-known result about the stability of conservative mechanical systems, which states that if the potential energy of a conservative system, currently at the position of an isolated equilibrium and perhaps subject to some simple constraints, has a minimum, then this equilibrium position of the system is stable [1]. The evolution of the fundamental concepts of system and trajectory stabilities then went through a long history, with many fruitful advances and developments,

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till the celebrated Ph.D. Thesis of A. M. Lyapunov, *The General Problem of Motion Stability*, finished in 1892 [2]. This monograph is so fundamental that its ideas and techniques are virtually leading all basic research and applications regarding stabilities of dynamical systems today. In fact, not only dynamical behavior analysis in modern physics but also controllers design in engineering systems depend on the principles of Lyapunov's stability theory. This chapter is devoted to a brief description of the basic stability theory, criteria, and methodologies of Lyapunov, as well as a few related important stability concepts, for nonlinear dynamical systems.

2 NONLINEAR SYSTEM PRELIMINARIES

2.1 Nonlinear Control Systems

A continuous-time nonlinear control system is generally described by a differential equation of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t; \mathbf{u}), \quad t \in [t_0, \infty) \tag{1}$$

where $\mathbf{x} = \mathbf{x}(t)$ is the *state* of the system belonging to a (usually bounded) region $\Omega_{\mathbf{x}} \subset \mathbf{R}^n$, $\mathbf{u}$ is the *control* input vector belonging to another (usually bounded) region $\Omega_{\mathbf{u}} \subset \mathbf{R}^m$ (oftentimes, $m \le n$), and $\mathbf{f}$ is a Lipschitz or continuously differentiable nonlinear function, so that the system has a unique solution for each admissible control input with a suitable initial condition $\mathbf{x}(t_0) = \mathbf{x}_0 \in \Omega_{\mathbf{x}}$. To indicate the time evolution and the dependence on the initial state $\mathbf{x}_0$, the *trajectory* (or *orbit*) of a system state, $\mathbf{x}(t)$, is sometimes denoted as $\varphi_t(\mathbf{x}_0)$.

In the control system (1), the initial time used is $t_0 \ge 0$, unless otherwise indicated. The entire space $\mathbf{R}^n$, to which the system states belong, is called the *state space*. Associated with the control system (1), there usually is an *observation* or *measurement* equation

$$\mathbf{y} = \mathbf{g}(\mathbf{x}, t; \mathbf{u}) \tag{2}$$

where $\mathbf{y} = \mathbf{y}(t) \in \mathbf{R}^{\ell}$ is the *output* of the system, $1 \le \ell \le n$, and $\mathbf{g}$ is a continuous or smooth nonlinear function. When both $n, \ell > 1$, the system is called a multi-input multi-output (MIMO) system; whereas if $n = \ell = 1$, it is called a single-input single-output (SISO) system. MISO and SIMO systems are similarly defined.

In the discrete-time setting, a nonlinear control system is described by a difference equation of the form

$$\begin{cases} \mathbf{x}_{k+1} = \mathbf{f}\left(\mathbf{x}_k, k; \mathbf{u}_k\right) \\ \mathbf{y}_k = \mathbf{g}\left(\mathbf{x}_k, k; \mathbf{u}_k\right), \end{cases} \quad k = 0, 1, \dots \tag{3}$$

where all notations are similarly defined.

Most of the time in this chapter, only the control system (1), or the first equation of (3), is discussed. In this case, the system state $\mathbf{x}$ is also considered as the system output for simplicity.

A special case of system (1), with or without control, is said to be *autonomous* if the time variable, t, does not appear separately (independently) from the state vector in the system function $\mathbf{f}$. For example, for a system function $\mathbf{f}(\mathbf{x}; \mathbf{u})$ with a state-feedback control $\mathbf{u}(t) = \mathbf{h}(\mathbf{x}(t))$, this is an autonomous system. In this case, the system is often written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{R}^n \tag{4}$$

Otherwise, as system (1) stands, it is said to be *nonautonomous*. The same terminology may be applied in the same way to discrete-time systems, although they have different characteristics.

An *equilibrium*, or *fixed point*, of system (4), if it exists, is a solution $\mathbf{x}^*$ of the algebraic equation

$$\mathbf{f}(\mathbf{x}^*) = 0 \tag{5}$$

It then follows from (4) and (5) that $\dot{\mathbf{x}}^* = 0$, which means that an equilibrium of a system must be a constant state. For the discrete-time case, an equilibrium of system

$$\mathbf{x}_{k+1} = \mathbf{f}\left(\mathbf{x}_k\right), \quad k = 0, 1, \ldots \tag{6}$$

is a solution, if it exists, of the equation

$$\mathbf{x}^* = \mathbf{f}(\mathbf{x}^*) \tag{7}$$

An equilibrium is *stable*, if all the nearby trajectories of the system states, starting from various initial states, approach it; it is *unstable*, if some or most nearby trajectories move away from it. The concept of system stability with respect to an equilibrium will be precisely introduced in Section 3.

A control system is *deterministic*, if there is a unique consequence to every change of the system parameters or initial states. It is *random* or *stochastic*, if there is more than one possible consequence for a change in its parameters or initial states according to some probability distribution [3]. This chapter only deals with deterministic systems.

2.2 Hyperbolic Equilibria and Their Manifolds

Consider the autonomous system (4). The Jacobian of this system is defined by

$$J(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \tag{8}$$

Clearly, this is a matrix-valued function of time. If the Jacobian is evaluated at a constant state, say an equilibrium $\mathbf{x}^*$ or initial state $\mathbf{x}_0$, then it becomes a constant matrix determined by $\mathbf{f}$ and $\mathbf{x}^*$ or $\mathbf{x}_0$.

An equilibrium, $\mathbf{x}^*$, of system (4) is said to be *hyperbolic*, if all eigenvalues of the system Jacobian, evaluated at this equilibrium, have nonzero real parts.

For a p-periodic solution of system (4), $\tilde{\mathbf{x}}(t)$, with a fundamental period $p > 0$, let $J(\tilde{\mathbf{x}}(t))$ be its Jacobian evaluated at $\tilde{\mathbf{x}}(t)$. Then, this Jacobian is also p-periodic:

$$J(\tilde{\mathbf{x}}(t+p)) = J(\tilde{\mathbf{x}}(t)) \quad \text{for all} \quad t \in [t_0, \infty)$$

In this case, there always exist a p-periodic nonsingular matrix, $M(t)$, and a constant matrix, Q, such that the fundamental solution matrix associated with the Jacobian $J(\tilde{\mathbf{x}}(t))$ is given by [4]

$$\Phi(t) = M(t)\, e^{tQ}$$

Here, the fundamental matrix $\Phi(t)$ consists of, as its columns, n linearly independent solution vectors of the linear equation $\dot{\mathbf{x}} = J(\tilde{\mathbf{x}}(t))\,\mathbf{x}$, with $\mathbf{x}(t_0) = \mathbf{x}_0$.

In the above, the eigenvalues of the constant matrix e^{pQ} are called the *Floquet multipliers* of the Jacobian. The p-periodic solution $\tilde{\mathbf{x}}(t)$ is called a *hyperbolic periodic orbit* of the system if all its corresponding Floquet multipliers have nonzero real parts.

Next, let $\mathcal{D}$ be a neighborhood of an equilibrium, $\mathbf{x}^*$, of the autonomous system (4). A *local stable* and *local unstable manifold* of $\mathbf{x}^*$ is defined, respectively, by

$$W^{\mathrm{s}}_{\mathrm{loc}}(\mathbf{x}^*) = \left\{ \mathbf{x} \in \mathcal{D} \,|\, \varphi_t(\mathbf{x}) \in \mathcal{D}\ \forall\, t \geq t_0 \text{ and } \varphi_t(\mathbf{x}) \to \mathbf{x}^* \text{ as}\, t \to \infty \right\} \tag{9}$$

and

$$W^{\mathrm{u}}_{\mathrm{loc}}(\mathbf{x}^*) = \left\{ \mathbf{x} \in \mathcal{D} \,|\, \varphi_t(\mathbf{x}) \in \mathcal{D}\ \forall\, t \leq t_0 \text{ and } \varphi_t(\mathbf{x}) \to \mathbf{x}^* \text{ as}\, t \to -\infty \right\} \tag{10}$$

Furthermore, a *stable manifold* and an *unstable manifold* of $\mathbf{x}^*$ are defined, respectively, by

$$W^{\mathrm{s}}(\mathbf{x}^*) = \left\{ \mathbf{x} \in \mathcal{D} \,|\, \varphi_t(\mathbf{x}) \cap W^{\mathrm{s}}_{\mathrm{loc}}(\mathbf{x}^*) \neq \phi \right\} \tag{11}$$

and

$$W^{\mathrm{u}}(\mathbf{x}^*) = \left\{ \mathbf{x} \in \mathcal{D} \,|\, \varphi_t(\mathbf{x}) \cap W^{\mathrm{u}}_{\mathrm{loc}}(\mathbf{x}^*) \neq \phi \right\} \tag{12}$$

where ϕ denotes the empty set.

For example, the autonomous system

$$\begin{cases} \dot{x} = y \\ \dot{y} = x\left(1 - x^2\right) \end{cases}$$

has a hyperbolic equilibrium $(x^*, y^*) = (0, 0)$. The local stable and unstable manifolds of this equilibrium are illustrated in Fig. 1a, and the corresponding stable and unstable manifolds are visualized in Fig. 1b.

A hyperbolic equilibrium only has stable and/or unstable manifolds since its associated Jacobian only has stable and/or unstable eigenvalues. The dynamics of an autonomous system in a neighborhood of a hyperbolic equilibrium is quite simple: it has either stable (convergent) or unstable (divergent) properties. Therefore, complex dynamical behaviors such as chaos are usually not associated with isolated hyperbolic equilibria or isolated hyperbolic periodic orbits [5–7] (also, see Theorem 16); they generally are confined within the so-called *center manifold*, $W^{\mathrm{c}}(\mathbf{x}^*)$, where $\dim(W^{\mathrm{s}}) + \dim(W^{\mathrm{c}}) + \dim(W^{\mathrm{u}}) = n$.

2.3 Open-Loop and Closed-Loop Systems

Let S be an MIMO system, which can be linear or nonlinear, continuous-time or discrete-time, deterministic or stochastic, or any well-defined input–output map. Let U and Y be the sets (sometimes, spaces) of the admissible input and corresponding output signals, respectively, both defined on the time domain $\mathcal{D} = [a, b]$, $-\infty \leq a < b \leq \infty$ (for control systems, usually $a = t_0 = 0$ and $b = \infty$). This simple relation is described by an open-loop map:

$$S : \mathbf{u} \to \mathbf{y} \quad \text{or} \quad \mathbf{y}(t) = S(\mathbf{u}(t)) \tag{13}$$

and its block diagram is shown in Fig. 2. Actually, every control system described by a differential or difference equation can be viewed as a map in this form. But, in such a situation, the map S can only be implicitly defined via the equation and the initial condition.

In the control system (1), or (3), if the control inputs are functions of the state vectors, $\mathbf{u} = \mathbf{h}(\mathbf{x}; t)$, then the control system can be implemented via a closed-loop configuration. A typical closed-loop system is shown in Fig. 3, where usually S_1 is the plant (described by $\mathbf{f}$) and S_2 is the controller (described by $\mathbf{h}$); of course, they can be reversed.

