Detailed Proof of Lemmas and Theorems

1 Proof of Lemma 2

Proof Define the formation errors $\tilde{x}_i(t) = x_i(t) - x_0(t) - x_i^*, i = 1, 2, \cdots N$, with $\tilde{x}_0(t) = -x_0^* = 0$. The Filippov solution of $\tilde{x}_i(t)$ is defined as the absolutely continuous solution of the differential inclusion

$$\dot{\tilde{x}}_i(t) \in \mathcal{K}\left[f_i(t, x_i(t)) - f_0(t, x_0(t)) - \alpha \, \mathrm{sgn}\left\{\sum_{j \in \mathcal{N}_i} a_{ij}[\tilde{x}_i(t) - \tilde{x}_j(t)]\right\}\right], \forall\, i = 1, 2, \cdots, N. \quad (1)$$

Based on Assumption 1, one follower must receive information from other followers or the leader, namely, it is connected with other followers or the leader. Define $\tilde{x}^+(t)$ as the maximal formation error component which is connected with non-maximal error components of the followers or connected with the component of the leader. Similarly, define $\tilde{x}^-(t)$ as the minimal formation error component which is connected with non-minimal error components of the followers or connected with the component of the leader. Suppose that, at any time t, $\tilde{x}^+(t)$ is the kth error component of agent i, and $\tilde{x}^-(t)$ is the lth error component of agent j, where $i, j \in \{1, 2, \cdots, N\}$, $k, l \in \{1, 2, \cdots, n\}$. The Filippov solutions of $\tilde{x}^+(t)$ and $\tilde{x}^-(t)$ can be described by

$$\begin{aligned} \dot{\tilde{x}}^+(t) \in& \mathcal{K}\left[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha \, \mathrm{sgn}\left\{\sum_{r \in \mathcal{N}_i} a_{ir}[\tilde{x}^+(t) - \tilde{x}_r^k(t)]\right\}\right], \\ \dot{\tilde{x}}^-(t) \in& \mathcal{K}\left[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) - \alpha \, \mathrm{sgn}\left\{\sum_{s \in \mathcal{N}_j} a_{js}[\tilde{x}^-(t) - \tilde{x}_s^l(t)]\right\}\right]. \end{aligned} \quad (2)$$

Based on Assumptions 2 and 3, for any $i = 1, 2, \cdots, N$ and each $t \in \boldsymbol{R}^+$, one has

$$
\begin{aligned}
& \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \| \\
&= \| f_i(t, x_i(t)) - f_i(t, x_0(t)) + f_i(t, x_0(t)) + f_0(t, x_0(t)) \| \\
&\leq \| f_i(t, x_i(t)) - f_i(t, x_0(t)) \| + \| f_i(t, x_0(t)) \| + \| f_0(t, x_0(t)) \| \\
&\leq \| f_i(t, x_i(t)) - f_i(t, x_0(t)) \| + \| f_i(t, x_0(t)) - f_i(t, x_i^E) \| + \| f_0(t, x_0(t)) - f_0(t, x_0^E) \| \\
&\leq L_J^F \| x_i(t) - x_0(t) \| + L_J^F \| x_0(t) - x_i^E \| + L_J^L \| x_0(t) - x_0^E \| \\
&\leq L_J^F \Big(\| x_i(t) - x_0(t) - x_i^* \| + \| x_i^* \| + \| x_0(t) - x_i^E \| \Big) + L_J^L \| x_0(t) - x_0^E \| \\
&\leq L_J^F \Big(\sqrt{n} \max\{| \tilde{x}^+(t) |, | \tilde{x}^-(t) |\} + \max_{i=1,2,\cdots,N} \{\| x_i^* \| + \| x_i^E \|\} + \beta \Big) + L_J^L \Big(\| x_0^E \| + \beta \Big). \qquad (3)
\end{aligned}
$$

Let

$$
P(t) = L_J^F \left(\sqrt{n} \max\{| \tilde{x}^+(t) |, | \tilde{x}^-(t) |\} + \max_{i=1,2,\cdots,N} \{\| x_i^* \| + \| x_i^E \|\} + \beta \right) + L_J^L \left(\| x_0^E \| + \beta \right). \qquad (4)
$$

If $\alpha > P(t), \forall t \in \boldsymbol{R}^+$, then $\alpha > \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$.

