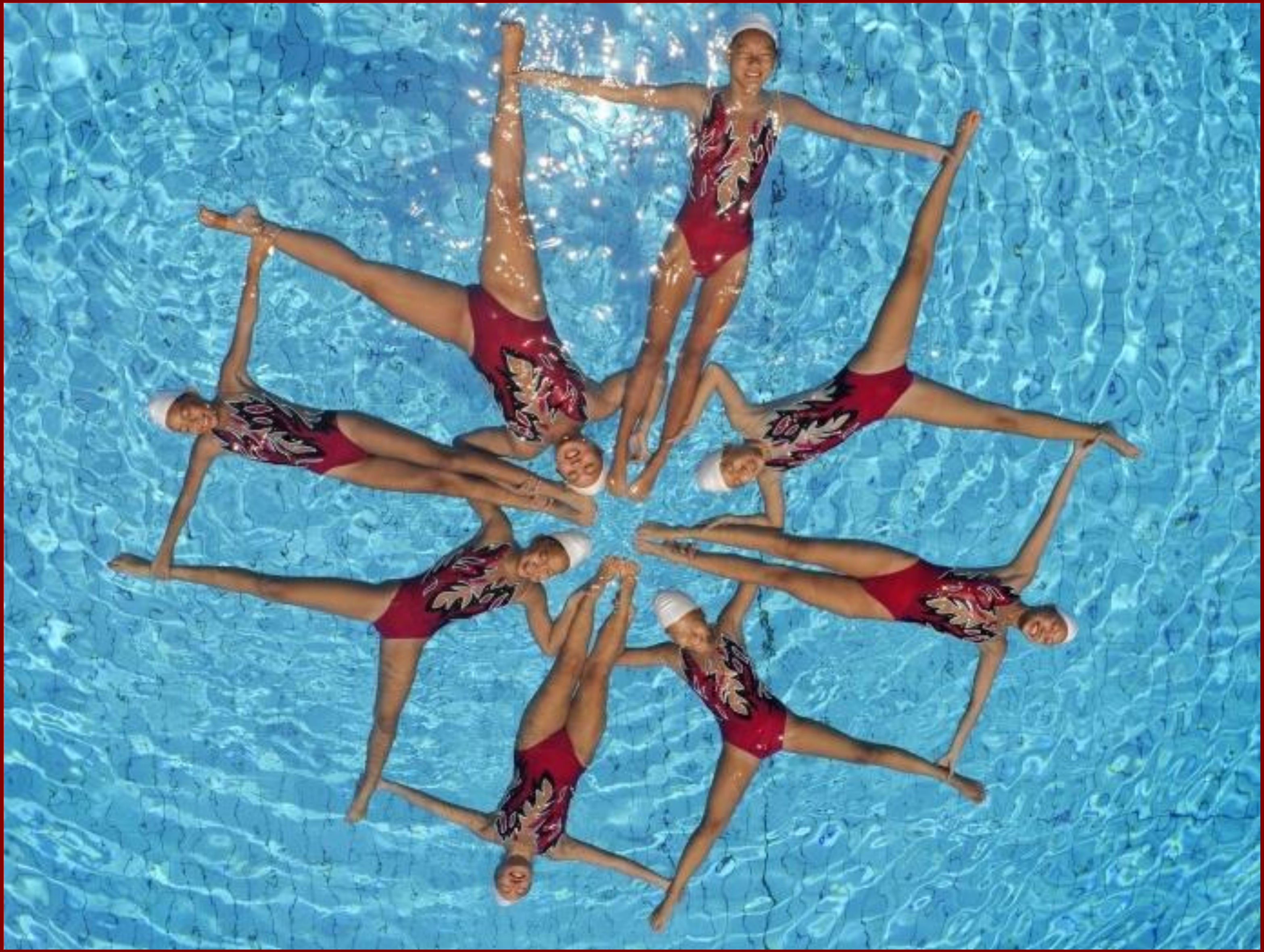
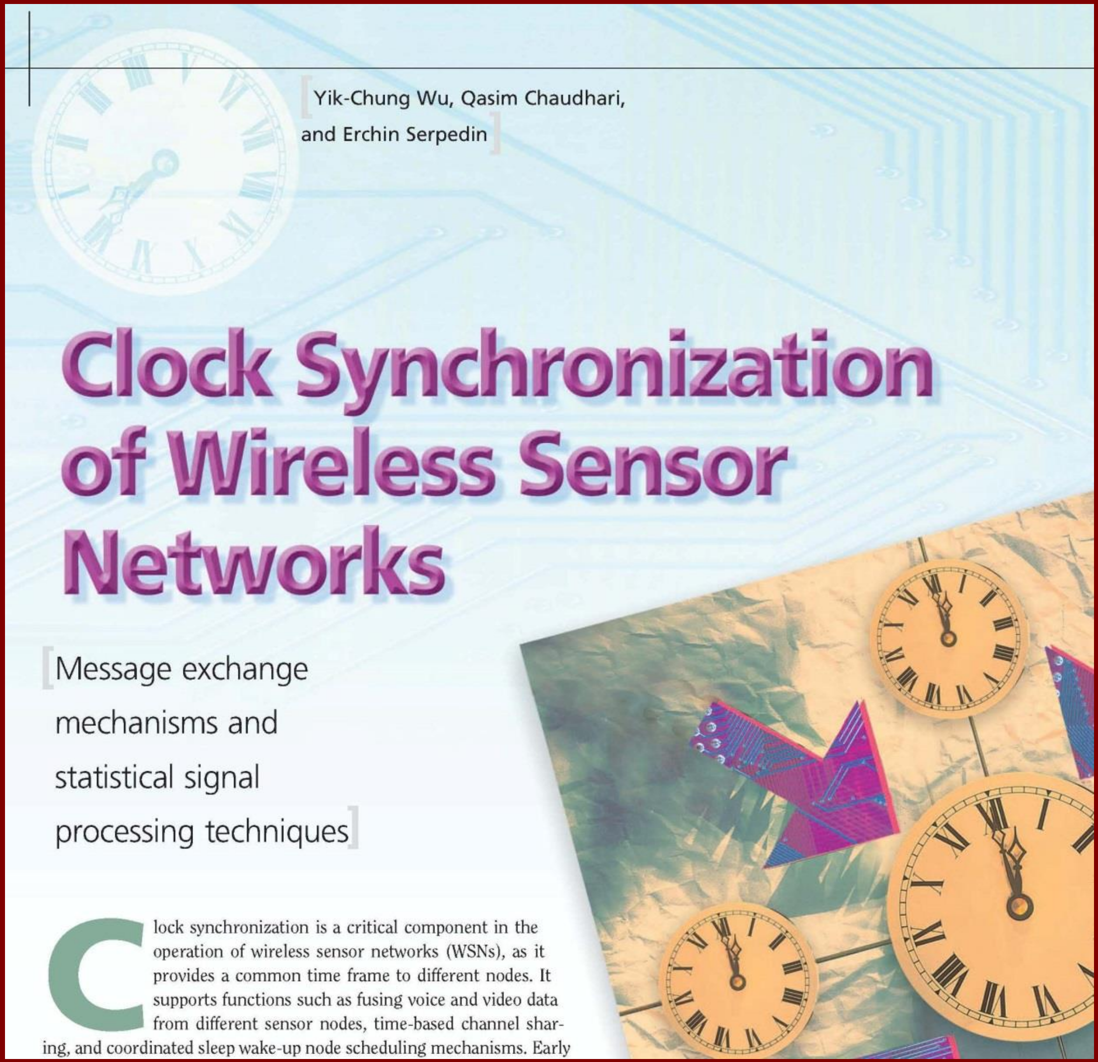
SYNC