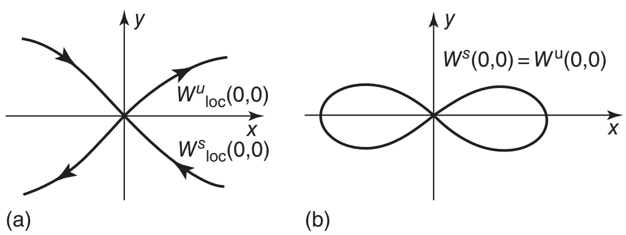

Figure 1 Stable and unstable manifolds.

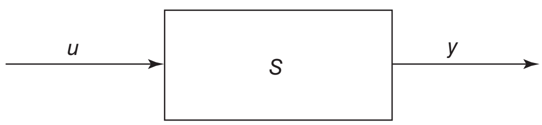

Figure 2 The block diagram of an open-loop system.

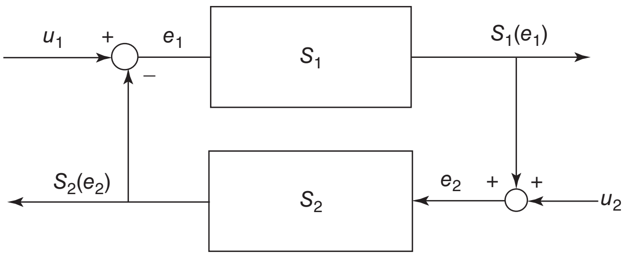

Figure 3 A typical closed-loop control systems.

2.4 Norms of Functions and Operators

This chapter only deals with finite-dimensional systems. For an n-dimensional vector-valued function, $\mathbf{x}(t) = [x_1(t) \cdots x_n(t)]^\top$, let $\|\cdot\|$ and $\|\cdot\|_p$ denote its *Euclidean norm* and L_p*-norm*, defined, respectively, by the "length"

$$\|\mathbf{x}(t)\| = \sqrt{x_1^2(t) + \cdots + x_n^2(t)}$$

and by

$$\|\mathbf{x}\|_p = \left(\int_a^b \|\mathbf{x}(t)\|^p \, dt \right)^{1/p}, \quad 1 \le p < \infty$$

$$\|\mathbf{x}\|_\infty = \operatorname{ess} \sup_{\substack{a \le t \le b \\ 1 \le i \le n}} |x_i(t)|$$

Here, a few remarks are in order:

(i) "ess sup" means "essential suprimum" (i.e., the suprimum except perhaps over a set of measure zero). For a piecewise continuous function, $f(t)$, they are actually the same:

$$\operatorname{ess} \sup_{a \le t \le b} |f(t)| = \sup_{a \le t \le b} |f(t)|$$

(ii) The main difference between the "sup" and the "max" is that max $|f(t)|$ is attainable but sup $|f(t)|$ may not. For example, $\max_{0 \le t < \infty} |\sin(t)| = 1$ and $\sup_{0 \le t < \infty} |1 - e^{-t}| = 1$.

(iii) The essential difference between the Euclidean norm and the L_p-norms is that the former is a function of time but the latter are all constants.

(iv) For a finite-dimensional vector $\mathbf{x}(t)$, with $n < \infty$, all the L_p-norms are equivalent in the sense that for any $p, q \in [1, \infty]$, there exist two positive constants α and β such that

$$\alpha \|\mathbf{x}\|_p \le \|\mathbf{x}\|_q \le \beta \|\mathbf{x}\|_p$$

For the input–output map (13), the so-called operator norm of the map S is defined to be the maximum gain from all possible inputs over the domain of the map to their corresponding outputs. More precisely, the *operator norm* of the map S in (13) is defined by

$$|||S||| = \sup_{\substack{\mathbf{u}_1, \mathbf{u}_2 \in U \\ \mathbf{u}_1 \neq \mathbf{u}_2}} \frac{\|\mathbf{y}_1 - \mathbf{y}_2\|_Y}{\|\mathbf{u}_1 - \mathbf{u}_2\|_U} \tag{14}$$

where $\mathbf{y}_i = S(\mathbf{u}_i) \in Y$, $i = 1, 2$, and $\|\cdot\|_U$ and $\|\cdot\|_Y$ are the norms of the functions defined on the input-output sets (or spaces) U and Y, respectively.

3 LYAPUNOV, ORBITAL, AND STRUCTURAL STABILITIES

Three different types of stabilities, namely, the Lyapunov stability of a system with respect to its equilibria, the orbital stability of a system output trajectory, and the structural stability of a system itself, are of fundamental importance in the studies of nonlinear dynamics and control systems.

Roughly speaking, the Lyapunov stability of a system with respect to its equilibrium of interest is about the behavior of the system outputs toward the equilibrium state—wandering nearby and around the equilibrium (stability in the sense of Lyapunov) or gradually approaches it (asymptotic stability); the orbital stability of a system output is the resistance of the trajectory to small perturbations to the trajectory; the structural stability of a system is the resistance of the system structure against small perturbations to the system structure [1, 8–17]. These three basic types of stabilities are introduced in this section, for dynamical systems without explicitly involving control inputs.

Consider the general nonautonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{R}^n \tag{15}$$

where the control input $\mathbf{u}(t) = \mathbf{h}(\mathbf{x}(t), t)$, if exists (see system (1)), has been combined into the system function $\mathbf{f}$ for simplicity of discussion. Without loss of generality, assume that the origin $\mathbf{x} = 0$ is the system equilibrium of interest. Lyapunov stability theory concerns various stabilities of the system orbits with respect to this equilibrium. When another equilibrium is discussed, the new equilibrium is first shifted to zero by a change of variables, and then the transformed system is studied in the same way.

3.1 Stability in the Sense of Lyapunov

System (15) is said to be *stable in the sense of Lyapunov* with respect to the equilibrium $\mathbf{x}^* = 0$ if, for any $\varepsilon > 0$ and any initial time $t_0 \geq 0$, there exists a constant, $\delta = \delta(\varepsilon, t_0) > 0$, such that

$$\|\mathbf{x}(t_0)\| < \delta \implies \|\mathbf{x}(t)\| < \varepsilon \quad \text{for all } t \geq t_0 \tag{16}$$

This stability is illustrated in Fig. 4.

It should be emphasized that the constant δ generally depends on both ε and t_0. It is particularly important to point out that, unlike autonomous systems, one cannot simply fix the initial time $t_0 = 0$ for a nonautonomous system in a general discussion of its stability. For example, consider the following linear time-varying system with a discontinuous coefficient:

$$\dot{x}(t) = \frac{1}{1-t}\, x(t), \quad x(t_0) = x_0$$

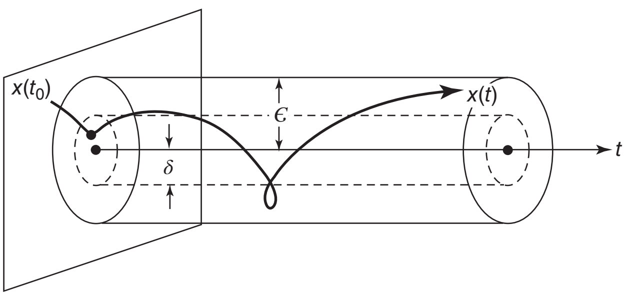

Figure 4 Geometric meaning of stability in the sense of Lyapunov.

It has an explicit solution

$$x(t) = x_0 \frac{1 - t_0}{1 - t}, \quad 0 \le t_0 \le t < \infty$$

which is stable in the sense of Lyapunov about the equilibrium $x^* = 0$ over the entire time domain $[0, \infty)$ if and only if $t_0 = 1$. This shows that the initial time, t_0, does play an important role in the stability of a nonautonomous system.

The above-defined stability, in the sense of Lyapunov, is said to be *uniform* with respect to the initial time, if the existing constant $\delta = \delta(\varepsilon)$ is indeed independent of t_0 over the entire time interval $[0, \infty)$. According to the above discussion, uniform stability is defined only for nonautonomous systems; it is not needed for autonomous systems because, for such systems, their stability is always uniform with respect to the initial time.

3.2 Asymptotic and Exponential Stabilities

System (15) is said to be *asymptotically stable* about its equilibrium $\mathbf{x}^* = 0$, if it is stable in the sense of Lyapunov and, furthermore, there exists a constant $\delta = \delta(t_0) > 0$ such that

$$\|\mathbf{x}(t_0)\| < \delta \implies \|\mathbf{x}(t)\| \to 0 \quad \text{as } t \to \infty \tag{17}$$

This stability is visualized in Fig. 5.

The asymptotic stability is said to be *uniform*, if the existing constant δ is independent of t_0 over $[0, \infty)$, and is said to be *global*, if the convergence, $\|\mathbf{x}\| \to 0$, is independent of the initial state $\mathbf{x}(t_0)$ over the entire spatial domain on which the system is defined (e.g., when $\delta = \infty$). If, furthermore,

$$\|\mathbf{x}(t_0)\| < \delta \implies \|\mathbf{x}(t)\| \le ce^{-\sigma t} \tag{18}$$

for two positive constants c and σ, then the equilibrium is said to be *exponentially stable*. The exponential stability is visualized in Fig. 6.

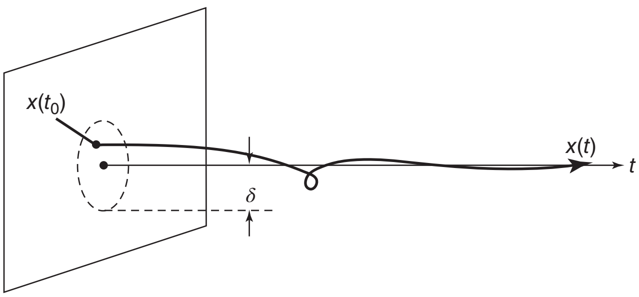

Figure 5 Geometric meaning of the asymptotic stability.

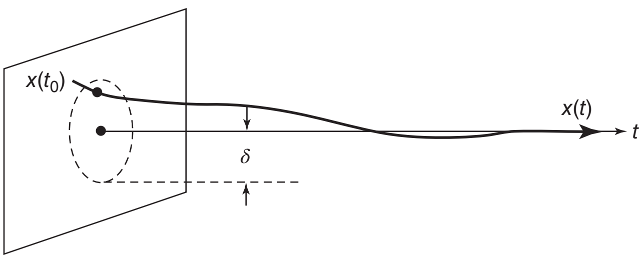

Figure 6 Geometric meaning of the exponential stability.

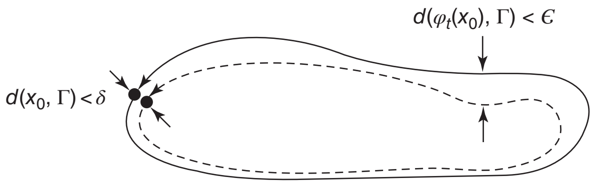

Figure 7 Geometric meaning of the orbital stability.

Clearly, exponential stability implies asymptotic stability, and asymptotic stability implies the stability in the sense of Lyapunov, but the reverse need not be true. For illustration, if a system has output trajectory $x_1(t) = x_0 \sin(t)$, then it is stable in the sense of Lyapunov about 0, but is not asymptotically stable; a system with output trajectory $x_2(t) = x_0(1 + t - t_0)^{-1}$ is asymptotically stable (so also is stable in the sense of Lyapunov) but is not exponentially stable about 0; however, a system with $x_3(t) = x_0 e^{-t}$ is exponentially stable (hence, is both asymptotically stable and stable in the sense of Lyapunov).