Now, it can be proved that if $\alpha > P(0)$ then $\alpha > P(t)$, $\forall t \in \boldsymbol{R}^+$. Because $\alpha > P(0)$ and $P(t)$ are continuously changing, suppose that $t_1 \in \boldsymbol{R}^+$ is the first time at which $\alpha = P(t)$. Since α, $\| x_0^E \|$, β, L_J^F, L_J^L and $\max_{i=1,2,\cdots,N} \{\| x_i^* \| + \| x_i^E \|\}$ are constants, one has $\max\{| \tilde{x}^+(t_1) |, | \tilde{x}^-(t_1) |\} > \max\{| \tilde{x}^+(0) |, | \tilde{x}^-(0) |\}$. So, there must exist a $t_2 \in [0, t_1)$ such that the derivative of $\max\{| \tilde{x}^+(t) |, | \tilde{x}^-(t) |\}$ is greater than zero.

Now, consider the following three cases.

- *Case* (*i*): $\{\tilde{x}^+(t) > 0, \tilde{x}^-(t) \geq 0\}$.

In this case, $\max\{| \tilde{x}^+(t) |, | \tilde{x}^-(t) |\} = \tilde{x}^+(t)$, and the derivative of $\max\{| \tilde{x}^+(t) |, | \tilde{x}^-(t) |\}$ is $\dot{\tilde{x}}^+(t)$. Since Assumption 1 holds and $\tilde{x}^+(t) > 0$, one has $\sum_{r \in \mathcal{N}_i} a_{ir} [\tilde{x}^+(t) - \tilde{x}_r^k(t)] > 0$. Thus,

$$
\dot{\tilde{x}}^+(t) \in \mathcal{K} \left[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha \right]
$$

If the derivative of $\max\{| \tilde{x}^+(t) |, | \tilde{x}^-(t) |\}$ is greater than zero at $t_2 \in [0, t_1)$, one has $\dot{\tilde{x}}^+(t_2) > 0$. Then, there must exist $i \in \{1, 2, \cdots, N\}$ and $k \in \{1, 2, \cdots, n\}$ such that $f_i^k(t_2, x_i(t_2)) - f_0^k(t_2, x_0(t_2)) > 0$ and the positive constant $\alpha < | f_i^k(t_2, x_i(t_2)) - f_0^k(t_2, x_0(t_2)) |$. Since $| f_i^k(t_2, x_i(t_2)) - f_0^k(t_2, x_0(t_2)) | \leq \| f_i(t_2, x_i(t_2)) - f_0(t_2, x_0(t_2)) \|$, one has $\alpha < \| f_i(t_2, x_i(t_2)) - f_0(t_2, x_0(t_2)) \|$. It follows that $\alpha < P(t_2)$ based on (3). Because $\alpha > P(0)$ and $P(t)$ are continuously changing, there must be a $t_3 \in [0, t_2)$ such that $\alpha = P(t_3)$. It contradicts the assumption that $t_1 \in \boldsymbol{R}^+$ is the first time at which $\alpha = P(t)$.

- *Case* (*ii*): $\{\tilde{x}^+(t) \leq 0, \tilde{x}^-(t) < 0\}$.

In this case, $\max\{\mid \tilde{x}^+(t) \mid, \mid \tilde{x}^-(t) \mid\} = -\tilde{x}^-(t)$, and the derivative of $\max\{\mid \tilde{x}^+(t) \mid, \mid \tilde{x}^-(t) \mid\}$ is $-\dot{\tilde{x}}^-(t)$. Since Assumption 1 holds and $\tilde{x}^-(t) < 0$, one has $\sum_{s \in \mathcal{N}_j} a_{js}[\tilde{x}^-(t) - \tilde{x}_s^l(t)] < 0$. Thus,

$$\dot{\tilde{x}}^-(t) \in \mathcal{K}\left[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha\right].$$

If the derivative of $\max\{\mid \tilde{x}^+(t) \mid, \mid \tilde{x}^-(t) \mid\}$ is greater than zero at $t_2 \in [0, t_1)$, one has $\dot{\tilde{x}}^-(t_2) < 0$. Then, there must exist $j \in \{1, 2, \cdots, N\}$ and $l \in \{1, 2, \cdots, n\}$ such that $f_j^l(t_2, x_j(t_2)) - f_0^l(t_2, x_0(t_2)) < 0$ and the positive constant $\alpha < \mid f_j^l(t_2, x_j(t_2)) - f_0^l(t_2, x_0(t_2)) \mid$. Since $\mid f_j^l(t_2, x_j(t_2)) - f_0^l(t_2, x_0(t_2)) \mid \leq \| f_j(t_2, x_j(t_2)) - f_0(t_2, x_0(t_2)) \|$, one has $\alpha < \| f_j(t_2, x_j(t_2)) - f_0(t_2, x_0(t_2)) \|$. It follows that $\alpha < P(t_2)$ based on (3). Because $\alpha > P(0)$ and $P(t)$ are continuously changing, there must be a $t_3 \in [0, t_2)$ such that $\alpha = P(t_3)$. It contradicts the assumption that $t_1 \in \boldsymbol{R}^+$ is the first time at which $\alpha = P(t)$.