GRChen / EE / CityU

Sync



Synchrony can be essential



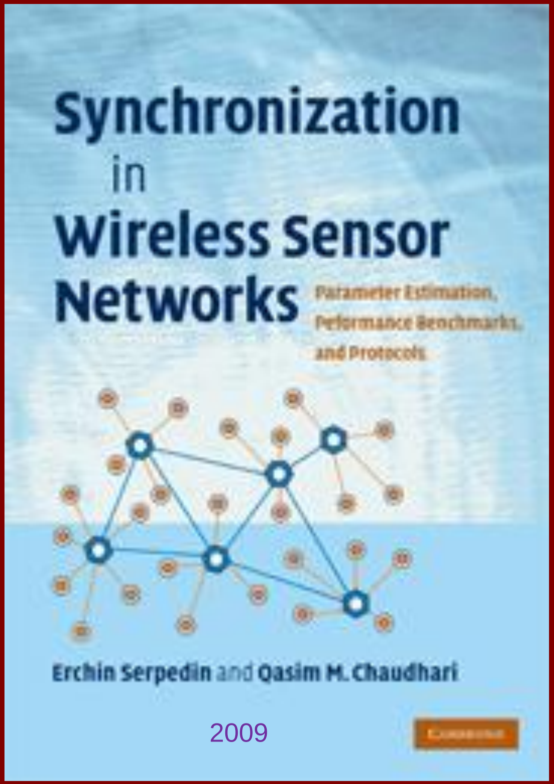
Yik-Chung Wu, Qasim Chaudhari, and Erchin Serpedin

Clock Synchronization of Wireless Sensor Networks

Message exchange mechanisms and statistical signal processing techniques

Clock synchronization is a critical component in the operation of wireless sensor networks (WSNs), as it provides a common time frame to different nodes. It supports functions such as fusing voice and video data from different sensor nodes, time-based channel sharing, and coordinated sleep wake-up node scheduling mechanisms. Early

IEEE Signal Processing Magazine (2012)



← "Clock synchronization is a critical component in the operation of wireless sensor networks, as it provides a common time frame to different nodes."

FULL SCREEN

MIT neuroscientists found that brain waves originating from the striatum (red) and from the prefrontal cortex (blue) become synchronized when an animal learns to categorize different patterns of dots.

Illustration: Jose-Luis Olivares/MIT

Synchronized brain waves enable rapid learning

MIT study finds neurons that hum together encode new information.

Anne Trafton | MIT News Office
June 12, 2014

Press Inquiries
PRESS MENTIONS

Contents

- Network synchronization and criteria
- Network spectra and synchronizability
- Networks with best synchronizability
- Networks with good controllability and strong robustness against attacks

A General Dynamical Network Model

An undirected network:

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij} H(x_j)$$

$x_i \in R^n$ $\quad i = 1,2,\ldots,N$

$f(.)$ – Lipschitz Coupling strength $c > 0$

$A = [a_{ij}]$ – Adjacency matrix H – Coupling matrix function

If there is a connection between node i and node j $(j \neq i)$,
then $a_{ij} = a_{ji} = 1$; otherwise, $a_{ij} = a_{ji} = 0$ and $a_{ii} = 0$, $i = 1, \ldots, N$

Laplacian $L = D - A$, $D = \mathrm{diag}\{d_1, \ldots, d_N\}$ (node degrees)

For connected networks, eigenvalues: $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$

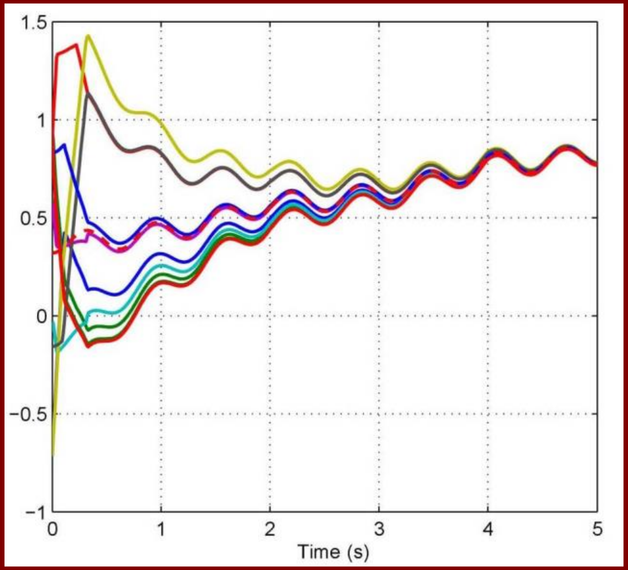
Network Synchronization

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij} H(x_j) \qquad x_i \in R^n \qquad i = 1,2,\ldots,N$$

(Complete state)

Synchronization: $$\lim_{t\to\infty} \| x_i(t) - x_j(t) \|_2 = 0, \qquad i, j = 1,2,\cdots,N$$

Numerical example:



Network Synchronization: Analysis

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij} H(x_j)$$

$x_i \in R^n$

$i = 1, 2, \ldots, N$

Put all equations together with $\mathrm{x} = [x_1^T, x_2^T, \ldots, x_N^T]^T$

Then linearize it at equilibrium s :

$$\dot{\mathrm{x}} = [I_N \otimes [\nabla f(s)]] - c[A \otimes [\nabla H(s)]]\mathrm{x}$$

$f(.)$ Lipschitz, or assume: $\| \nabla f(s) \| \leq M$ ⇒ Only cA or $\{c\lambda_i\}$ is important

After linearization, perform local analysis

Network Synchronization: Criteria

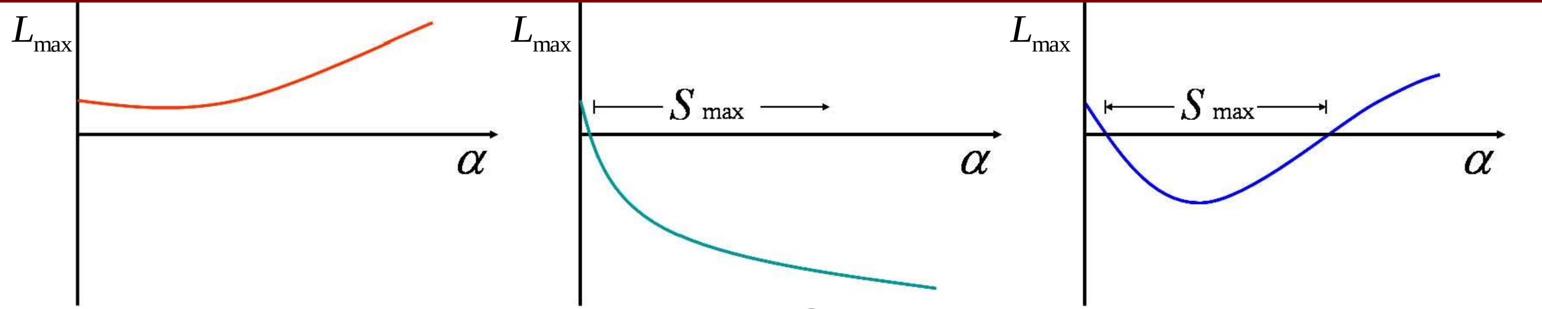
$$\dot{\mathrm{x}} = [I_N \otimes [\nabla f(s)]] - c[A \otimes [\nabla H(s)]]\mathrm{x}$$

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$$

Master stability equation: (L.M. Pecora and T. Carroll, 1998)

$$\dot{\mathrm{y}} = [[\nabla f(s)] - \alpha[\nabla H(s)]]\mathrm{y}, \qquad \alpha = \inf\{c\lambda_i(s), i = 2,3,\ldots,N\}$$