3.3 Orbital Stability

The orbital stability differs from the Lyapunov stabilities in that it concerns with the stability of a system output (or state) trajectory under small external perturbations.

Let $\varphi_t(\mathbf{x}_0)$ be a p-periodic solution, with $p > 0$, of the autonomous system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{R}^n \tag{19}$$

and let Γ represent the closed orbit of $\varphi_t(\mathbf{x}_0)$ in the state space, namely,

$$\Gamma = \left\{ \mathbf{y} \,|\, \mathbf{y} = \varphi_t(\mathbf{x}_0), \ 0 \le t < p \right\}$$

If, for any $\varepsilon > 0$, there exists a constant $\delta = \delta(\varepsilon) > 0$ such that for any $\mathbf{x}_0$ satisfying

$$d(\mathbf{x}_0, \Gamma) := \inf_{\mathbf{y} \in \Gamma} \|\mathbf{x}_0 - \mathbf{y}\| < \delta$$

the solution of the system, $\varphi_t(\mathbf{x}_0)$, satisfies

$$d(\varphi_t(\mathbf{x}_0), \Gamma) < \varepsilon, \quad \text{for all } t \ge t_0$$

then this p-periodic solution trajectory, $\varphi_t(\mathbf{x}_0)$, is said to be *orbitally stable*.

Orbital stability is visualized in Fig. 7. For a simple example, a stable periodic solution, particularly a stable equilibrium of a system, is orbitally stable. This is because all nearby trajectories approach it and, as such, it becomes a nearby orbit after a small perturbation and so will move back to its original position (or stay nearby). On the contrary, unstable and semi-stable (saddle type of) periodic orbits are orbitally unstable.

A more precise concept of orbital stability is given in the sense of Zhukovskij [18].

A solution $\varphi_t(\mathbf{x}_0)$ of system (19) is said to be stable in the sense of Zhukovskij if, for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that for any $\mathbf{y}_0 \in B_\delta(\mathbf{x}_0)$, a ball of radius δ centered at $\mathbf{x}_0$, there exist two functions, $\tau_1 = \tau_1(t)$ and $\tau_2 = \tau_2(t)$, satisfying

$$\|\mathbf{x}_{\tau_1}(\mathbf{x}_0) - \mathbf{x}_{\tau_2}(\mathbf{x}_0)\| < \varepsilon$$

for all $t \ge t_0$, where τ_1 and τ_2 are homeomorphisms from $[0, \infty)$ to $[0, \infty)$ with $\tau_1(0) = \tau_2(0) = 0$.

Furthermore, a Zhukovskij-stable solution $\varphi_t(\mathbf{x}_0)$ of system (19) is said to be asymptotically stable in the sense of Zhukovskij if, for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that for any $\mathbf{y}_0 \in B_\delta(\mathbf{x}_0)$, a ball of radius δ centered at $\mathbf{x}_0$, there exist two functions, $\tau_1 = \tau_1(t)$ and $\tau_2 = \tau_2(t)$, satisfying

$$\|\mathbf{x}_{\tau_1}(\mathbf{x}_0) - \mathbf{x}_{\tau_2}(\mathbf{x}_0)\| \to 0$$

as $t \to \infty$, where τ_1 and τ_2 are homeomorphisms from $[0, \infty)$ to $[0, \infty)$ with $\tau_1(0) = \tau_2(0) = 0$.

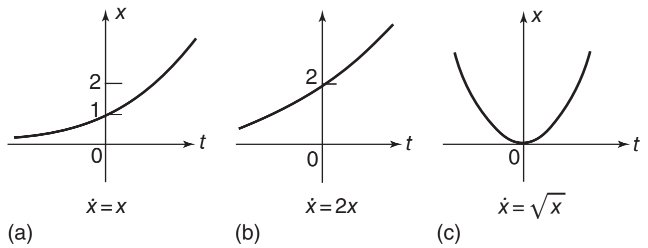

Figure 8 Trajectories of three systems for comparison.

It can be verified that (asymptotic) Zhukovskij stability implies (asymptotic) Lyapunov stability about an equilibrium. However, the converse may not be true. Nevertheless, these two types of stabilities are equivalent if the orbit $\varphi_t(\mathbf{x}_0)$ is an equilibrium of the system.

3.4 Structural Stability

Two systems are said to be *topologically orbitally equivalent*, if there exists a homeomorphism (i.e., a continuous map whose inverse exists and is also continuous) that transforms the family of trajectories of the first system to that of the second while preserving their motion directions. Roughly, this means that the geometrical pictures of the orbit families of the two systems are similar (no one has extra knots, sharp corners, bifurcating branches, etc.). For instance, systems $\dot{x} = x$ and $\dot{x} = 2x$ are topologically orbitally equivalent but are not so between $\dot{x} = x$ and $\dot{x} = \sqrt{x}$. These three system trajectories are shown in Fig. 8.

Return to the autonomous system (19). If the dynamics of the system in the state space changes radically, for example, by the appearance of a new equilibrium or a new periodic orbit, due to small external perturbations, then the system is considered to be structurally unstable.

To be more precise, consider the following set of functions:

$$\mathcal{S} = \left\{ \mathbf{g}(\mathbf{x}) \,\middle|\, \|\mathbf{g}(\mathbf{x})\| < \infty\,, \ \left\| \frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}} \right\| < \infty \ \text{for all } \mathbf{x} \in \mathbf{R}^n \right\}$$

If, for any $\mathbf{g} \in \mathcal{S}$, there exists an $\varepsilon > 0$ such that the orbits of the two systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad \text{and} \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \varepsilon\, \mathbf{g}(\mathbf{x})$$

are topologically orbitally equivalent, then the autonomous system (19), namely, the first (unperturbed) system above, is said to be *structurally stable*.

For example, $\dot{x} = x$ is structurally stable but $\dot{x} = x^2$ is not, in a neighborhood of the origin. This is because when the second system is slightly perturbed, to become say $\dot{x} = x^2 + \varepsilon$, where $\varepsilon > 0$, then the resulting system has two equilibria, $x_1^* = \sqrt{\varepsilon}$ and $x_2^* = -\sqrt{\varepsilon}$, which has more equilibria than the original system that possesses only one, $x^* = 0$.

4 VARIOUS STABILITY THEOREMS

Consider the general nonautonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)\,, \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{R}^n \tag{20}$$

where $\mathbf{f} : \mathcal{D} \times [0, \infty) \to \mathbf{R}^n$ is continuously differentiable in a neighborhood of the origin, $\mathcal{D} \subseteq \mathbf{R}^n$, with a given initial state $\mathbf{x}_0 \in \mathcal{D}$. Again, without loss of generality, assume that $\mathbf{x}^* = 0$ is a system equilibrium of interest.

4.1 Lyapunov Stability Theorems

First, for the autonomous system (19), an important special case of (20), with a continuously differentiable $\mathbf{f} : \mathcal{D} \to \mathbf{R}^n$, the following criterion of stability, called the *first* (or *indirect*) *method of Lyapunov*, is very convenient to use.

Theorem 1 ***(First Method of Lyapunov)*** (for continuous-time autonomous systems)
For system (19), let $J = \left[\partial \mathbf{f}/\partial \mathbf{x}\right]_{\mathbf{x}=\mathbf{x}^*=0}$ be its Jacobian evaluated at the zero equilibrium. If all the eigenvalues of J have negative real parts, then the system is asymptotically stable about $\mathbf{x}^* = 0$.

First, note that this and the following Lyapunov theorems apply to linear systems as well, for linear systems are merely a special case of nonlinear systems. When $\mathbf{f}(\mathbf{x}) = A\,\mathbf{x}$, the linear time-invariant system $\dot{\mathbf{x}} = A\,\mathbf{x}$ has the only equilibrium $\mathbf{x}^* = 0$. If A has all eigenvalues with negative real parts, Theorem 1 implies that the system is asymptotically stable about its equilibrium since the system Jacobian is simply $J = A$. This is consistent with the familiar linear stability results.

Note also that the region of asymptotic stability given by Theorem 1 is local, which can be quite large for some nonlinear systems but may be very small for some others. However, there is no general criterion for determining the boundaries of such local stability regions when this and the following Lyapunov methods are applied.

Moreover, it is important to note that this theorem cannot be applied to a *general* nonautonomous system, since for general nonautonomous systems, this theorem is either necessary nor sufficient [19]. A simple counterexample is the following linear time-varying system [11, 20]:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 + 1.5\cos^2(t) & 1 - 1.5\sin(t)\cos(t) \\ -1 - 1.5\sin(t)\cos(t) & -1 + 1.5\sin^2(t) \end{bmatrix} \mathbf{x}(t)$$

This system has eigenvalues $\lambda_{1,2} = -0.25 \pm j\,0.25\sqrt{7}$, both having negative real parts and being independent of the time variable t. If the above theorem is used to judge this system, the conclusion would be that the system is asymptotically stable about its equilibrium 0. However, the solution of this system is precisely

$$\mathbf{x}(t) = \begin{bmatrix} e^{0.5t}\cos(t) & e^{-t}\sin(t) \\ -e^{0.5t}\sin(t) & e^{-t}\cos(t) \end{bmatrix} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix}$$

which is unstable, for any initial conditions with a bounded and nonzero value of $x_1(t_0)$, no matter how small this initial value is. This example shows that by using the Lyapunov first method *alone* to determine the stability of a general time-varying system, the conclusion can be wrong.

This type of counterexamples can be easily found [17]. On the one hand, this demonstrates the necessity of other general criteria for asymptotic stability of nonautonomous systems. On the other hand, however, a word of caution is that this type of counterexamples do not completely rule out the possibility of applying the first method of Lyapunov to *some* special nonautonomous systems in case studies. The reason is that there is no theorem saying that "the Lyapunov first method cannot be applied to *all* nonautonomous systems." Due to the complexity of nonlinear dynamical systems, oftentimes they have to be studied class by class, or even case by case. It has been widely experienced that the first method of Lyapunov does work for some, perhaps not too many, *specific* nonautonomous systems in case studies (e.g., in the study of some chaotic systems [5]). The point is that one has to be very careful when this method is applied to a particular nonautonomous system; the stability conclusion must be verified by some other means at the same time.

Here, it is emphasized that a rigorous approach for asymptotic stability analysis of general nonautonomous systems is provided by the second method of Lyapunov, for which the following set of class-$\mathcal{K}$ functions is useful:

$$\mathcal{K} = \left\{ g(t) : g(t_0) = 0, \quad g(t) > 0 \text{ if } t > t_0,\ g(t) \text{ is continuous and nondecreasing on } [t_0, \infty) \right\}$$

Theorem 2 ***(Second Method of Lyapunov)*** (for continuous-time nonautonomous systems)
The system (20) is globally (over the entire domain $\mathcal{D}$), uniformly (with respect to the initial time over the entire time interval $[t_0, \infty)$), and asymptotically stable about its zero equilibrium, if there exist a scalar-valued function, $V(\mathbf{x}, t)$, defined on $\mathcal{D} \times [t_0, \infty)$, and three functions $\alpha(\cdot), \beta(\cdot), \gamma(\cdot) \in \mathcal{K}$, such that

(i) $V(0, t_0) = 0$;
(ii) $V(\mathbf{x}, t) > 0$ for all $\mathbf{x} \neq 0$ in $\mathcal{D}$ and all $t \geq t_0$;
(iii) $\alpha(\|\mathbf{x}\|) \leq V(\mathbf{x}, t) \leq \beta(\|\mathbf{x}\|)$ for all $t \geq t_0$;
(iv) $\dot{V}(\mathbf{x}, t) \leq -\gamma(\|\mathbf{x}\|) < 0$ for all $t \geq t_0$.