- *Case* (*iii*): $\{\tilde{x}^+(t) > 0, \tilde{x}^-(t) < 0\}$.

(i) If $\{\tilde{x}^+(t) \geq -\tilde{x}^-(t)\}$, then $\max\{\mid \tilde{x}^+(t) \mid, \mid \tilde{x}^-(t) \mid\} = \tilde{x}^+(t)$. So, the proof is the same as that in Case (i).

(ii) If $\{\tilde{x}^+(t) < -\tilde{x}^-(t)\}$, then $\max\{\mid \tilde{x}^+(t) \mid, \mid \tilde{x}^-(t) \mid\} = -\tilde{x}^-(t)$. So, the proof is the same as that in Case (ii).

Combining the above three cases, it can be concluded that the derivative of $\max\{\mid \tilde{x}^+(t) \mid, \mid \tilde{x}^-(t) \mid\}$ will not be greater than zero. Hence, if $\alpha > P(0)$, i.e., Assumption 4 holds, then $\alpha > P(t), \forall t \in \boldsymbol{R}^+$. It follows that $\alpha > \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$, based on (3).

The proof is now completed.

2 Proof of Lemma 3

Proof Six cases are discussed as follows:

- *Case (i)*: $(\tilde{x}^+(t), \tilde{x}^-(t)), (\tilde{x}^+(t)', \tilde{x}^-(t)') \in D_1$.

$$\begin{aligned} & \parallel V(\tilde{x}^+(t), \tilde{x}^-(t)) - V(\tilde{x}^+(t)', \tilde{x}^-(t)') \parallel \\ = & \parallel \tilde{x}^+(t) - \tilde{x}^+(t)' \parallel \\ \leq & \parallel \tilde{x}^+(t) - \tilde{x}^+(t)' \parallel + \parallel \tilde{x}^-(t) - \tilde{x}^-(t)' \parallel \\ \leq & \sqrt{2} \parallel (\tilde{x}^+(t), \tilde{x}^-(t))^T - (\tilde{x}^+(t)', \tilde{x}^-(t)')^T \parallel . \end{aligned}$$

- *Case (ii)*: $(\tilde{x}^+(t), \tilde{x}^-(t)), (\tilde{x}^+(t)', \tilde{x}^-(t)') \in D_2$.

$$\begin{aligned} & \parallel V(\tilde{x}^+(t), \tilde{x}^-(t)) - V(\tilde{x}^+(t)', \tilde{x}^-(t)') \parallel \\ = & \parallel (\tilde{x}^+(t) - \tilde{x}^-(t)) - (\tilde{x}^+(t)' - \tilde{x}^-(t)') \parallel \\ \leq & \parallel \tilde{x}^+(t) - \tilde{x}^+(t)' \parallel + \parallel \tilde{x}^-(t) - \tilde{x}^-(t)' \parallel \\ \leq & \sqrt{2} \parallel (\tilde{x}^+(t), \tilde{x}^-(t))^T - (\tilde{x}^+(t)', \tilde{x}^-(t)')^T \parallel . \end{aligned}$$

- *Case (iii)*: $(\tilde{x}^+(t), \tilde{x}^-(t)), (\tilde{x}^+(t)', \tilde{x}^-(t)') \in D_3$.

$$\begin{aligned} & \parallel V(\tilde{x}^+(t), \tilde{x}^-(t)) - V(\tilde{x}^+(t)', \tilde{x}^-(t)') \parallel \\ = & \parallel -\tilde{x}^-(t) - (-\tilde{x}^-(t)') \parallel \\ \leq & \parallel \tilde{x}^+(t) - \tilde{x}^+(t)' \parallel + \parallel \tilde{x}^-(t) - \tilde{x}^-(t)' \parallel \\ \leq & \sqrt{2} \parallel (\tilde{x}^+(t), \tilde{x}^-(t))^T - (\tilde{x}^+(t)', \tilde{x}^-(t)')^T \parallel . \end{aligned}$$