Maximum Lyapunov exponent $L_{\max}$ is a function of α

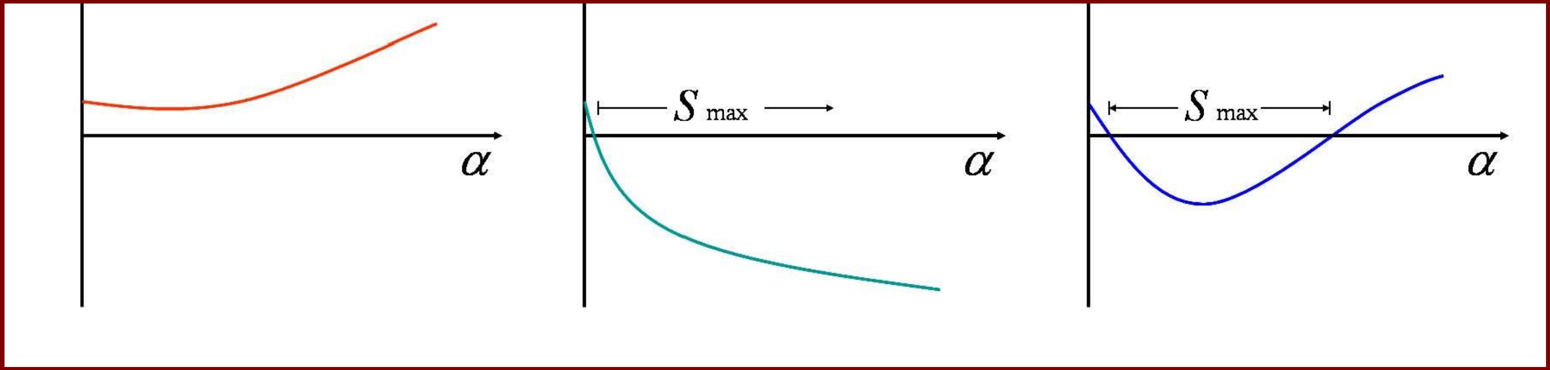


$S_{\max}$ -- sync region

Synchronizing: if $S_1 = (\alpha_1, \infty)$, $0 \leq \alpha_1 < \alpha < \infty$ or if $S_2 = (\alpha_2, \alpha_3)$, $0 \leq \alpha_2 < \alpha < \alpha_3 < \infty$

Network Synchronization: Criteria

Recall: Laplacian eigenvalues: $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N$



synchronizing if $0 \le \alpha_1 < c\lambda_2 < \infty$ or if $0 < \alpha_2 < \dfrac{\lambda_2}{\lambda_N} < \alpha_3$

Case I:
No sync

Case II:
Sync region
$S_1 = (\alpha_1, \infty)$

Case III:
Sync region
$S_2 = (\alpha_2, \alpha_3)$

Case IV:
Union of intervals

λ_2 bigger is better

$\dfrac{\lambda_2}{\lambda_N}$ bigger is better

In retrospect

Synchronizability characterized by Laplacian eigenvalues:

1. unbounded region (X.F. Wang and GRC, 2002)

$$\lambda_2 , \qquad 0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N \qquad S_1 = (\alpha_1, \infty)$$

2. bounded region (M. Banahona and L.M. Pecora, 2002)

$$\lambda_2 / \lambda_N , \quad 0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N \qquad S_2 = (\alpha_2, \alpha_3)$$

3. union of several disconnected regions

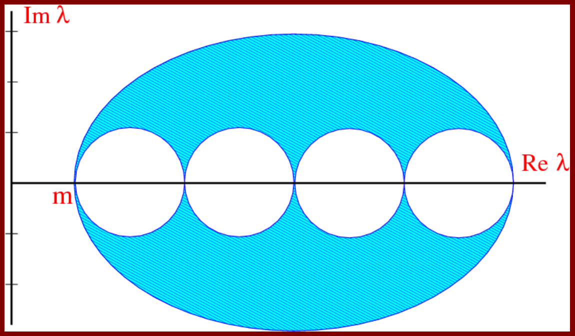
$$S_m = (\alpha_1, \alpha_2) \cup (\alpha_3, \alpha_4) \cup \cdots \cup (\alpha_m, \infty)$$

(A. Stefanski, P. Perlikowski, and T. Kapitaniak, 2007)
(Z.S. Duan, C. Liu, GRC, and L. Huang, 2007 - 2009)

Spectra of Networks

- Some theoretical results
- Relation with network topology
- Role in network synchronizability

Spectrum



Theoretical Bounds of Laplacian Eigenvalues

λ_2 bigger is better $\frac{\lambda_2}{\lambda_N}$ bigger is better

Concern: upper and lower bounds of Laplacian eigenvalues

There are many classical results in graph special analysis

Graph Theory Textbooks

For example:

P. V. Mieghem, Graph Spectra for Complex Networks (2011)

Theoretical Bounds of Laplacian Eigenvalues

Node-degree sequence eigenvalue sequence (both in increasing order)

$$d = (d_1, d_2, \ldots, d_N)^T \qquad \lambda = \left(\lambda_1, \lambda_2, \ldots, \lambda_N\right)^T$$

$$\frac{\| \lambda - d \|_2}{\| d \|_2} \leq \frac{\sqrt{\| d \|_1}}{\| d \|_2} \leq \sqrt{\frac{N}{\| d \|_1}}$$

Distribution:

For any node-degree d_i there exists a $\lambda_* \in \left\{\lambda_j \mid j = 1,2,\ldots,N\right\}$

such that

$$d_i - \sqrt{d_i} \leq \lambda_* \leq d_i + \sqrt{d_i}, \quad i = 1,2,\ldots,N$$

C. Zhan, GRC and L. Yeung (2010)

$$\frac{\| \lambda - d \|_2}{\| d \|_2} \le \frac{\sqrt{\| d \|_1}}{\| d \|_2} \le \sqrt{\frac{N}{\| d \|_1}}$$

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)^T$$

$$d = (d_1, d_2, \ldots, d_N)^T$$

Lemma (Hoffman-Wielanelt, 1953)

For matrices $B - C = A$:
$$\sum_{i=1}^{n} | \lambda_i(B) - \lambda_i(C) |^2 \le \| A \|_F^2$$

Frobenius Norm

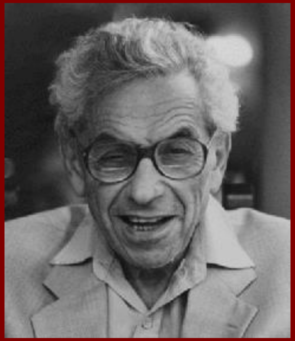
→ For Laplacian $L = D - A$:
$$\sum_{i=1}^{n} | \lambda_i(L) - d_i |^2 \le \| A \|_F^2$$

$$\| A \|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} | a_{ij} |^2 = \sum_{i=1}^{n} d_i$$

→
$$\left(\frac{\| \lambda - d \|_2}{\| d \|_2} \right)^2 \le \frac{\sum d_i}{\sum d_i^2} \left(= \frac{\sqrt{\| d \|_1}}{\| d \|_2} \right) \le \frac{\sum d_i}{\left(\sum d_i\right)^2 / N} = \frac{N}{\| d \|_1}$$