In Theorem 2, the function V is called a *Lyapunov function*. The method of constructing a Lyapunov function for stability determination is called the *second* (or *direct*) *method of Lyapunov*.

The geometric meaning of a Lyapunov function used for determining the system stability about the zero equilibrium may be illustrated in Fig. 9. In this figure, assuming that a Lyapunov function, $V(\mathbf{x})$, has been found, which has a bowl-shape as shown based on conditions (i) and (ii). Then, condition (iv) is

$$\dot{V}(\mathbf{x}) = \left[\frac{\partial V}{\partial \mathbf{x}}\right] \dot{\mathbf{x}} < 0 \tag{21}$$

where $\left[\frac{\partial V}{\partial \mathbf{x}}\right]$ is the gradient of V along the trajectory $\mathbf{x}$. It is known from Calculus that, if the inner product of this gradient and the tangent vector $\dot{\mathbf{x}}$ is constantly negative, as guaranteed by condition (21), then the angle between these two vectors is larger than 90°, so that the surface of $V(\mathbf{x})$ is monotonically decreasing to zero (as visualized in Fig. 9). Consequently, the system trajectory $\mathbf{x}$, the projection on the domain as shown in the figure, converges to zero as time tends to infinity.

As an example, consider the following nonautonomous system:

$$\dot{\mathbf{x}} = A\,\mathbf{x} + \mathbf{g}(\mathbf{x}, t)$$

where A is a stable constant matrix and $\mathbf{g}$ is a nonlinear function satisfying $\mathbf{g}(0, t) = 0$ and $\|\mathbf{g}(\mathbf{x}, t)\| \leq c\,\|\mathbf{x}\|$ for a constant $c > 0$ for all $t \in [t_0, \infty)$. Since A is stable, the following *Lyapunov equation*

$$PA + A^{\top}P + I = 0$$

has a unique positive definite and symmetric matrix solution, P. Using the Lyapunov function $V(\mathbf{x}, t) = \mathbf{x}^{\top} P\,\mathbf{x}$, it can be easily verified that

$$\dot{V}(\mathbf{x}, t) = \mathbf{x}^{\top}\left[PA + A^{\top}P\right]\mathbf{x} + 2\,\mathbf{x}^{\top}P\mathbf{g}(\mathbf{x}, t) \leq -\mathbf{x}^{\top}\mathbf{x} + 2\,\lambda_{\max}(P)\,c\,\|\mathbf{x}\|^2$$

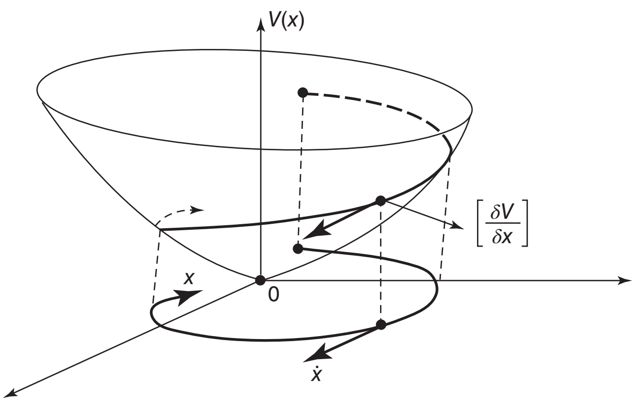

Figure 9 Geometric meaning of the Lyapunov function.

where $\lambda_{\max}(P)$ is the largest eigenvalue of P. Therefore, if the constant $c < 1/(2\lambda_{\max}(P))$ and if the three class-$\mathcal{K}$ functions are chosen as

$$\alpha(\zeta) = \lambda_{\min}(P)\,\zeta^2, \quad \beta(\zeta) = \lambda_{\max}(P)\,\zeta^2, \quad \gamma(\zeta) = \left[1 - 2c\,\lambda_{\max}(P)\right]\,\zeta^2$$

then conditions (iii) and (iv) of Theorem 2 are satisfied. As a result, the above system is globally, uniformly, and asymptotically stable about its zero equilibrium. This example shows that the linear part of a weakly nonlinear nonautonomous system can indeed dominate the system stability.

Note that, in Theorem 2, the uniform stability is guaranteed by the class-$\mathcal{K}$ functions α, β, γ stated in conditions (iii) and (iv), which is necessary since the solution of a nonautonomous system may sensitively depend on the initial time, as seen from the numerical example discussed in Section 3.1. For autonomous systems, these class-$\mathcal{K}$ functions (hence, condition (iii)) are not needed. In this case, Theorem 2 reduces to the following simple form.

Theorem 3 ***(Second Method of Lyapunov)*** (for continuous-time autonomous systems)
The autonomous system (19) is globally (over the entire domain $\mathcal{D}$) and asymptotically stable about its zero equilibrium, if there exists a scalar-valued function, $V(\mathbf{x})$, defined on $\mathcal{D}$, such that

(i) $V(0) = 0$;
(ii) $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$ in $\mathcal{D}$;
(iii) $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$ in $\mathcal{D}$.

Note that if condition (iv) in Theorem 3 is replaced by

(iv) $\dot{V}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \mathcal{D}$

then the resulting stability is only in the sense of Lyapunov but may not be asymptotic. For example, consider a simple model of an undamped pendulum of length ℓ described by

$$\begin{cases} \dot{x} = -\frac{g}{\ell}\,\sin(y) \\ \dot{y} = x \end{cases}$$

where $x = \theta$ is the angular variable defined on $-\pi < \theta < \pi$, with the vertical axis as its reference, and g is the gravity constant. Since the system Jacobian at the zero equilibrium has a pair of purely imaginary eigenvalues, $\lambda_{1,2} = \pm\sqrt{-g/\ell}$, Theorem 1 is not applicable. However, if one uses the Lyapunov function

$$V = \frac{g}{\ell}\,(1 - \cos(y)) + \frac{1}{2}x^2$$

then it can be easily verified that $\dot{V} = 0$ over the entire domain. Thus, the conclusion is that the undamped pendulum is stable in the sense of Lyapunov but not asymptotically, consistent with the physics of the undamped pendulum.

Theorem 4 ***(Krasovskii Theorem)*** (for continuous-time autonomous systems)
For the autonomous system (19), let $J(\mathbf{x}) = \left[\partial\mathbf{f}/\partial\mathbf{x}\right]$ be its Jacobian. A sufficient condition for the system to be asymptotically stable about its zero equilibrium is that there exist two real positive definite and symmetric constant matrices, P and Q, such that the matrix

$$J^{\top}(\mathbf{x})\,P + PJ(\mathbf{x}) + Q$$

is semi-negative definite for all $\mathbf{x} \neq 0$ in a neighborhood $\mathcal{D}$ of the origin. For this case, a Lyapunov function is given by

$$V(\mathbf{x}) = \mathbf{f}^{\top}(\mathbf{x}) P \mathbf{f}(\mathbf{x})$$

Furthermore, if $\mathcal{D} = \mathbf{R}^n$ and $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$, then this asymptotic stability is global.

Similar stability criteria can be established for discrete-time systems. Two main results are summarized as follows.

Theorem 5 ***(First Method of Lyapunov)*** (for discrete-time "autonomous" systems)
Let $\mathbf{x}^* = 0$ be an equilibrium of the discrete-time "autonomous" system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) \tag{22}$$

where $\mathbf{f} : \mathcal{D} \to \mathbf{R}^n$ is continuously differentiable in a neighborhood of the origin, $\mathcal{D} \subseteq \mathbf{R}^n$, and let $J = \left[\partial \mathbf{f}/\partial \mathbf{x}_k\right]_{\mathbf{x}_k = \mathbf{x}^* = 0}$ be the Jacobian of the system evaluated at this equilibrium. If all the eigenvalues of J are strictly less than one in absolute value, then the system is asymptotically stable about its zero equilibrium.

Theorem 6 ***(Second Method of Lyapunov)*** (for discrete-time "nonautonomous" systems)
Let $\mathbf{x}^* = 0$ be an equilibrium of the "nonautonomous" system

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k) \tag{23}$$

where $\mathbf{f}_k : \mathcal{D} \to \mathbf{R}^n$ is continuously differentiable in a neighborhood of the origin, $\mathcal{D} \subseteq \mathbf{R}^n$. Then, the system (22) is globally (over the entire domain $\mathcal{D}$) and asymptotically stable about its zero equilibrium, if there exists a scalar-valued function, $V(\mathbf{x}_k, k)$, defined on $\mathcal{D}$ and continuous in $\mathbf{x}_k$, such that

(i) $V(0, k) = 0$ for all $k \geq k_0$;
(ii) $V(\mathbf{x}_k, k) > 0$ for all $\mathbf{x}_k \neq 0$ in $\mathcal{D}$ and for all $k \geq k_0$;
(iii) $\Delta V(\mathbf{x}_k, k) := V\left(\mathbf{x}_k, k\right) - V\left(\mathbf{x}_{k-1}, k-1\right) < 0$ for all $\mathbf{x}_k \neq 0$ in $\mathcal{D}$ and all $k \geq k_0 + 1$;
(iv) $0 < W\left(\|\mathbf{x}_k\|\right) < V\left(\mathbf{x}_k, k\right)$ for all $k \geq k_0 + 1$, where $W(\tau)$ is a positive continuous function defined on $\mathcal{D}$, satisfying $W(\|\mathbf{x}_{k_0}\|) = 0$ and $\lim_{\tau \to \infty} W(\tau) = \infty$ monotonically.

As a special case, for discrete-time "autonomous" systems, Theorem 6 reduces to the following simple form.

Theorem 7 ***(Second Method of Lyapunov)*** (for discrete-time "autonomous" systems)
Let $\mathbf{x}^* = 0$ be an equilibrium for the "autonomous" system (22). Then, the system is globally (over the entire domain $\mathcal{D}$) and asymptotically stable about this zero equilibrium if there exists a scalar-valued function, $V(\mathbf{x}_k)$, defined on $\mathcal{D}$ and continuous in $\mathbf{x}_k$, such that

(i) $V(0) = 0$;
(ii) $V(\mathbf{x}_k) > 0$ for all $\mathbf{x}_k \neq 0$ in $\mathcal{D}$;
(iii) $\Delta V(\mathbf{x}_k) := V\left(\mathbf{x}_k\right) - V\left(\mathbf{x}_{k-1}\right) < 0$ for all $\mathbf{x}_k \neq 0$ in $\mathcal{D}$;
(iv) $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$.

To this end, it is important to emphasize that all the Lyapunov theorems stated above only offer *sufficient* conditions for asymptotic stability. On the other hand, usually more than one Lyapunov function may be constructed for the same system. For a given system, one choice of a Lyapunov function may yield a less conservative result (e.g., with a larger stability region) than other choices. However, no conclusion regarding stability may be drawn if, for technical reasons, a satisfactory Lyapunov function cannot be found. Nevertheless, there is a reverse theorem regarding the existence of a Lyapunov function [7].