- *Case (iv)*: $(\tilde{x}^+(t), \tilde{x}^-(t)) \in D_1, (\tilde{x}^+(t)', \tilde{x}^-(t)') \in D_2$.

$$\begin{aligned} & \parallel V(\tilde{x}^+(t), \tilde{x}^-(t)) - V(\tilde{x}^+(t)', \tilde{x}^-(t)') \parallel \\ = & \parallel \tilde{x}^+(t) - (\tilde{x}^+(t)' - \tilde{x}^-(t)') \parallel \\ \leq & \parallel \tilde{x}^+(t) - \tilde{x}^+(t)' \parallel + \parallel \tilde{x}^-(t)' \parallel . \end{aligned}$$

For $(\tilde{x}^+(t), \tilde{x}^-(t)) \in D_1, (\tilde{x}^+(t)', \tilde{x}^-(t)') \in D_2$, one has $\tilde{x}^-(t) \geq 0, \tilde{x}^-(t)' < 0$, thus

$$\parallel \tilde{x}^-(t)' \parallel \leq \parallel \tilde{x}^-(t) - \tilde{x}^-(t)' \parallel .$$

Hence,

$$\begin{aligned} & \parallel V(\tilde{x}^+(t), \tilde{x}^-(t)) - V(\tilde{x}^+(t)', \tilde{x}^-(t)') \parallel \\ \leq & \parallel \tilde{x}^+(t) - \tilde{x}^+(t)' \parallel + \parallel \tilde{x}^-(t)' \parallel \\ \leq & \parallel \tilde{x}^+(t) - \tilde{x}^+(t)' \parallel + \parallel \tilde{x}^-(t) - \tilde{x}^-(t)' \parallel \\ \leq & \sqrt{2} \parallel (\tilde{x}^+(t), \tilde{x}^-(t))^T - (\tilde{x}^+(t)', \tilde{x}^-(t)')^T \parallel . \end{aligned}$$

- *Case* (v): $(\tilde{x}^+(t), \tilde{x}^-(t)) \in D_1, (\tilde{x}^+(t)', \tilde{x}^-(t)') \in D_3$.

$$\begin{aligned} & \parallel V(\tilde{x}^+(t), \tilde{x}^-(t)) - V(\tilde{x}^+(t)', \tilde{x}^-(t)') \parallel \\ = & \parallel \tilde{x}^+(t) - (-\tilde{x}^-(t)') \parallel \\ \leq & \parallel \tilde{x}^+(t) \parallel + \parallel \tilde{x}^-(t)' \parallel . \end{aligned}$$

For $(\tilde{x}^+(t), \tilde{x}^-(t)) \in D_1, (\tilde{x}^+(t)', \tilde{x}^-(t)') \in D_3$, one has $\tilde{x}^+(t) \geq 0, \tilde{x}^-(t) \geq 0, \tilde{x}^+(t)' \leq 0, \tilde{x}^-(t)' < 0$, thus

$$\parallel \tilde{x}^+(t) \parallel \leq \parallel \tilde{x}^+(t) - \tilde{x}^+(t)' \parallel,$$

and

$$\parallel \tilde{x}^-(t)' \parallel \leq \parallel \tilde{x}^-(t) - \tilde{x}^-(t)' \parallel .$$

Hence,

$$\begin{aligned} & \parallel V(\tilde{x}^+(t), \tilde{x}^-(t)) - V(\tilde{x}^+(t)', \tilde{x}^-(t)') \parallel \\ \leq & \parallel \tilde{x}^+(t) \parallel + \parallel \tilde{x}^-(t)' \parallel \\ \leq & \parallel \tilde{x}^+(t) - \tilde{x}^+(t)' \parallel + \parallel \tilde{x}^-(t) - \tilde{x}^-(t)' \parallel \\ \leq & \sqrt{2} \parallel (\tilde{x}^+(t), \tilde{x}^-(t))^T - (\tilde{x}^+(t)', \tilde{x}^-(t)')^T \parallel . \end{aligned}$$

- *Case* (vi): $(\tilde{x}^+(t), \tilde{x}^-(t)) \in D_2, (\tilde{x}^+(t)', \tilde{x}^-(t)') \in D_3$.

$$\begin{aligned} & \parallel V(\tilde{x}^+(t), \tilde{x}^-(t)) - V(\tilde{x}^+(t)', \tilde{x}^-(t)') \parallel \\ = & \parallel (\tilde{x}^+(t) - \tilde{x}^-(t)) - (-\tilde{x}^-(t)') \parallel \\ \leq & \parallel \tilde{x}^+(t) \parallel + \parallel \tilde{x}^-(t) - \tilde{x}^-(t)' \parallel . \end{aligned}$$