Cauchy Inequality

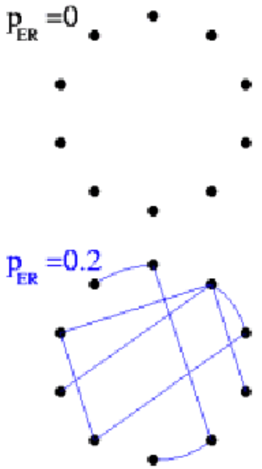
Random Graphs



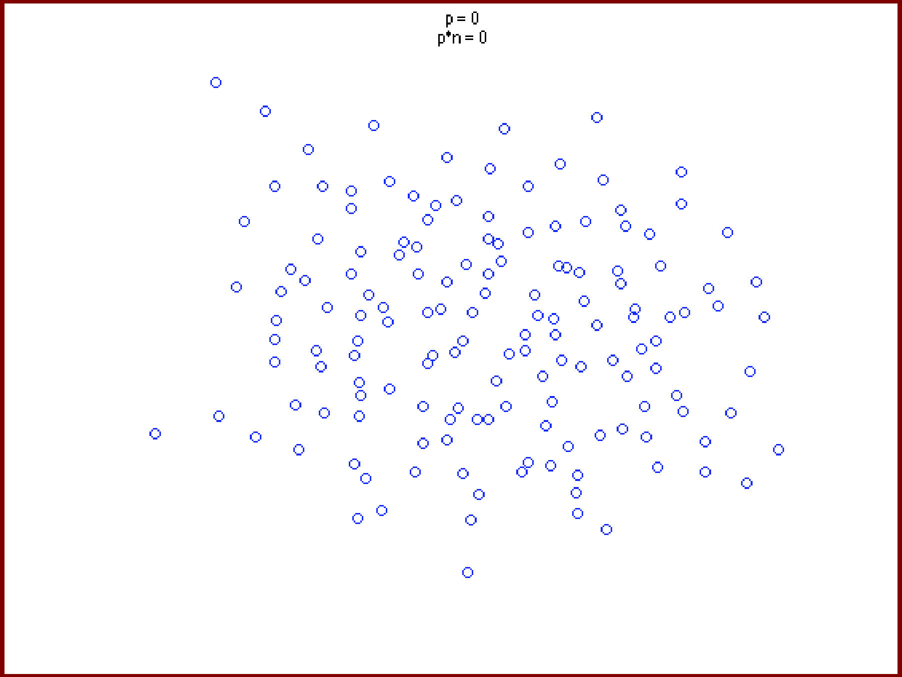
Erdős-Rényi

(Publ. Math. Inst. Hung. Acd. Sci. **5,** 17 (1960))

N nodes, each pair of node is connected with probability p



Example →



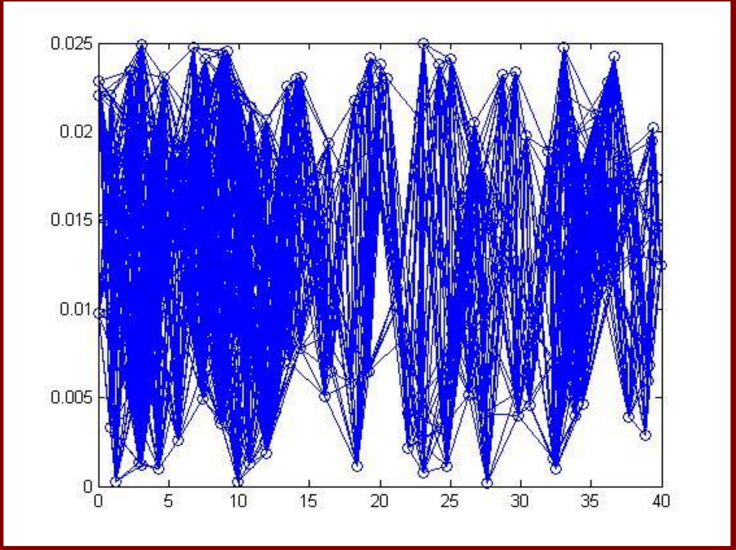
Rectangular Random Graphs

N nodes are randomly uniformly and independently distributed in a unit rectangle $[a,b]^2 \subset R^2$ with $a \cdot b = 1$

(It can be generalized to higher-dimensional setting)

Two nodes are connected by an edge if they are inside a disc of radius $r > 0$

Example:
$N = 200$
$a = 40$
$r = 2.5$



Theorem: Eigenvalue ratio is bounded by

$$\frac{1}{(N-1)N^2} \leq \frac{\lambda_2}{\lambda_N} \leq \frac{8(ar)^2}{a^4+1} \log_2^2 N$$

Lower bound:

The worst case: all nodes are located on the diagonal

$$\lambda_2 \geq \frac{1}{ND} \geq \frac{1}{N(N-1)} \qquad (D - \text{diameter})$$

and

$$\lambda_N \leq N$$

E. Estrada and GRC (2015)

$$\frac{1}{(N-1)N^2} \le \frac{\lambda_2}{\lambda_N} \le \frac{8(ar)^2}{a^4+1}\log_2^2 N$$

Upper bound:

Lemma 1: Diameter D = diagonal length / r

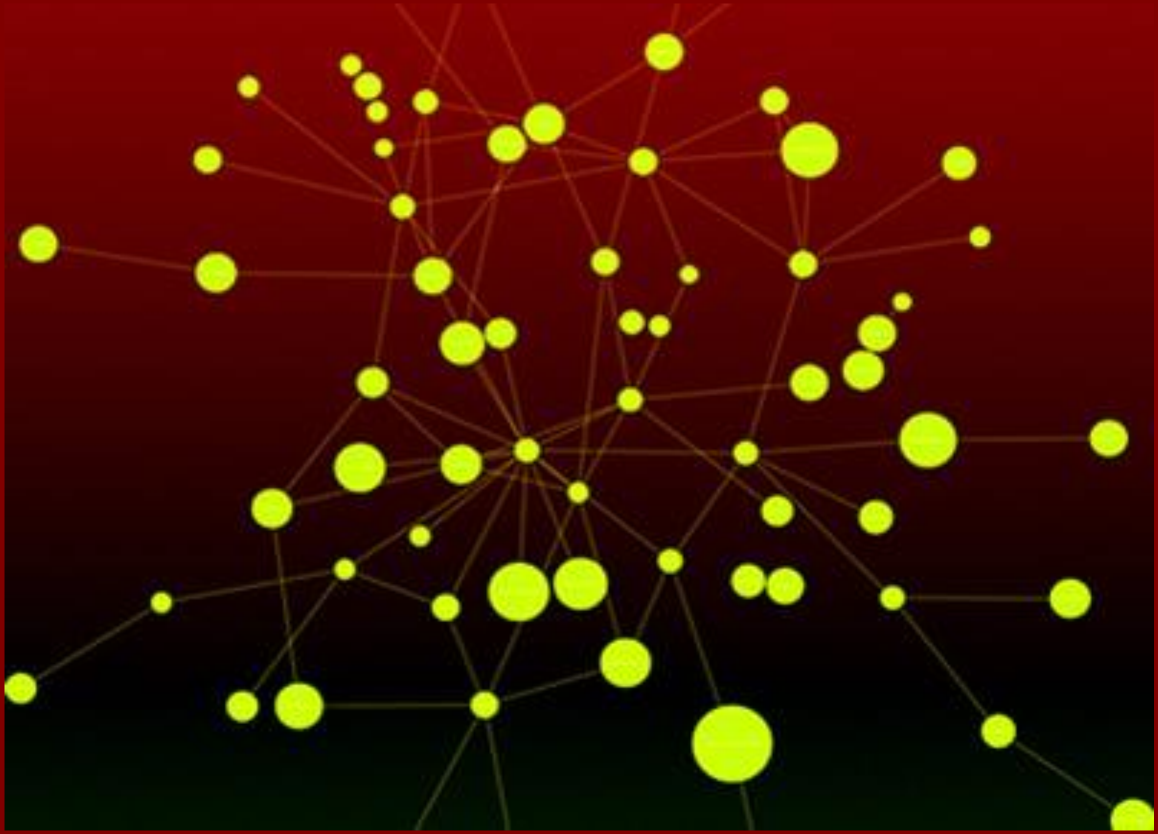
→ $$D \ge \left\lfloor \frac{\sqrt{a^4+1}}{ar} \right\rfloor$$

Lemma 2: Based on a result of Alon-Milman (1985)