Theorem 8 ***(Massera Inverse Theorem)*** Suppose that the autonomous system (19) is asymptotically stable about its equilibrium $\mathbf{x}^*$ and $\mathbf{f}$ is continuously differentiable with respect to $\mathbf{x}$ for all $t \in [t_0, \infty)$. Then, a Lyapunov function exists for this system.

4.2 Some Instability Theorems

Once again, consider a general autonomous system,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{R}^n \tag{24}$$

with an equilibrium $\mathbf{x}^* = 0$. To disprove the stability of this system, the following instability theorems are useful.

Theorem 9 ***(A Linear Instability Theorem)*** For system (24), let $J = \left[\partial\mathbf{f}/\partial\mathbf{x}\right]_{\mathbf{x}=\mathbf{x}^*=0}$ be its Jacobian evaluated at $\mathbf{x}^* = 0$. If at least one of the eigenvalues of J has a positive real part, then $\mathbf{x}^* = 0$ is unstable.

For discrete-time systems, there is a similar result: A discrete-time "autonomous" system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k), \quad k = 0, 1, 2, \ldots$$

is unstable about its equilibrium $\mathbf{x}^* = 0$ if at least one of the eigenvalues of the system Jacobian is larger than 1 in absolute value.

The following two negative theorems can be easily extended to nonautonomous systems in an obvious way.

Theorem 10 ***(A General Instability Theorem)*** For system (24), let $V(\mathbf{x})$ be a positive and continuously differentiable function defined on a neighborhood $\mathcal{D}$ of the origin, satisfying $V(0) = 0$. Assume that in any subset of $\mathcal{D}$, containing the origin, there is an $\tilde{\mathbf{x}}$ such that $V(\tilde{\mathbf{x}}) > 0$. If, moreover,

$$\frac{d}{dt} V(\mathbf{x}) > 0 \quad \text{for all} \quad \mathbf{x} \neq 0 \text{ in } \mathcal{D}$$

then the system is unstable about the equilibrium $\mathbf{x}^* = 0$.

One example is the system

$$\begin{cases} \dot{x} = y + x\left(x^2 + y^4\right) \\ \dot{y} = -x + y\left(x^2 + y^4\right) \end{cases}$$

which has equilibrium $(x^*, y^*) = (0, 0)$. The system Jacobian at the equilibrium has a pair of imaginary eigenvalues, $\lambda_{1,2} = \pm\sqrt{-1}$, so Theorem 1 is not applicable. On the contrary, the Lyapunov function

$$V = \frac{1}{2}\left(x^2 + y^2\right)$$

leads to $\dot{V} = (x^2 + y^2)(x^2 + y^4) > 0$ for all $(x, y) \neq (0, 0)$. Therefore, the conclusion is that this system is unstable about its zero equilibrium.

Theorem 11 ***(Chetaev Instability Theorem)*** For system (24), let $V(\mathbf{x})$ be a positive and continuously differentiable function defined on $\mathcal{D}$, and let Ω be a subset of $\mathcal{D}$, containing the origin (i.e., $0 \in \mathcal{D} \cap \Omega$). If

(i) $V(\mathbf{x}) > 0$ and $\dot{V}(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$ in $\mathcal{D}$,
(ii) $V(\mathbf{x}) = 0$ for all $\mathbf{x}$ on the boundary of Ω,

then the system is unstable about the equilibrium $\mathbf{x}^* = 0$.

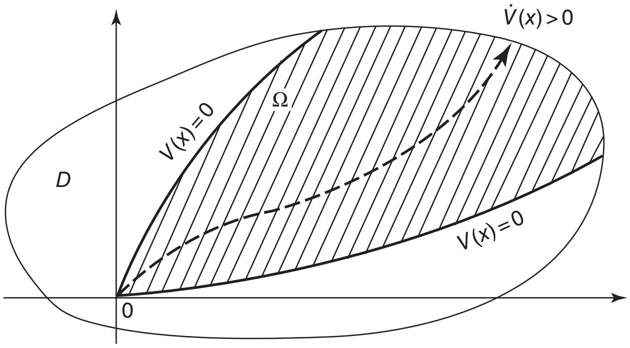

Figure 10 Illustration of the Chetaev theorem.

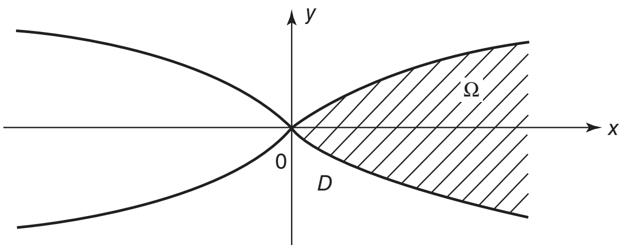

Figure 11 The defining region of a Lyapunov function.

This instability theorem is illustrated in Fig. 10, which graphically shows that if the theorem conditions are satisfied, then there is a gap within any neighborhood of the origin, so that a system trajectory can escape from the neighborhood of the origin along a path in this gap [1].

As an example, consider the system

$$\begin{cases} \dot{x} = x^2 + 2y^5 \\ \dot{y} = xy^2 \end{cases}$$

with the Lyapunov function

$$V = x^2 - y^4$$

which is positive inside the region defined by

$$x = y^2 \quad \text{and} \quad x = -y^2$$

Let $\mathcal{D}$ be the right-half plane and Ω be the shaded area shown in Fig. 11. Clearly, $V = 0$ on the boundary of Ω, and $V > 0$ and $\dot{V} = 2x^3 > 0$ for all $(x, y) \in \mathcal{D}$. According to the Chetaev theorem, this system is unstable about its zero equilibrium.

4.3 LaSalle Invariance Principle

Consider again the autonomous system (24) with an equilibrium $\mathbf{x}^* = 0$. Let $V(\mathbf{x})$ be a Lyapunov function defined on a neighborhood $\mathcal{D}$ of the origin. Let $\varphi_t(\mathbf{x}_0)$ be a bounded solution orbit of the system, with the initial state $\mathbf{x}_0$ and all its limit states being confined in $\mathcal{D}$. Moreover, let

$$E = \left\{ \mathbf{x} \in \mathcal{D} \,|\, \dot{V}(\mathbf{x}) = 0 \right\} \tag{25}$$

and $M \subset E$ be the largest *invariant subset* of E in the sense that if the initial state $\mathbf{x}_0 \in M$ then the entire orbit $\varphi_t(\mathbf{x}_0) \subset M$ for all $t \geq t_0$.

Theorem 12 ***(LaSalle Invariance Principle)*** Under the above assumptions, for any initial state $\mathbf{x}_0 \in \mathcal{D}$, the solution orbit satisfies

$$\varphi_t(\mathbf{x}_0) \to M \quad \text{as} \quad t \to \infty$$

This invariance principle is consistent with the Lyapunov theorems when they both are applicable to the same problem [6, 12]. Sometimes, when $\dot{V} = 0$ over a subset of the domain of V, a Lyapunov theorem is not easy to directly apply, but the LaSalle invariance principle may be convenient to use. For instance, consider the system

$$\begin{cases} \dot{x} = -x + \frac{1}{3}x^3 + y \\ \dot{y} = -x \end{cases}$$

The Lyapunov function $V = x^2 + y^2$ yields

$$\dot{V} = \frac{1}{2}x^2\left(\frac{1}{3}x^2 - 1\right)$$

which is negative for $x^2 < 3$ but is zero for $x = 0$ and $x^2 = 3$, regardless of variable y. Thus, Lyapunov theorems do not seem to be applicable, at least not directly. However, observe that the set E defined above has only three straight lines: $x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$, and that all trajectories that intersect the line $x = 0$ will remain on the line only if $y = 0$. This means that the largest invariant subset M containing the points with $x = 0$ is the only point $(0, 0)$. It then follows from the LaSalle invariance principle that starting from any initial state located in a neighborhood of the origin bounded within the two stripes $x = \pm\sqrt{3}$, say located inside the disk

$$\mathcal{D} = \left\{ (x, y) \,|\, x^2 + y^2 < 3 \right\}$$

the solution orbit will always be attracted to the point $(0, 0)$. This means that the system is locally asymptotically stable about its zero equilibrium.

4.4 Comparison Principle and Vector Lyapunov Functions

For large-scale and interconnected nonlinear (control) systems, or systems described by differential inequalities rather than differential equations, the above stability criteria may not be directly applicable. In many such cases, the comparison principle and vector Lyapunov function methods turn out to be advantageous [21–23].

To introduce the comparison principle, consider the general 2-dimensional nonautonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{26}$$

where $\mathbf{f}(0, t) = 0$ is continuous on a neighborhood $\mathcal{D}$ of the origin, $t_0 \leq t < \infty$.

In this case, since $\mathbf{f}$ is only continuous, but not necessarily satisfying the Lipschitz condition, this differential equation may have more than one solution [24]. Let $\mathbf{x}_{\max}(t)$ and $\mathbf{x}_{\min}(t)$ be its maximum and minimum solutions, respectively, in the sense that

$$\mathbf{x}_{\min}(t) \leq \mathbf{x}(t) \leq \mathbf{x}_{\max}(t) \quad \text{componentwise,} \quad \text{for all } t \in [t_0, \infty)$$

where $\mathbf{x}(t)$ is any solution of the equation, and $\mathbf{x}_{\min}(t_0) = \mathbf{x}(t_0) = \mathbf{x}_{\max}(t_0) = \mathbf{x}_0$.

Theorem 13 ***(Comparison Principle)*** Let $\mathbf{y}(t)$ be a solution of the following differential inequality:

$$\dot{\mathbf{y}}(t) \leq \mathbf{f}(\mathbf{y}, t) \quad \text{with} \quad \mathbf{y}(t_0) \leq \mathbf{x}_0 \quad \text{componentwise}$$

If $\mathbf{x}_{\max}(t)$ is the maximum solution of system (26), then

$$\mathbf{y}(t) \leq \mathbf{x}_{\max}(t) \quad \text{componentwise} \quad \text{for all} \quad t \in [t_0, \infty)$$

The next theorem is established based on this comparison principle.

A vector-valued function, $\mathbf{g}(\mathbf{x}, t) = [g_1(\mathbf{x}, t) \cdots g_n(\mathbf{x}, t)]^\top$ is said to be *quasi-monotonic*, if

$$x_i = \tilde{x}_i \quad \text{and} \quad x_j \geq \tilde{x}_j \ (j \neq i) \quad \Longrightarrow \quad g_i(\mathbf{x}, t) \geq g_i(\tilde{\mathbf{x}}, t), \quad i = 1, \ldots, n$$

Theorem 14 ***(Vector Lyapunov Function Theorem)*** Let $\mathbf{v}(\mathbf{x}, t)$ be a vector Lyapunov function associated with the nonautonomous system (26), with $\mathbf{v}(\mathbf{x}, t) = \left[V_1(\mathbf{x}, t) \cdots V_n(\mathbf{x}, t)\right]^\top$, in which each V_i is a continuous Lyapunov function for the system, $i = 1, \ldots, n$, satisfying $\|\mathbf{v}(\mathbf{x}, t)\| > 0$ for $\mathbf{x} \neq 0$. Assume that

$$\dot{\mathbf{v}}(\mathbf{x}, t) \leq \mathbf{g}(\mathbf{v}(\mathbf{x}, t), t) \quad \text{componentwise}$$

for a continuous and quasi-monotonic function $\mathbf{g}$ defined on $\mathcal{D}$. Then

(i) if the system

$$\dot{\mathbf{y}}(t) = \mathbf{g}(\mathbf{y}, t)$$

is stable in the sense of Lyapunov (or asymptotically stable) about its zero equilibrium $\mathbf{y}^* = 0$, then so is the nonautonomous system (26);

(ii) if, moreover, $\|\mathbf{v}(\mathbf{x}, t)\|$ is monotonically decreasing with respect to t and the above stability (or asymptotic stability) is uniform, then so is the nonautonomous system (26);

(iii) if, furthermore, $\|\mathbf{v}(\mathbf{x}, t)\| \geq c \, \|\mathbf{x}\|^\sigma$ for two positive constants c and σ, and the above stability (or asymptotic stability) is exponential, then so is the nonautonomous system (26).