For $(\tilde{x}^+(t), \tilde{x}^-(t)) \in D_2, (\tilde{x}^+(t)', \tilde{x}^-(t)') \in D_3$, one has $\tilde{x}^+(t) > 0, \tilde{x}^+(t)' \leq 0$, thus

$$\parallel \tilde{x}^+(t) \parallel \leq \parallel \tilde{x}^+(t) - \tilde{x}^+(t)' \parallel .$$

Hence,

$$\begin{aligned} & \parallel V(\tilde{x}^+(t), \tilde{x}^-(t)) - V(\tilde{x}^+(t)', \tilde{x}^-(t)') \parallel \\ \leq & \parallel \tilde{x}^+(t) \parallel + \parallel \tilde{x}^-(t) - \tilde{x}^-(t)' \parallel \\ \leq & \parallel \tilde{x}^+(t) - \tilde{x}^+(t)' \parallel + \parallel \tilde{x}^-(t) - \tilde{x}^-(t)' \parallel \\ \leq & \sqrt{2} \parallel (\tilde{x}^+(t), \tilde{x}^-(t))^T - (\tilde{x}^+(t)', \tilde{x}^-(t)')^T \parallel . \end{aligned}$$

Combining the above six cases, it can be concluded that, for every $(\tilde{x}^+(t), \tilde{x}^-(t)), (\tilde{x}^+(t)', \tilde{x}^-(t)') \in D$, one has

$$\begin{aligned} &\| V(\tilde{x}^+(t), \tilde{x}^-(t)) - V(\tilde{x}^+(t)', \tilde{x}^-(t)') \| \\ \leq &\sqrt{2} \| (\tilde{x}^+(t), \tilde{x}^-(t))^T - (\tilde{x}^+(t)', \tilde{x}^-(t)')^T \| . \end{aligned}$$

Therefore, V is a locally Lipschitz function on D.

The proof is now completed.

3 Proof of Lemma 4

Proof If a function is continuously differentiable at x, it is regular at x. Since V is continuously differentiable everywhere except for $\{\tilde{x}^+(t) > 0, \tilde{x}^-(t) = 0\}$, $\{\tilde{x}^+(t) = 0, \tilde{x}^-(t) < 0\}$ and $\{\tilde{x}^+(t) = 0, \tilde{x}^-(t) = 0\}$, it needs to show that V is regular on these three sets.

Let $y = (\tilde{x}^+(t), \tilde{x}^-(t))^T$ and $v = (v_1, v_2)^T$. The right directional derivative of V at $y \in \boldsymbol{R}^2$ in the direction $v \in \boldsymbol{R}^2$ is defined as

$$V'(y; v) = \lim_{h \to 0^+} \frac{V(\tilde{x}^+(t) + hv_1, \tilde{x}^-(t) + hv_2) - V(\tilde{x}^+(t), \tilde{x}^-(t))}{h}.$$

The general directional derivative of V at y in the direction v is defined as

$$V^o(y; v) = \lim_{\substack{\delta \to 0^+ \\ \epsilon \to 0^+}} \sup_{\substack{z \in B(y,\delta) \\ h \in [0,\epsilon)}} \frac{V(z_1 + hv_1, z_2 + hv_2) - V(z_1, z_2)}{h}.$$

- *Case* (i): $\{\tilde{x}^+(t) > 0, \tilde{x}^-(t) = 0\}$.

If $v_1 \geq 0, v_2 \geq 0$, then $(\tilde{x}^+(t) + hv_1, hv_2)_{h \to 0^+} \in D_1$, hence

$$\begin{aligned} V'(y; v) &= \lim_{h \to 0^+} \frac{(\tilde{x}^+(t) + hv_1) - \tilde{x}^+(t)}{h} \\ &= v_1. \end{aligned}$$

For $z \in B(y, \delta)$, when $\delta \to 0^+$, $z \in D_1$ and $z \in D_2$ are possible, hence

$$\begin{aligned} V^o(y; v) &= \lim_{\substack{\delta \to 0^+ \\ \epsilon \to 0^+}} \sup_{\substack{z \in B(y,\delta) \\ h \in [0,\epsilon)}} \left\{ \frac{(z_1 + hv_1) - z_1}{h}, \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h} \right\} \\ &= v_1. \end{aligned}$$

So, $V'(y;v) = V^o(y;v)$.