→ $$\lambda_2 \le \frac{8k_{\max}}{D^2}\log_2^2 N$$

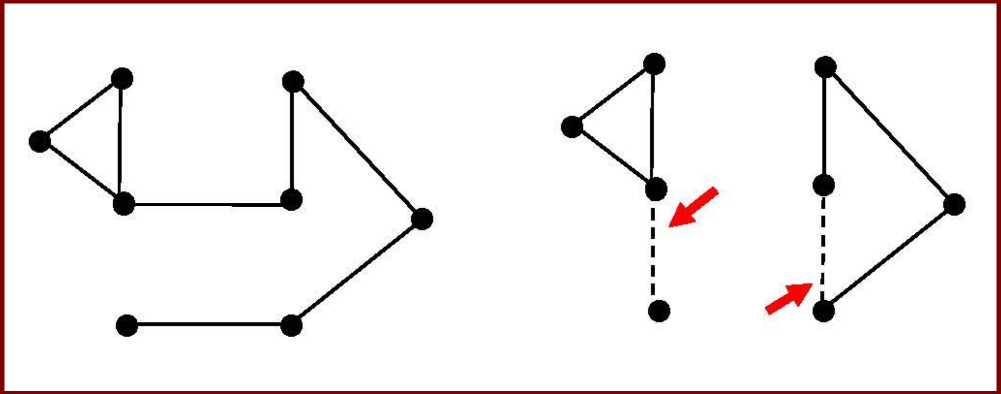
E. Estrada and GRC (2015)

Network Topology and Synchronizability



Topology Determines Synchronizability?

Answer: ***Yes*** **or** ***No***



This makes the situation complicated and the study difficult

More Edges → Better Synchronizability ?

Answer: *Yes* or *No*

- **Lemma:** For any given connected undirected graph G , by adding any new edge e , one has

$$\lambda_i(G+e) \geq \lambda_i(G), \qquad i = 1,2,..,N$$

- Note:

$$\lambda_2 \longrightarrow$$

$$\frac{\lambda_2 \rightarrow}{\lambda_N \longrightarrow} \quad \Downarrow \qquad\qquad \frac{\lambda_2 \longrightarrow}{\lambda_N \rightarrow} \quad \Uparrow$$

This also makes the situation complicated and the study difficult

Z.S. Duan, GRC and L. Huang (2007)

What Topology → Good Synchronizability ?

Example:

Given Laplacian

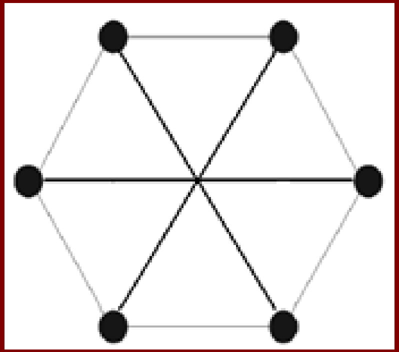
$$L = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & -1 & 3 & 0 & -1 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & -1 & -1 & 4 & -1 \\ -1 & 0 & -1 & -1 & -1 & 4 \end{bmatrix}$$

Q: How to replace 0 and -1 (while keeping the connectivity and all row-sums = 0), such that λ_2 / λ_N = maximum ?

Answer:

$$L^* = \begin{bmatrix} 3 & -1 & 0 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 0 & -1 & 3 \end{bmatrix}$$

$\rightarrow$ λ_2 / λ_N = maximum



Observation:

Homogeneity + Symmetry

Problem

- With the same numbers of node and edges, while keeping the connectivity, **what kind of network has the best possible synchronizability?**

$$\underset{A\in A^*}{Max}\frac{\lambda_2}{\lambda_N} \qquad A^*\text{- set of } N\times N \text{ adjacency matrices}$$

Such that $\sum_{i=1}^{N} d_i = N\bar{k}$ and $\lambda_2 > 0$

(total degree = constant) (connected)

- Computationally, this is NP-hard:

$$\max_{A\in A^*}\frac{\lambda_2}{\lambda_N} = \max_{A\in A^*}\left\{ \min_{x^T e=0, x\neq 0} \frac{x^T[D-A]x}{x^T x} \Big/ \max_{x^T e=0, x\neq 0} \frac{x^T[D-A]x}{x^T x} \right\}$$

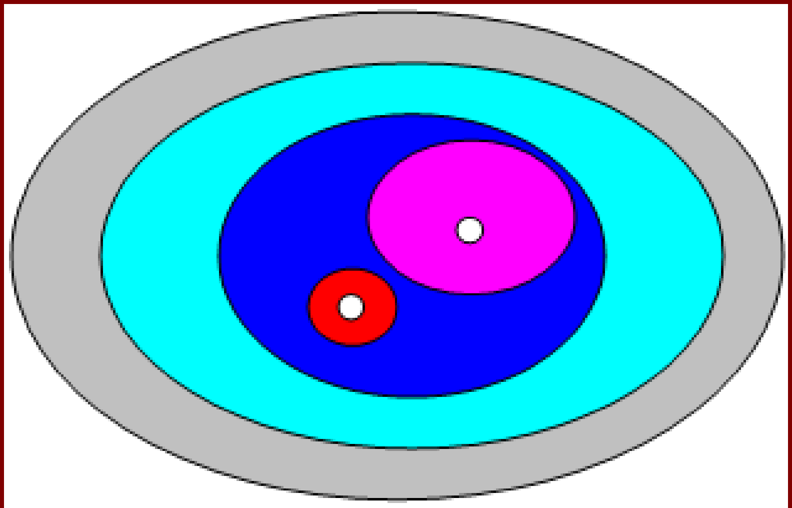
Our Approach

- **Homogeneity + Symmetry**
- **Same node degree** $d_1 = \cdots = d_N$
- **Shortest average path length** $l_1 = \cdots = l_N$
- **Shortest path-sum** $l_i = \sum_{j \neq i} l_{ij}$
- **Longest girth** $g_1 = \cdots = g_N$

D.H. Shi, GRC, W.W.K. Thong and X. Yan (2013)

Non-Convex Optimization

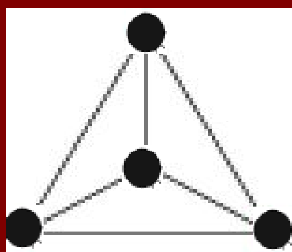
Illustration:

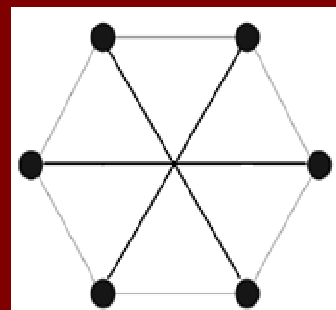


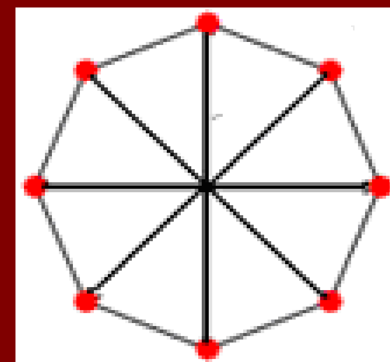
White：Optimal solution location

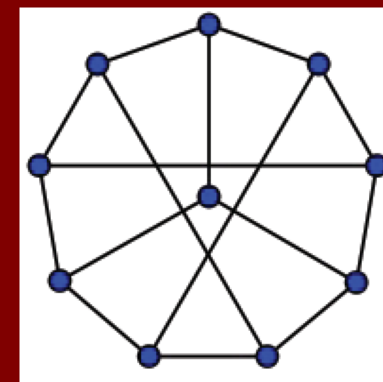
Grey：networks with same numbers of nodes and edges
Green：degree-homogeneous networks
Blue：networks with maximum girths
Pink：possible optimal networks
Red：near homogenous networks