A simple and frequently used comparison function is

$$\mathbf{g}(\mathbf{y}, t) = A\,\mathbf{y} + \mathbf{h}(\mathbf{y}, t), \qquad \lim_{\|\mathbf{y}\| \to 0} \frac{\|\mathbf{h}(\mathbf{y}, t)\|}{\|\mathbf{y}\|} = 0$$

where A is a stable M matrix (Metzler matrix) defined as follows: $A = [a_{ij}]$ is an M matrix if

$$a_{ii} < 0 \quad \text{and} \quad a_{ij} \geq 0 \quad (i \neq j), \quad i, j = 1, \ldots, n$$

4.5 Orbital and Structural Stability Theorems

Theorem 15 ***(Orbital Stability Theorem)*** Let $\tilde{\mathbf{x}}(t)$ be a p-periodic solution of an autonomous system. Suppose that the system has Floquet multipliers λ_i, with $\lambda_1 = 0$ and $|\lambda_i| < 1$ for $i = 2, \ldots, n$. Then, this periodic solution $\tilde{\mathbf{x}}(t)$ is orbitally stable.

Theorem 16 ***(Peixoto Structural Stability Theorem)*** Consider a two-dimensional autonomous system. Suppose that $\mathbf{f}$ is twice differentiable on a compact and connected subset $\mathcal{D}$ bounded by a simple closed curve, Γ, with an outward normal vector, $\vec{n}$. Assume that $\mathbf{f} \cdot \vec{n} \neq 0$ on Γ. Then, the system is structurally stable on $\mathcal{D}$ if and only if

(i) all equilibria are hyperbolic;

(ii) all periodic orbits are hyperbolic;

(iii) if x and y are hyperbolic saddles (probably, $x = y$), then $W^s(x)$ and $W^u(y)$ are transversal.

5 LINEAR STABILITY OF NONLINEAR SYSTEMS

The first method of Lyapunov provides a linear stability analysis for nonlinear autonomous systems. In this section, the following general nonautonomous system is considered:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)\,, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{R}^n \tag{27}$$

which is assumed to have an equilibrium $\mathbf{x}^* = 0$.

5.1 Linear Stability of Nonautonomous Systems

In system (27), Taylor-expanding the function $\mathbf{f}$ about $\mathbf{x}^* = 0$ gives

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) = J(t)\,\mathbf{x} + \mathbf{g}(\mathbf{x}, t) \tag{28}$$

where $J(t) = \left[\partial \mathbf{f}/\partial \mathbf{x}\right]_{\mathbf{x}=0}$ is the Jacobian and $\mathbf{g}(\mathbf{x}, t)$ is the residual of the expansion, which is assumed to satisfy

$$\|\mathbf{g}(\mathbf{x}, t)\| \leq a\,\|\mathbf{x}\|^2 \quad \text{for all} \quad t \in [t_0, \infty)$$

where $a > 0$ is a constant. It is known, from the theory of elementary ordinary differential equations [24], that the solution of Eq. (28) is given by

$$\mathbf{x}(t) = \Phi(t, t_0)\,\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t, \tau)\mathbf{g}(\mathbf{x}(\tau), \tau)d\tau \tag{29}$$

where $\Phi(t, \tau)$ is the fundamental matrix associated with matrix $J(t)$.

Theorem 17 ***(A General Linear Stability Theorem)*** For the nonlinear nonautonomous system (28), if there are two positive constants c and σ such that

$$\|\Phi(t, \tau)\| \leq c\,e^{-\sigma(t-\tau)} \quad \text{for all} \quad t_0 \leq \tau \leq t < \infty$$

and if

$$\lim_{\|\mathbf{x}\| \to 0} \frac{\|\mathbf{g}(\mathbf{x}, t)\|}{\|\mathbf{x}\|} = 0$$

uniformly with respect to $t \in [t_0, \infty)$, then there are two positive constants, γ and δ, such that

$$\|\mathbf{x}(t)\| \leq c\,\|\mathbf{x}_0\|\,e^{-\gamma(t-t_0)}$$

for all $\|\mathbf{x}_0\| \leq \delta$ and all $t \in [t_0, \infty)$.

This result implies that under the theorem conditions, the system is locally, uniformly, and exponentially stable about its equilibrium $\mathbf{x}^* = 0$.

In particular, if the system matrix $J(t) = J$ is a stable constant matrix, then the following simple criterion is convenient to use.

Theorem 18 ***(A Special Linear Stability Theorem)*** Suppose that, in system (28), the matrix $J(t) = J$ is a stable constant matrix, namely all its eigenvalues have negative real parts, with $\mathbf{g}(0, t) = 0$. Let P be a positive definite and symmetric matrix solution of the Lyapunov equation

$$PJ + J^\top P + Q = 0$$

where Q is a positive definite and symmetric constant matrix. If

$$\|\mathbf{g}(\mathbf{x}, t)\| \leq a\,\|\mathbf{x}\|$$

for a constant $a < \frac{1}{2}\lambda_{\max}(P)$ uniformly on $[t_0, \infty)$, where $\lambda_{\max}(P)$ is the maximum eigenvalue of P, then system (28) is globally, uniformly, and asymptotically stable about its equilibrium $\mathbf{x}^* = 0$.

This actually is the example for illustration of Theorem 2 discussed in Section 4.1.

Again, consider system (27) with an equilibrium $\mathbf{x}^* = 0$.

Theorem 19 ***(The Lyapunov Converse Theorem)*** Suppose that $\mathbf{f}$ is continuously differentiable in a neighborhood of the origin, and its Jacobian $J(t)$ is bounded and Lipschitz in the neighborhood, uniformly in t. Then, the system is exponentially stable about its equilibrium if and only its linearized system $\dot{\mathbf{x}} = J(t)\mathbf{x}$ is exponentially stable about the origin.

5.2 Linear Stability of Nonlinear Systems with Periodic Linearity

Consider a nonlinear nonautonomous system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) = J(t)\,\mathbf{x} + \mathbf{g}(\mathbf{x}, t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{R}^n \tag{30}$$

where $\mathbf{g}(0, t) = 0$ and $J(t)$ is a p-periodic matrix ($p > 0$):

$$J(t + p) = J(t) \qquad \text{for all} \quad t \in [t_0, \infty)$$

Theorem 20 ***(Floquet Theorem)*** For system (30), assume that $\mathbf{g}(\mathbf{x}, t)$ and $\partial\mathbf{g}(\mathbf{x}, t)/\partial\mathbf{x}$ are both continuous in a bounded region $\mathcal{D}$ containing the origin. Assume, moreover, that

$$\lim_{\|\mathbf{x}\| \to 0} \frac{\|\mathbf{g}(\mathbf{x}, t)\|}{\|\mathbf{x}\|} = 0$$

uniformly on $[t_0, \infty)$. If the system Floquet multipliers satisfy

$$|\,\lambda_i\,| < 1\,, \quad i = 1, \ldots, n\,, \quad \text{for all } t \in [t_0, \infty) \tag{31}$$

then system (30) is globally, uniformly, and asymptotically stable about its equilibrium $\mathbf{x}^* = 0$.

Note that if $\mathbf{g}(\mathbf{x}, t) = 0$ in system (28) or (30), all the above linear stability results still hold and are consistent with the familiar results from the linear systems theory.

6 TOTAL STABILITY: STABILITY UNDER PERSISTENT DISTURBANCES

Consider a nonautonomous system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{h}(\mathbf{x}, t)\,, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{R}^n \tag{32}$$

where $\mathbf{f}$ is continuously differentiable, with $\mathbf{f}(0, t) = 0$, and $\mathbf{h}$ is a *persistent perturbation* in the sense that, for any $\varepsilon > 0$, there are two positive constants, δ_1 and δ_2, such that if $\|\mathbf{h}(\tilde{\mathbf{x}}, t)\| < \delta_1$ for all $t \in [t_0, \infty)$ and if $\|\tilde{\mathbf{x}}(t_0)\| < \delta_2$ then $\|\tilde{\mathbf{x}}(t)\| < \varepsilon$.

The equilibrium $\mathbf{x}^* = 0$ of the unperturbed system (system (32) with $\mathbf{h} = 0$ therein) is said to be *totally stable*, if the persistently perturbed system (32) remains to be stable in the sense of Lyapunov.

As the next theorem states, all uniformly and asymptotically stable systems with persistent perturbations are totally stable, namely, a stable orbit starting from a neighborhood of another orbit will stay nearby [7, 9].

Theorem 21 ***(Malkin Theorem)*** If the unperturbed system (32), with $\mathbf{h} = 0$ therein, is uniformly and asymptotically stable about its equilibrium $\mathbf{x}^* = 0$, then it is totally stable, namely, the persistently perturbed system (32) remains to be stable in the sense of Lyapunov.

Next, consider an autonomous system with persistent perturbations:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{h}(\mathbf{x}, t), \qquad \mathbf{x} \in \mathbf{R}^n \tag{33}$$

Theorem 22 ***(Perturbed Orbital Stability Theorem)*** If $\varphi_t(\mathbf{x}_0)$ is an orbitally stable solution of the unperturbed autonomous system (33) (with $\mathbf{h} = 0$ therein), then it is totally stable, that is, the perturbed system remains to be orbitally stable under persistent perturbations.

7 ABSOLUTE STABILITY AND FREQUENCY DOMAIN CRITERIA

Consider a feedback system in the Lur'e form:

$$\begin{cases} \dot{\mathbf{x}} = A\,\mathbf{x} + B\,\mathbf{h}(\mathbf{y}) \\ \mathbf{y} = C\,\mathbf{x} \end{cases} \tag{34}$$

where A, B, C are constant matrices, in which A is nonsingular but B and C are not necessarily square, and $\mathbf{h}$ is a vector-valued nonlinear function.

By taking the Laplace transforms with zero initial conditions, and denoting the transforms, with $\hat{\mathbf{x}} = \mathcal{L}\{\mathbf{x}\}$, the state vector is obtained as

$$\hat{\mathbf{x}} = [sI - A]^{-1} B\,\mathcal{L}\{\mathbf{h}(\mathbf{y})\} \tag{35}$$

so that the output is given by

$$\hat{\mathbf{y}} = C\,G(s)\,\mathcal{L}\{\mathbf{h}(\mathbf{y})\} \tag{36}$$

where the system transfer matrix

$$G(s) = [sI - A]^{-1} B \tag{37}$$

This can be implemented via the block diagram shown in Fig. 12 where, for notational convenience, both time- and frequency-domain symbols are mixed.