If $v_1 \leq 0, v_2 < 0$, then $(\tilde{x}^+(t) + hv_1, hv_2)_{h \to 0^+} \in D_2$, hence

$$\begin{aligned} V'(y;v) &= \lim_{h \to 0^+} \frac{((\tilde{x}^+(t) + hv_1) - hv_2) - \tilde{x}^+(t)}{h} \\ &= v_1 - v_2. \end{aligned}$$

For $z \in B(y,\delta)$, when $\delta \to 0^+$, $z \in D_1$ and $z \in D_2$ are possible, hence

$$\begin{aligned} V^o(y;v) &= \lim_{\substack{\delta \to 0^+ \\ \epsilon \to 0^+}} \sup_{\substack{z \in B(y,\delta) \\ h \in [0,\epsilon)}} \left\{ \frac{(z_1 + hv_1) - z_1}{h}, \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h} \right\} \\ &= v_1 - v_2. \end{aligned}$$

So, $V'(y;v) = V^o(y;v)$.

If $v_1 < 0, v_2 \geq 0$, then $(\tilde{x}^+(t) + hv_1, hv_2)_{h \to 0^+} \in D_1$, hence

$$\begin{aligned} V'(y;v) &= \lim_{h \to 0^+} \frac{(\tilde{x}^+(t) + hv_1) - \tilde{x}^+(t)}{h} \\ &= v_1. \end{aligned}$$

For $z \in B(y,\delta)$, when $\delta \to 0^+$, $z \in D_1$ and $z \in D_2$ are possible, hence

$$\begin{aligned} V^o(y;v) &= \lim_{\substack{\delta \to 0^+ \\ \epsilon \to 0^+}} \sup_{\substack{z \in B(y,\delta) \\ h \in [0,\epsilon)}} \left\{ \frac{(z_1 + hv_1) - z_1}{h}, \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h} \right\} \\ &= v_1. \end{aligned}$$

So, $V'(y;v) = V^o(y;v)$.

If $v_1 > 0, v_2 < 0$, then $(\tilde{x}^+(t) + hv_1, hv_2)_{h \to 0^+} \in D_2$, hence

$$\begin{aligned} V'(y;v) &= \lim_{h \to 0^+} \frac{((\tilde{x}^+(t) + hv_1) - hv_2) - \tilde{x}^+(t)}{h} \\ &= v_1 - v_2. \end{aligned}$$

For $z \in B(y,\delta)$, when $\delta \to 0^+$, $z \in D_1$ and $z \in D_2$ are possible, hence

$$\begin{aligned} V^o(y;v) &= \lim_{\substack{\delta \to 0^+ \\ \epsilon \to 0^+}} \sup_{\substack{z \in B(y,\delta) \\ h \in [0,\epsilon)}} \left\{ \frac{(z_1 + hv_1) - z_1}{h}, \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h} \right\} \\ &= v_1 - v_2. \end{aligned}$$

So, $V'(y;v) = V^o(y;v)$.

- *Case (ii)*: $\{\tilde{x}^+(t) = 0, \tilde{x}^-(t) < 0\}$.

If $v_1 > 0, v_2 \geq 0$, then $(hv_1, \tilde{x}^-(t) + hv_2)_{h\to 0^+} \in D_2$, hence

$$\begin{aligned} V'(y;v) &= \lim_{h\to 0^+} \frac{(hv_1 - (\tilde{x}^-(t) + hv_2)) - (-\tilde{x}^-(t))}{h} \\ &= v_1 - v_2. \end{aligned}$$

For $z \in B(y,\delta)$, when $\delta \to 0^+$, $z \in D_2$ and $z \in D_3$ are possible, hence

$$\begin{aligned} V^o(y;v) &= \lim_{\substack{\delta\to 0^+ \\ \epsilon\to 0^+}} \sup_{\substack{z\in B(y,\delta) \\ h\in[0,\epsilon)}} \left\{ \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h}, \frac{-((z_2 + hv_2)) - (-(z_2))}{h} \right\} \\ &= v_1 - v_2. \end{aligned}$$

So, $V'(y;v) = V^o(y;v)$.