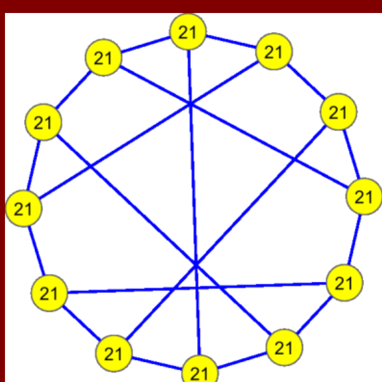
Optimal 3-Regular Networks

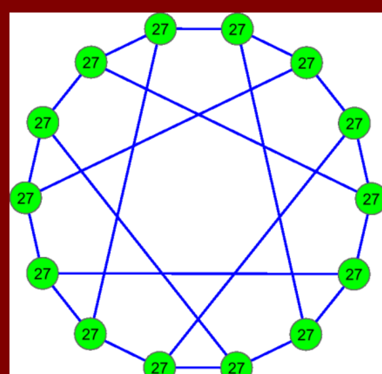


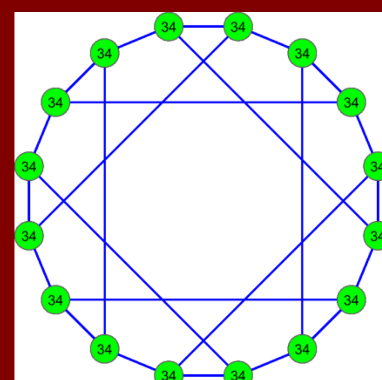


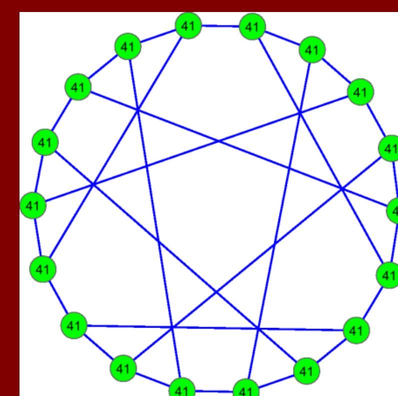




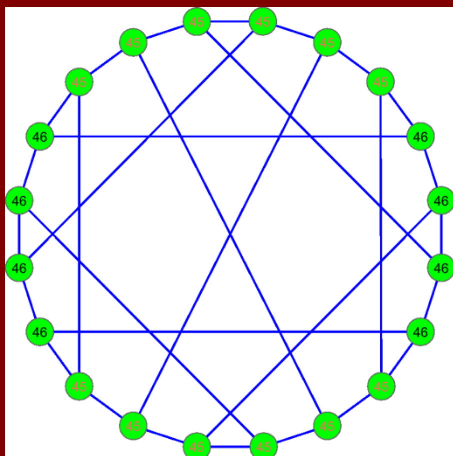


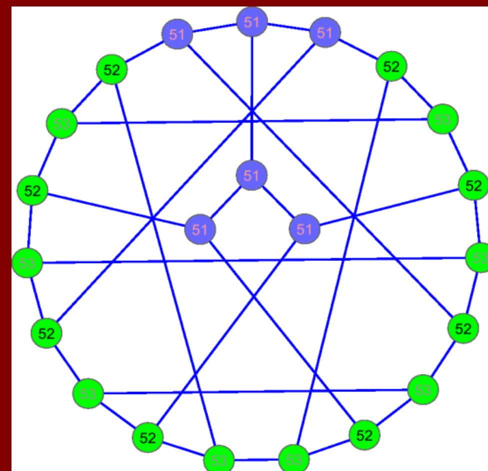


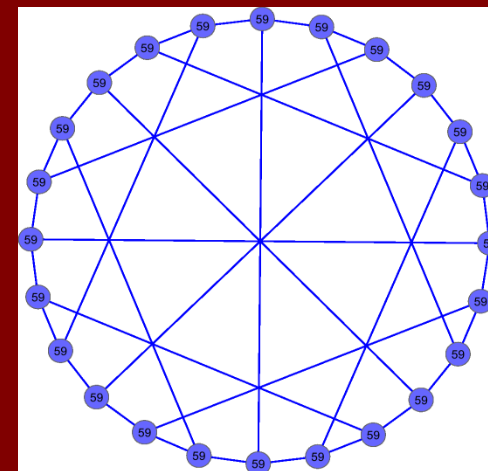


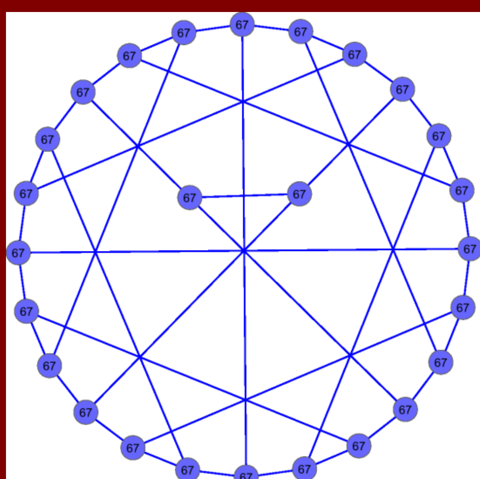


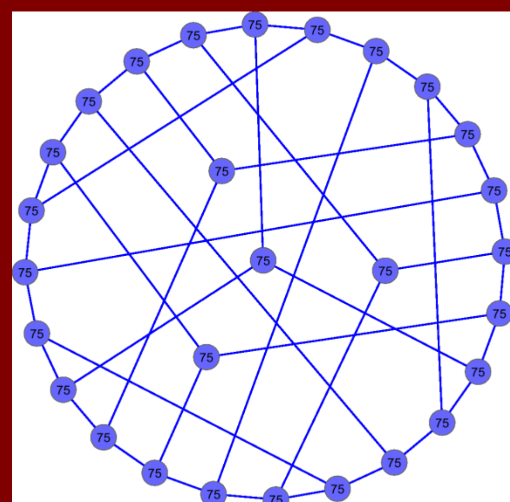
Optimal 3-Regular Networks

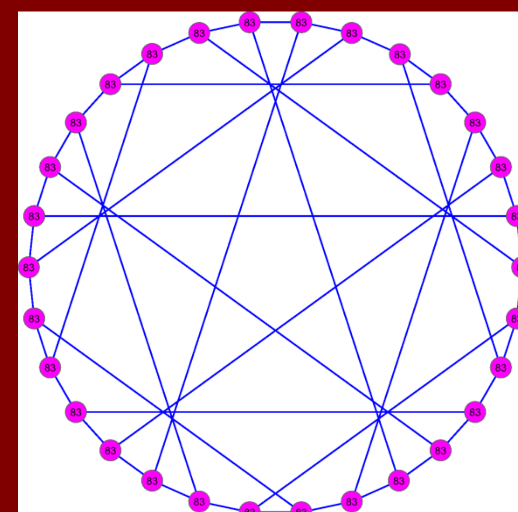












Open Problems

Looking for optimal solutions:

Given: $\begin{bmatrix} d_1 & -1 & 0 & & -1 \\ -1 & d_2 & & & 0 \\ 0 & & \ddots & & \\ & & & \ddots & 0 \\ -1 & 0 & & 0 & d_N \end{bmatrix}$ Move which -1 $\begin{bmatrix} d_1 & -1 & 0 & & -1 \\ -1 & d_2 & & & 0 \\ 0 & & \ddots & & \\ & & & \ddots & 0 \\ -1 & 0 & & 0 & d_N \end{bmatrix}$ can maximize $\dfrac{\lambda_2}{\lambda_N}$?

Constraints:
keeping the graph connectivity and all row-sums = 0

Where to add -1 $\begin{bmatrix} d_1 & -1 & 0 & & -1 \\ -1 & d_2 & & \boxed{-1} & 0 \\ 0 & & \ddots & & \\ & \boxed{-1} & & \ddots & 0 \\ -1 & 0 & & 0 & d_N \end{bmatrix}$ can maximize $\dfrac{\lambda_2}{\lambda_N}$?

Where to delete -1 …………………… can maximize $\dfrac{\lambda_2}{\lambda_N}$?

And so on …… ???