The Lur'e system shown in Fig. 12 is a closed-loop configuration, where the feedback loop is usually considered as a "controller." Thus, this system is sometimes written in the following equivalent form:

$$\begin{cases} \dot{\mathbf{x}} = A\,\mathbf{x} + B\,\mathbf{u} \\ \mathbf{y} = C\,\mathbf{x} \\ \mathbf{u} = \mathbf{h}(\mathbf{y}) \end{cases} \tag{38}$$

7.1 SISO Lur'e Systems

First, SISO Lur'e systems are discussed, where $u = h(y)$ and $y = \mathbf{c}^\top \mathbf{x}$ are both scalar-valued functions:

$$\begin{cases} \dot{\mathbf{x}} = A\,\mathbf{x} + \mathbf{b}\,u \\ y = \mathbf{c}^\top \mathbf{x} \\ u = h(y) \end{cases} \tag{39}$$

Assume that $h(0) = 0$; thus, $\mathbf{x}^* = 0$ is an equilibrium of the system.

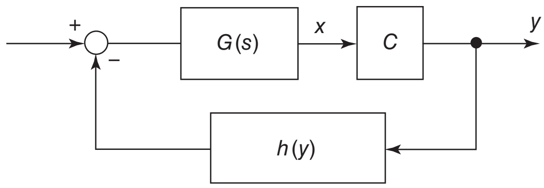

Figure 12 Configuration of the Lur'e system.

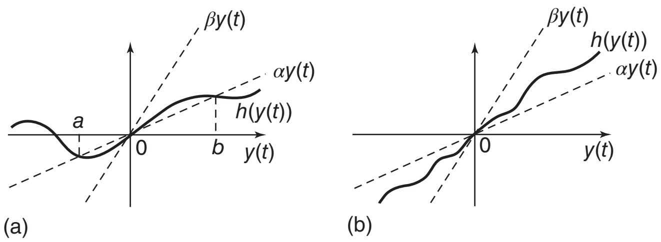

Figure 13 Local and global sector conditions.

The Sector Condition: The Lur'e system (39) is said to satisfy the (*local* or *global*) *sector condition* on the nonlinear function $h(\cdot)$, if there exist two constants, $\alpha < \beta$, such that

(i) local sector condition:

$$\alpha y^2(t) \leq y(t)h(y(t)) \leq \beta y^2(t) \qquad \text{for all} \quad a \leq y(t) \leq b \text{ and } t \in [t_0, \infty) \tag{40}$$

(ii) global sector condition:

$$\alpha y^2(t) \leq y(t)h(y(t)) \leq \beta y^2(t) \qquad \text{for all} -\infty < y(t) < \infty \text{ and } t \in [t_0, \infty) \tag{41}$$

Here, $[\alpha, \beta]$ is called the *sector* for the nonlinear function $h(\cdot)$. Moreover, the system (39) is said to be *absolutely stable within the sector* $[\alpha, \beta]$ if the system is globally asymptotically stable about its equilibrium $\mathbf{x}^* = 0$ for any nonlinear function $h(\cdot)$ satisfying the global sector condition. These local and global sector conditions are visualized in Figs. 13a and 13b, respectively.

Theorem 23 ***(Popov Criterion)*** Suppose that the SISO Lur'e system (39) satisfies the following conditions:

(i) A is stable and $\{A, \mathbf{b}\}$ is controllable;
(ii) the system satisfies the global sector condition with $\alpha = 0$ therein;
(iii) for any $\varepsilon > 0$, there is a constant $\gamma > 0$ such that

$$\text{Re}\{\,(1 + j\gamma\,\omega)G(j\,\omega)\,\} + \frac{1}{\beta} \geq \varepsilon \qquad \text{for all} \quad \omega \geq 0 \tag{42}$$

where $G(s)$ is the transfer function defined by (37), and $\text{Re}\{\cdot\}$ denotes the real part of a complex number (or function). Then, the system is globally asymptotically stable about its equilibrium $\mathbf{x}^* = 0$ within the sector.

The Popov criterion has the following geometric meaning: Separate the complex function $G(s)$ into its real and imaginary parts, namely,

$$G(j\,\omega) = G_r(\omega) + j\,G_i(\omega)$$

and rewrite condition (iii) as

$$\frac{1}{\beta} > -G_r(\omega) + \gamma\,\omega\,G_i(\omega) \qquad \text{for all } \omega \geq 0$$

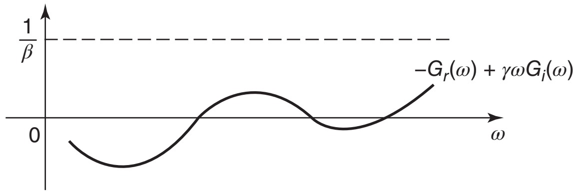

Figure 14 Geometric meaning of the Popov criterion.

Then, the graphical situation of the Popov criterion shown in Fig. 14 implies the global asymptotic stability of the system about its zero equilibrium.

The Popov criterion has a natural connection to the linear Nyquist criterion [11, 15, 16, 25]. A more direct generalization of the Nyquist criterion to nonlinear systems is the following.

Theorem 24 ***(Circle Criterion)*** Suppose that the SISO Lur'e system (39) satisfies the following conditions:

(i) A has no purely imaginary eigenvalues and has κ eigenvalues with positive real parts.
(ii) The system satisfies the global sector condition.
(iii) One of the following situations holds:
(iv) $0 < \alpha < \beta$: the Nyquist plot of $G(j\omega)$ encircles the disk $D\left(-\frac{1}{\alpha}, -\frac{1}{\beta}\right)$ counterclockwise for κ times but does not enter it;
(v) $0 = \alpha < \beta$: the Nyquist plot of $G(j\omega)$ stays within the open half-plane $\mathrm{Re}\{s\} > -\frac{1}{\beta}$;
(vi) $\alpha < 0 < \beta$: the Nyquist plot of $G(j\omega)$ stays within the open disk $D\left(-\frac{1}{\beta}, -\frac{1}{\alpha}\right)$;
(vii) $\alpha < \beta < 0$: the Nyquist plot of $-G(j\omega)$ encircles the disk $D\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)$ counterclockwise for κ times but does not enter it.

Then, the system is globally asymptotically stable about its equilibrium $\mathbf{x}^* = 0$.

Here, the disk $D\left(-\frac{1}{\alpha}, -\frac{1}{\beta}\right)$, for the case of $0 < \alpha < \beta$, is shown in Fig. 15.

7.2 MIMO Lur'e Systems

Consider a MIMO Lur'e system, as shown in Fig. 12, namely,

$$\begin{cases} \mathbf{x}(s) = G(s)\,\mathbf{u}(s) \\ \mathbf{u}(t) = -\mathbf{h}\,(\mathbf{y}(t)) \end{cases} \tag{43}$$

with $G(s)$ as defined in (37). If this system satisfies the following Popov inequality:

$$\int_{t_0}^{t_1} \mathbf{y}^{\top}(\tau)\mathbf{x}(\tau)d\tau \geq -\gamma \qquad \text{for all} \quad t_1 \geq t_0 \tag{44}$$

for a constant $\gamma \geq 0$ independent of t, then it is said to be *hyperstable*.

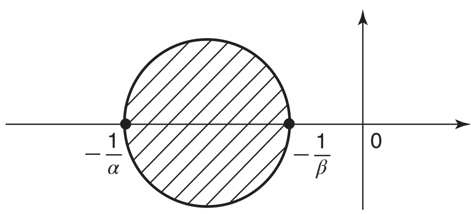

Figure 15 The disk $D\left(-\frac{1}{\alpha}, -\frac{1}{\beta}\right)$.

The linear part of this MIMO system is described by the transfer matrix $G(s)$, which is said to be *positive real* if

(i) there are no poles of $G(s)$ located inside the open half-plane $\text{Re}\{s\} > 0$;
(ii) poles of $G(s)$ on the imaginary axis are simple, and the residues form a semi-positive definite matrix;
(iii) the matrix $\left[G(j\omega) + G^{\top}(j\omega)\right]$ is a semi-positive definite matrix for all real values of ω that are not poles of $G(s)$.

Theorem 25 ***(Hyperstability Theorem)*** The MIMO Lur'e system (43) is hyperstable if and only if its transfer matrix $G(s)$ is positive real.

7.3 Describing Function Method

Return to the SISO Lur'e system (39) and consider its periodic output, $y(t)$. Assume that the nonlinear function $h(\cdot)$ therein is a time-invariant odd function and satisfies the property that for $y(t) = \alpha \sin(\omega t)$, with real constants ω and $\alpha \neq 0$, only the first-order harmonic of $-h(y)$ in its Fourier series expansion is significant. Under this setup, the specially defined function

$$\Psi(\alpha) = -\frac{2\omega}{\alpha\pi} \int_0^{\pi/\omega} h\left(\alpha \sin(\omega t)\right) \sin(\omega t)\, dt \tag{45}$$

is called the *describing function* of the nonlinearity $-h(\cdot)$, or of the system [15, 25, 26].

Theorem 26 ***(First-Order Harmonic Balance Approximation)*** Under the above conditions, if furthermore the first-order harmonic balance equations

$$G_r(j\omega)\,\Psi(\alpha) = 1 \quad \text{and} \quad G_i(j\omega) = 0$$

have solutions ω and $\alpha \neq 0$, then

$$y^{<1>}(t) = \frac{j\alpha}{2}\, e^{-j\omega t} - \frac{j\alpha}{2}\, e^{j\omega t}$$

is the first-order approximation of a possible periodic orbit of the output of system (39); but if the above harmonic balance equations have no solutions, then likely the system will not have any periodic output.

When solving the equation $G_r(j\omega)\Psi(\alpha) = 1$ graphically, one can sketch two curves in the complex plane, $G_r(j\omega)$ and $-1/\Psi(\alpha)$, by increasing gradually ω and α, respectively, to find their crossing points:

(i) if the two curves are (almost) tangent, as illustrated in Fig. 16a, then a conclusion drawn from the describing function method will not be satisfactory in general;
(ii) if the two curves are (almost) transversal, as illustrated in Fig. 16b, then a conclusion drawn from the describing function analysis will generally be reliable.

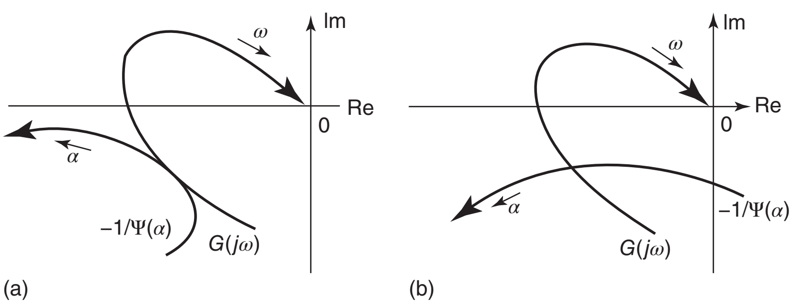

Figure 16 Graphical describing function analysis.

Theorem 27 ***(Graphical Stability Criterion for a Periodic Orbit)*** Each intersection point of the above two curves, $G_r(j\omega)$ and $-1/\Psi(\alpha)$, corresponds to a periodic orbit, $y^{<1>}(t)$, of the output of system (39). If the points, near the intersection and on one side of the curve $-1/\Psi(\alpha)$ where $-\alpha$ is increasing, are not encircled by the curve $G_r(j\omega)$, then the corresponding periodic output is stable; otherwise, it is unstable.