If $v_1 \leq 0, v_2 < 0$, then $(hv_1, \tilde{x}^-(t) + hv_2)_{h\to 0^+} \in D_3$, hence

$$\begin{aligned} V'(y;v) &= \lim_{h\to 0^+} \frac{(-(\tilde{x}^-(t) + hv_2)) - (-\tilde{x}^-(t))}{h} \\ &= -v_2. \end{aligned}$$

For $z \in B(y,\delta)$, when $\delta \to 0^+$, $z \in D_2$ and $z \in D_3$ are possible, hence

$$\begin{aligned} V^o(y;v) &= \lim_{\substack{\delta\to 0^+ \\ \epsilon\to 0^+}} \sup_{\substack{z\in B(y,\delta) \\ h\in[0,\epsilon)}} \left\{ \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h}, \frac{-((z_2 + hv_2)) - (-(z_2))}{h} \right\} \\ &= -v_2. \end{aligned}$$

So, $V'(y;v) = V^o(y;v)$.

If $v_1 \leq 0, v_2 \geq 0$, then $(hv_1, \tilde{x}^-(t) + hv_2)_{h\to 0^+} \in D_3$, hence

$$\begin{aligned} V'(y;v) &= \lim_{h\to 0^+} \frac{(-(\tilde{x}^-(t) + hv_2)) - (-\tilde{x}^-(t))}{h} \\ &= -v_2. \end{aligned}$$

For $z \in B(y,\delta)$, when $\delta \to 0^+$, $z \in D_2$ and $z \in D_3$ are possible, hence

$$\begin{aligned} V^o(y;v) &= \lim_{\substack{\delta\to 0^+ \\ \epsilon\to 0^+}} \sup_{\substack{z\in B(y,\delta) \\ h\in[0,\epsilon)}} \left\{ \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h}, \frac{-((z_2 + hv_2)) - (-(z_2))}{h} \right\} \\ &= -v_2. \end{aligned}$$

So, $V'(y;v) = V^o(y;v)$.

If $v_1 > 0, v_2 < 0$, then $(hv_1, \tilde{x}^-(t) + hv_2)_{h \to 0^+} \in D_2$, hence

$$V'(y;v) = \lim_{h \to 0^+} \frac{(hv_1 - (\tilde{x}^-(t) + hv_2)) - (-\tilde{x}^-(t))}{h}$$
$$= v_1 - v_2.$$

For $z \in B(y, \delta)$, when $\delta \to 0^+$, $z \in D_2$ and $z \in D_3$ are possible, hence

$$V^o(y;v) = \lim_{\substack{\delta \to 0^+ \\ \epsilon \to 0^+}} \sup_{\substack{z \in B(y,\delta) \\ h \in [0,\epsilon)}} \left\{ \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h}, \frac{-((z_2 + hv_2)) - (-(z_2))}{h} \right\}$$
$$= v_1 - v_2.$$

So, $V'(y;v) = V^o(y;v)$.

- *Case (iii)*: $\{\tilde{x}^+(t) = 0, \tilde{x}^-(t) = 0\}$.

If $v_1 \geq 0, v_2 \geq 0$, then $(hv_1, hv_2)_{h \to 0^+} \in D_1$, hence

$$V'(y;v) = \lim_{h \to 0^+} \frac{hv_1 - 0}{h}$$
$$= v_1.$$

For $z \in B(y, \delta)$, when $\delta \to 0^+$, $z \in D_1$, $z \in D_2$ and $z \in D_3$ are all possible, hence

$$V^o(y;v) = \lim_{\substack{\delta \to 0^+ \\ \epsilon \to 0^+}} \sup_{\substack{z \in B(y,\delta) \\ h \in [0,\epsilon)}} \left\{ \frac{(z_1 + hv_1) - z_1}{h}, \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h}, \frac{-((z_2 + hv_2)) - (-(z_2))}{h} \right\}$$
$$= v_1.$$

So, $V'(y;v) = V^o(y;v)$.

If $v_1 \leq 0, v_2 < 0$, then $(hv_1, hv_2)_{h \to 0^+} \in D_3$, hence

$$V'(y;v) = \lim_{h \to 0^+} \frac{-hv_2 - 0}{h}$$
$$= -v_2.$$

For $z \in B(y,\delta)$, when $\delta \to 0^+$, $z \in D_1$,$z \in D_2$ and $z \in D_3$ are all possible, hence

$$\begin{aligned} V^o(y;v) &= \lim_{\substack{\delta \to 0^+ \\ \epsilon \to 0^+}} \sup_{\substack{z \in B(y,\delta) \\ h \in [0,\epsilon)}} \left\{ \frac{(z_1 + hv_1) - z_1}{h}, \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h}, \right. \\ &\qquad \left. \frac{-((z_2 + hv_2)) - (-(z_2))}{h} \right\} \\ &= -v_2. \end{aligned}$$

So, $V'(y;v) = V^o(y;v)$.