Multiplex Congruence Networks of Natural Numbers

- Good Sync-Controllability
- Strong Robustness against Attacks

Chinese Remainder Theorem

三三数之剩二，五五数之剩三，七七数之剩二。

$$\begin{cases} x \equiv 2 \pmod{3} \\ x \equiv 3 \pmod{5} \\ x \equiv 2 \pmod{7} \end{cases}$$

(Congruence Equations)

《孙子歌诀》

三人同行七十稀，五树梅花廿一枝，七子
团圆正半月，除百零五便得知。

$N = 3\times5\times7 = 105$

$(2\times70+3\times21+2\times15)/105$ with remainder 23 （Answer： $x = 23$）

Notation: Let $n\ (> 1)$, $x\ (> n)$, $a\ (< n)$ be integers

If n is divisible by $(x - a)$, then x and a are *congruent modulo* n, denoted as $x \equiv a \pmod{n}$

Chinese Remainder Theorem

Let $n_1, \ldots, n_k$ (all >1) be integers

If n_i are pairwise coprime, then for integers $a_1, \ldots, a_k$, there exist infinitely many integers x satisfying

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ &\vdots \\ x &\equiv a_k \pmod{n_k} \end{aligned}$$

(congruence)

And, any two such x are congruent modulo $N = n_1 \times \ldots \times n_k$

(Only natural numbers $a_1, \ldots, a_k$ are discussed here)

Congruence Networks

Given a natural number r, there exist infinitely many pairs of natural numbers (a,m) satisfying $a \equiv r \pmod{m}$

Example：

For $r = 2$, one has $(a,m) = (3,5), (5,7), (7,11), \ldots$
namely, $3 \equiv 2 \pmod{5}, 5 \equiv 2 \pmod{7}, 7 \equiv 2 \pmod{11}, \ldots$

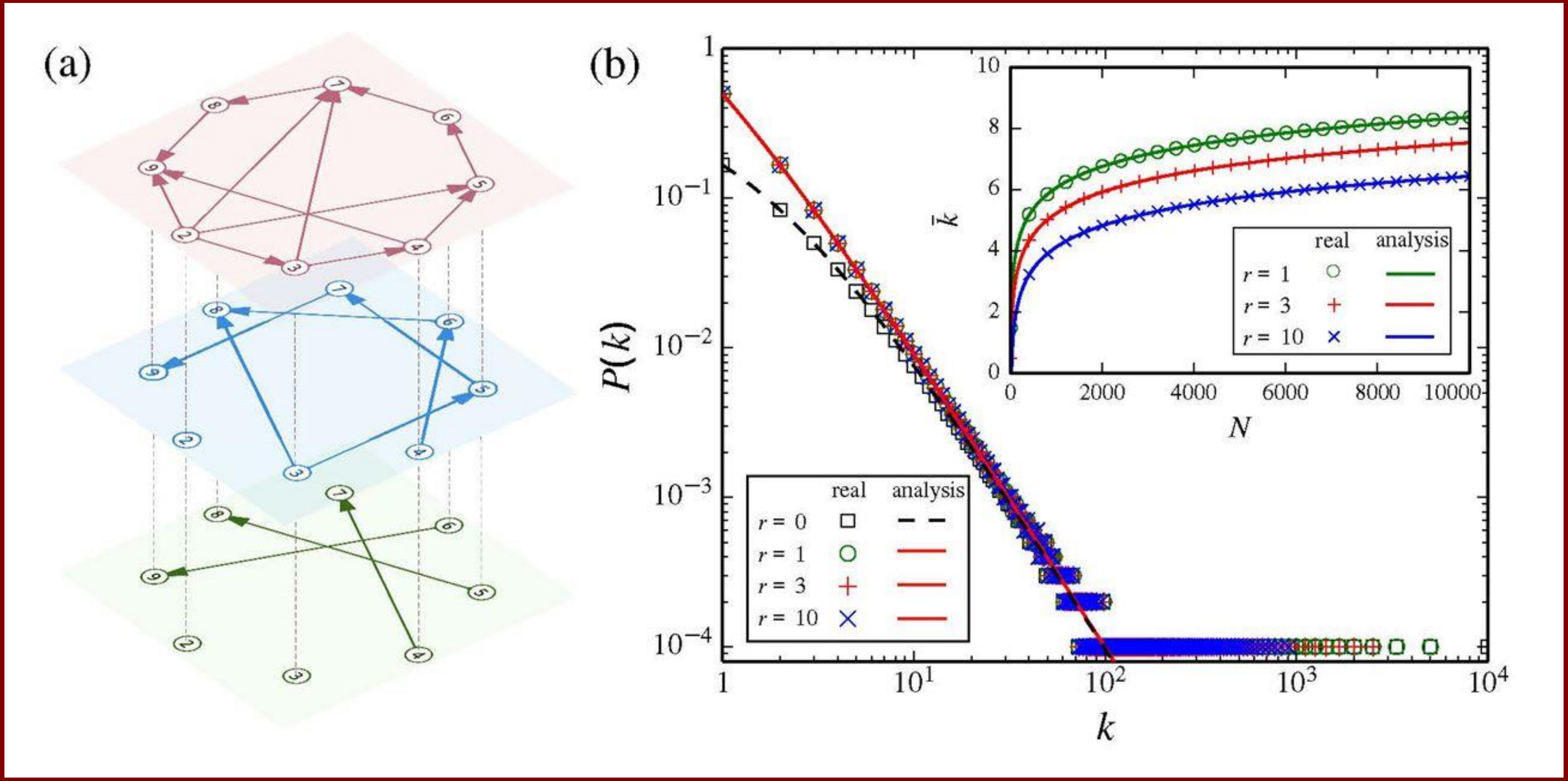
Connecting $3 \rightarrow 5, 5 \rightarrow 7, 7 \rightarrow 11, \ldots, \ldots \rightarrow N$
or $3 \leftarrow 5, 5 \leftarrow 7, 7 \leftarrow 11, \ldots, \ldots \leftarrow N$

yields a directed congruence network of N nodes

Different r yields different Multiplex Congruence Network (MCN), denoted as $G(r, N)$

X.Y. Yan, W.X. Wang, GRC and D.H. Shi (2015)