8 BIBO STABILITY

A relatively simple and also relatively weak notion of stability is discussed in this section. This is the *bounded-input bounded-output* (*BIBO*) *stability*, which refers to the property of a system that any bounded input to the system produces a bounded output through the system [11, 27, 28].

It can be verified that the linear system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ is BIBO stable if the matrix A is asymptotically stable.

Here, the focus is on the input–output map (13), with its configuration shown in Fig. 2.

Definition 1 The system S is said to be BIBO stable from the input set U to the output set Y, if for each admissible input $\mathbf{u} \in U$ and the corresponding output $\mathbf{y} \in Y$, there exist two nonnegative constants, b_i and b_o, such that

$$\|\mathbf{u}\|_U \le b_i \quad \Longrightarrow \quad \|\mathbf{y}\|_Y \le b_o \tag{46}$$

Note that since all norms are equivalent for a finite-dimensional vector, it is generally insignificant to distinguish under what kind of norms for the input and output signals the BIBO stability is defined and achieved. Moreover, it is important to note that in the above definition, even if b_i is small and b_o is large, the system is still considered to be BIBO stable. Therefore, this stability may not be very practical for some systems in certain applications.

8.1 Small Gain Theorem

A convenient criterion for verifying the BIBO stability of a closed-loop control system is the small gain theorem [11, 27, 28], which applies to almost all kinds of systems (linear and nonlinear, continuous-time and discrete-time, deterministic and stochastic, and time-delayed, of any dimensions), as long as the mathematical setup is appropriately formulated to meet the theorem conditions. The main disadvantage of this criterion is its over-conservativity.

Return to the typical closed-loop system shown in Fig. 3, where the inputs, outputs, and internal signals are related via the following equations:

$$\begin{cases} S_1(e_1) = e_2 - u_2 \\ S_2(e_2) = u_1 - e_1 \end{cases} \tag{47}$$

It is important to note that the individual BIBO stability of S_1 and S_2 is not sufficient for the BIBO stability of the connected closed-loop system. For instance, in the discrete-time setting of Fig. 3, suppose that $S_1 \equiv 1$ and $S_2 \equiv -1$, with $u_1(k) \equiv 1$ for all $k = 0, 1, \ldots$. Then, S_1 and S_2 are BIBO stable individually, but it can be easily verified that $y_1(k) = k \to \infty$ as the discrete-time variable k evolves. Therefore, a stronger condition describing the interaction of S_1 and S_2 is necessary.

Theorem 28 ***(Small Gain Theorem)*** If there exists four constants, L_1, L_2, M_1, M_2, satisfying $L_1L_2 < 1$, such that

$$\begin{cases} \|S_1(e_1)\| \le M_1 + L_1\|e_1\| \\ \|S_2(e_2)\| \le M_2 + L_2\|e_2\| \end{cases} \tag{48}$$

then

$$\begin{cases} \|e_1\| \le (1 - L_1L_2)^{-1}\left(\|u_1\| + L_2\|u_2\| + M_2 + L_2M_1\right) \\ \|e_2\| \le (1 - L_1L_2)^{-1}\left(\|u_2\| + L_1\|u_1\| + M_1 + L_1M_2\right) \end{cases} \tag{49}$$

where the norms $\|\cdot\|$ are defined over the spaces that the signals belong. Consequently, (48) and (49) together imply that if the system inputs (u_1 and u_2) are bounded then the corresponding outputs ($S_1(e_1)$ and $S_2(e_2)$) are bounded.

Note that the four constants, L_1, L_2, M_1, M_2, can be somewhat arbitrary (e.g., either L_1 or L_2 can be large) provided that $L_1L_2 < 1$, which is the key condition for the theorem to hold, which is used to obtain $(1 - L_1L_2)^{-1}$ in the bounds (49).

In the special case where the input–output spaces, U and Y, are both the L_2-space, a similar criterion based on the system passivity property can be obtained [11, 28]. In this case, an inner product between any two vectors in the space is defined by

$$\langle \xi, \eta \rangle = \int_{t_0}^{\infty} \xi^{\top}(\tau)\eta(\tau)d\tau$$

Theorem 29 ***(Passivity Stability Theorem)*** If there exists four constants, L_1, L_2, M_1, M_2, with $L_1 + L_2 > 0$, such that

$$\begin{cases} \langle e_1, S_1(e_1) \rangle \geq L_1 \|e_1\|_2^2 + M_1 \\ \langle e_2, S_2(e_2) \rangle \geq L_2 \|S_2(e_2)\|_2^2 + M_2 \end{cases} \tag{50}$$

then the closed-loop system (47) is BIBO stable.

As mentioned, the main disadvantage of this criterion is its over-conservativity in providing some sufficient conditions for the BIBO stability. One resolution is to transform the system into the Lur'e structure, and then apply the circle or Popov criterion under the sector condition (if it can be satisfied), which usually can lead to less-conservative stability conditions.

8.2 Contraction Mapping Theorem

The small gain theorem discussed above by nature is a kind of Contraction Mapping Theorem. The Contraction Mapping Theorem can be used to determine the BIBO stability property of a system described by a map in various forms, provided that the system (or the map) is appropriately formulated. The following is a typical (global) Contraction Mapping Theorem.

Theorem 30 ***(Contraction Mapping Theorem)*** If the operator norm of the input-output map S, defined on $\mathbf{R}^n$, satisfies $|||S||| < 1$, then the system equation

$$\mathbf{y}(t) = S(\mathbf{y}(t)) + \mathbf{c}$$

has a unique solution for any constant vector $\mathbf{c} \in \mathbf{R}^n$. This solution satisfies

$$\|\mathbf{y}\| \leq (1 - |||S|||)^{-1}\|\mathbf{c}\|$$

Moreover, the solution of the equation

$$\mathbf{y}_{k+1} = S(\mathbf{y}_k), \quad \mathbf{y}_0 \in \mathbf{R}^n, \quad k = 0, 1, \ldots$$

satisfies

$$\|\mathbf{y}_k\| \to 0 \quad \text{as} \quad k \to \infty$$

9 CONCLUDING REMARKS

This chapter has offered a brief introduction and description of the basic theory and methodology of the Lyapunov stability, orbital stability, structural stability, and input–output stability for nonlinear dynamical systems. More subtle details for stability analysis of general dynamical systems can be found in, for example [1, 6, 8–13, 15, 16, 24, 25, 28–30]. When control is explicitly involved, stability and stabilization issues are studied in [11, 14, 27, 31, 32], to name just a few.

Several important classes of nonlinear (control) systems have been left out in the above discussion of various stability issues: some general functional systems such as systems with time delays [33], measure ordinary differential equations such as systems with impulses [34, 35], and some weakly nonlinear systems like piecewise linear and switching (non)linear systems. Moreover, discussions on more advanced nonlinear systems, such as singular nonlinear systems (perhaps with time delays), infinite-dimensional (non)linear systems, spatiotemporal systems described by nonlinear partial differential equations, and nonlinear stochastic (control) systems, are beyond the scope of this elementary expository chapter.

Finally, for more recent studies of system stability theories, the reader is referred to [36, 37].

BIBLIOGRAPHY

1 D. R. Merkin, *Introduction to the Theory of Stability*, Springer-Verlag, New York, 1997.
2 A. M. Lyapunov, *The General Problem of Stability of Motion, 100 Anniversary*, Taylor & Francis, London, 1992.
3 G. Chen, G. Chen, and S. H. Hsu, *Linear Stochastic Control Systems*, CRC Press, Boca Raton, FL, 1995.
4 F. Brauer and J. A. Nohel, *The Qualitative Theory of Ordinary Differential Equations: An Introduction*, Dover Pub, New York, 1989.
5 G. Chen and X. Dong, *From Chaos to Order: Methodologies, Perspectives and Applications*, World Scientific Pub. Co., Singapore, 1998.
6 P. Glendinning, *Stability, Instability and Chaos*, Cambridge University Press, New York, 1994.
7 F. C. Hoppensteadt, *Analysis and Simulation of Chaotic Systems*, Springer-Verlag, New York, 1993.
8 N. P. Bhatia and G. P. Szegö, *Stability Theory of Dynamical Systems*, Springer-Verlag, Berlin, 1970.
9 W. Hahn, *Stability of Motion*, Springer-Verlag, Berlin, 1967.
10 D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations*, 2nd ed., Oxford University Press, New York, 1987.
11 H. K. Khalil, *Nonlinear Systems*, 2nd ed., Prentice-Hall, Upper Saddle River, NJ, 1996.
12 J. P. LaSalle, *The Stability of Dynamical Systems*, SIAM, Philadelphia, 1976.
13 S. Lefschetz, *Differential Equations: Geometric Theory*, 2nd ed., Dover Pub, New York, 1977.
14 N. Minorsky, *Theory of Nonlinear Control Systems*, McGraw-Hill, New York, 1969.
15 R. R. Mohler, *Nonlinear Systems: Vol. 1, Dynamics and Control*, Prentice-Hall, Englewood Cliffs, NJ, 1991.
16 P. C. Parks and V. Hahn, *Stability Theory*, Prentice-Hall, New York, 1981.
17 M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed., Prentice-Hall, Englewood Cliffs, NJ, 1993.
18 N. E. Zhukovskij, On solidity of motion, in *Collection of Works*, Vol. 1, Leningrad Press, Moscow, 1948, p. 67.
19 M. Y. Wu, A note on stability of linear time-varying systems, *IEEE Trans. on Auto. Contr.*: 162 (Apr. 1974).
20 F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, Springer-Verlag, Berlin, 1990.
21 V. Lakshmikantham, V. M. Matrosov, and S. Sivasundaram, *Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems*, Kluwer Academic Pub, Boston, 1991.
22 A. N. Michel and R. K. Miller, *Qualitative Analysis of Large Scale Dynamical Systems*, Academic Press, New York, 1977.
23 D. D. Siljack, *Large Scale Dynamic Systems: Stability and Structure*, North-Holland, Amsterdam, 1978.

24 R. Grimshaw, *Nonlinear Ordinary Differential Equations*, CRC Press, Boca Raton, FL, 1993.
25 K. S. Narendra and J. H. Taylor, *Frequency Domain Criteria for Absolute Stability*, Academic Press, New York, 1973.
26 D. P. Atherton, *Nonlinear Control Engineering*, Student ed., Van Nostrand Reinhold, New York, 1982.
27 R. J. P. de Figueiredo and G. Chen, *Nonlinear Feedback Control Systems: An Operator Theory Approach*, Academic Press, San Diego, 1993.
28 C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, Academic Press, New York, 1975.
29 L. Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, Springer-Verlag, New York, 1971.
30 S. Y. Pilyugin, *Introduction to Structurally Stable Systems of Differential Equations*, Birkhäser, Boston, 1992.
31 H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*, Springer-Verlag, New York, 1990.
32 E. D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, 2nd ed., Springer-Verlag, New York, 1998.
33 J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.
34 D. D. Bainov and P. S. Simeonov, *Systems with Impulse Effect: Stability, Theory and Applications*, Ellis Horwood Ltd., Chichester, 1989.
35 A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific Pub. Co., Singapore, 1995.
36 A. A. Martynyuk, ed., *Advances in Stability Theory at the End of the 20th Century*, Taylor & Francis, London, 2003.
37 G. Chen, *Nonlinear systems: Stability, Dynamics and Control*, World Scientific Pub. Co., Singapore, 2024.