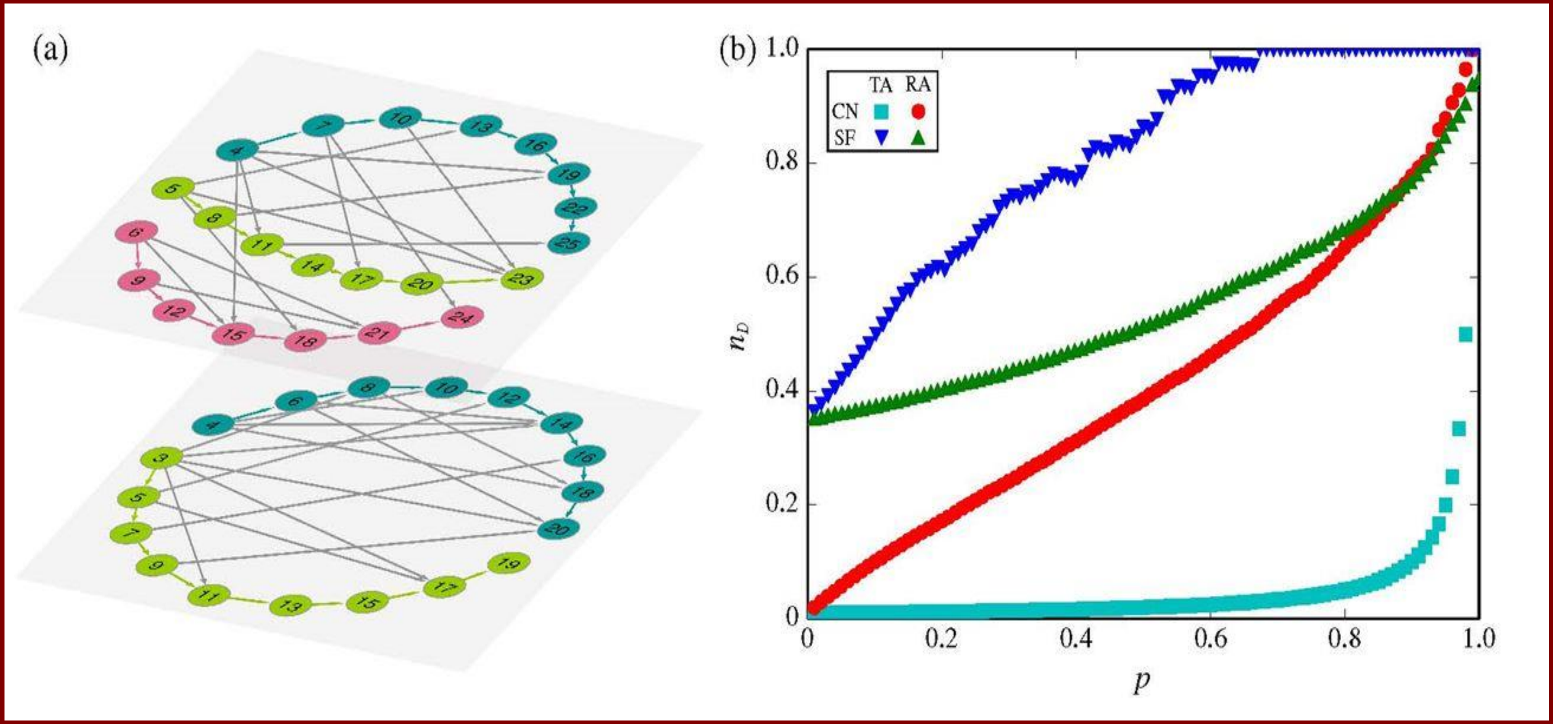
Example



(a) MCN: $G\,(r = \{1,2,3\},9)$ (b) Degree distribution, $N = 10000$

MCNs are scale-free networks with node-degree distribution $P(k) \sim k^{-2}$

Sync-Controllability and Robustness



(a) MCNs have chain-structures (b) Number of control nodes needed: n_D

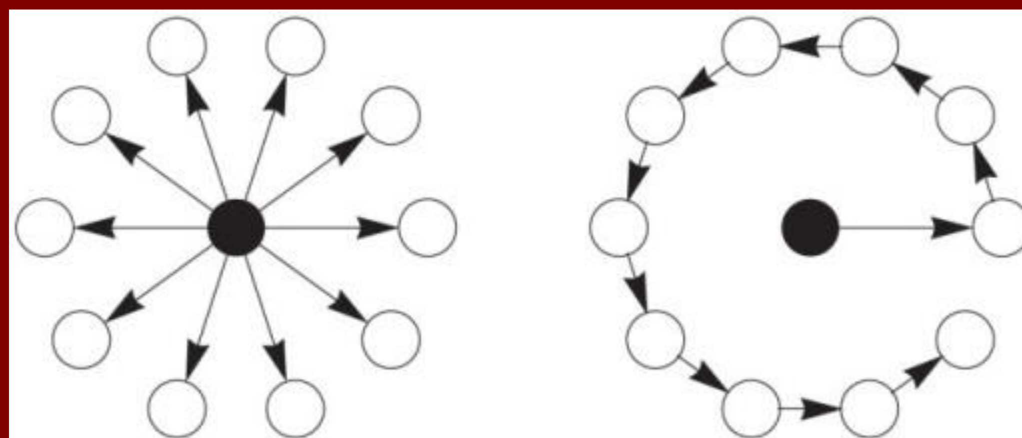
Robustness against attacks:

CN = Congruence Network (N = 100); SF = Scale-free Network (N = 100)

TA = Targeted Attack; RA = Random Attack

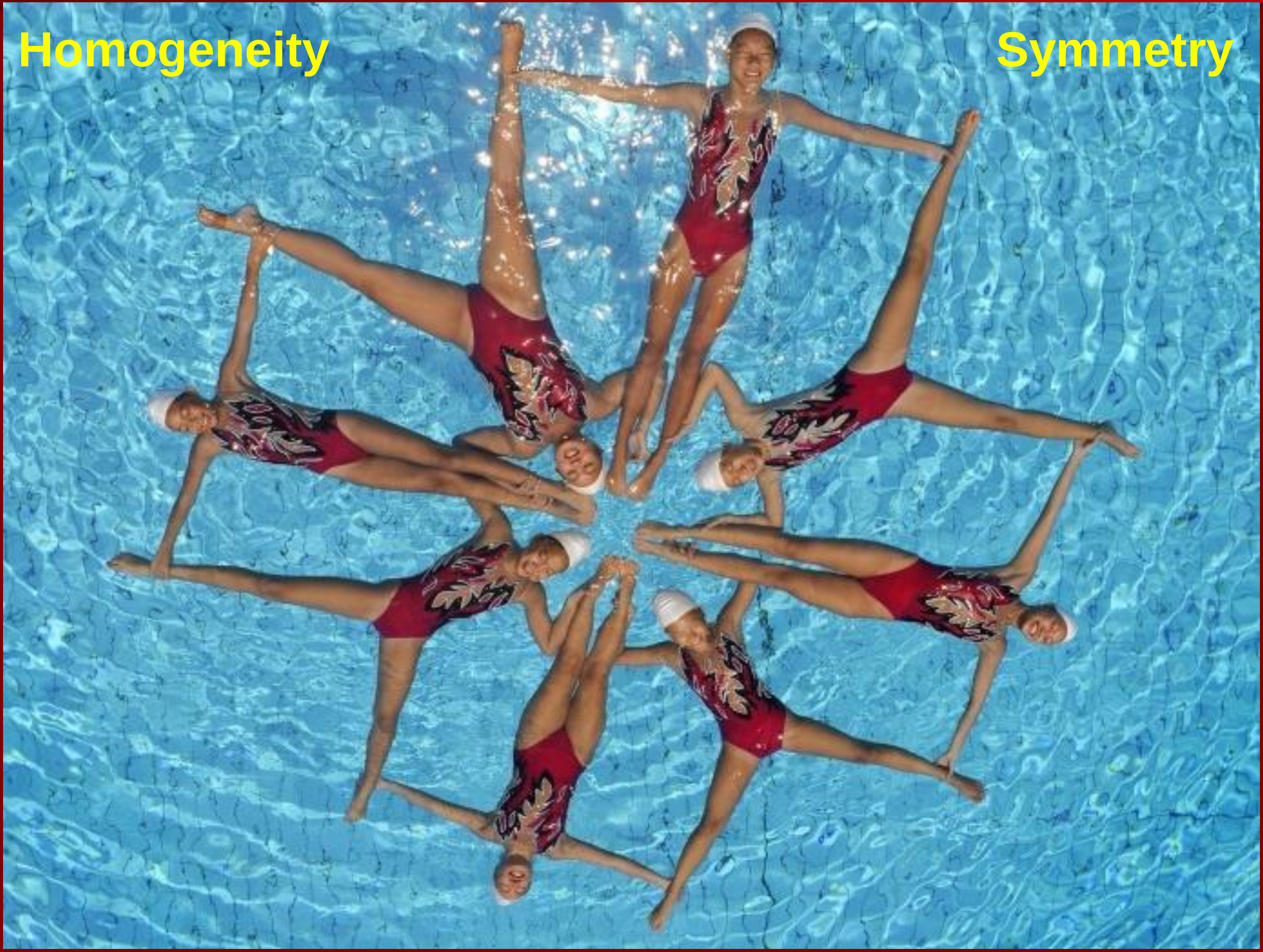
Graphical Explanation

Chain structure is good for both controllability and robustness against attacks



Controllability: ● **is Controller** (Driver Node)

Thank You !



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