The case of $v_1 < 0, v_2 \geq 0$ is impossible for $\tilde{x}^+(t) > \tilde{x}^-(t)$.

If $v_1 > 0, v_2 < 0$, then $(hv_1, hv_2)_{h \to 0^+} \in D_2$, hence

$$\begin{aligned} V'(y;v) &= \lim_{h \to 0^+} \frac{hv_1 - hv_2}{h} \\ &= v_1 - v_2. \end{aligned}$$

For $z \in B(y,\delta)$, when $\delta \to 0^+$, $z \in D_1$,$z \in D_2$ and $z \in D_3$ are all possible, hence

$$\begin{aligned} V^o(y;v) &= \lim_{\substack{\delta \to 0^+ \\ \epsilon \to 0^+}} \sup_{\substack{z \in B(y,\delta) \\ h \in [0,\epsilon)}} \left\{ \frac{(z_1 + hv_1) - z_1}{h}, \frac{((z_1 + hv_1) - (z_2 + hv_2)) - (z_1 - z_2)}{h}, \right. \\ &\qquad \left. \frac{-((z_2 + hv_2)) - (-(z_2))}{h} \right\} \\ &= v_1 - v_2. \end{aligned}$$

So, $V'(y;v) = V^o(y;v)$.

For all the cases, the right directional derivative of V is equal to the generalized directional derivative of V, i.e., $V'(y;v) = V^o(y;v)$. Therefore, the function V is regular on D.

The proof is now completed.

4 Proof of Lemma 5

Proof If $\tilde{x}^+(t) = 0$ and $\tilde{x}^-(t) = 0$, then $V = 0$. If $\tilde{x}^+(t) > 0$ and $\tilde{x}^-(t) \geq 0$, i.e., $(\tilde{x}^+(t), \tilde{x}^-(t)) \in D_1 \setminus \{(0,0)\}$, then $V = \tilde{x}^+(t) > 0$. If $\tilde{x}^+(t) > 0$ and $\tilde{x}^-(t) < 0$, i.e., $(\tilde{x}^+(t), \tilde{x}^-(t)) \in D_2$, then $V = \tilde{x}^+(t) - \tilde{x}^-(t) > 0$. If $\tilde{x}^+(t) \leq 0$ and $\tilde{x}^-(t) < 0$, i.e., $(\tilde{x}^+(t), \tilde{x}^-(t)) \in D_3$, then $V = -\tilde{x}^-(t) > 0$. So, V is globally positive definite.

If $(\tilde{x}^+(t), \tilde{x}^-(t)) \in D_1$, then as either $\tilde{x}^+ \to \infty$ or both $\tilde{x}^+, \tilde{x}^- \to \infty$, one has $V = \tilde{x}^+(t) \to \infty$. If $(\tilde{x}^+(t), \tilde{x}^-(t)) \in D_2$, then as either $\tilde{x}^+ \to \infty$ or $\tilde{x}^- \to -\infty$, or both, one has $V = \tilde{x}^+(t) - \tilde{x}^-(t) \to \infty$. If $(\tilde{x}^+(t), \tilde{x}^-(t)) \in D_3$, then as either $\tilde{x}^+ \to -\infty$ or both $\tilde{x}^+, \tilde{x}^- \to -\infty$, one has $V = -\tilde{x}^-(t) \to \infty$. So, V is radially unbounded.

The proof is now completed.

5 Proof of Lemma 6

Proof If Assumptions 1 - 4 hold, then Lemma 2 holds, i.e., $\alpha > \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$. Five cases are discussed as follows:

- *Case* (i): $\tilde{x}^+(t) > 0$ and $\tilde{x}^-(t) > 0$.

Since $\tilde{x}^+(t) > 0$ and Assumption 1 holds, one has $\sum_{r \in \mathcal{N}_i} a_{ir}[\tilde{x}^+(t) - \tilde{x}_r^k(t)] > 0$, and for

$$\partial V(\tilde{x}^+(t), \tilde{x}^-(t)) = \{(1, 0)\},$$

one has

$$\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha].$$

Since $| f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) | \leq \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, it follows from Lemma 2 that

$$\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0.$$

- *Case* (ii): $\tilde{x}^+(t) > 0$ and $\tilde{x}^-(t) < 0$.

Since $\tilde{x}^+(t) > 0, \tilde{x}^-(t) < 0$ and Assumption 1 holds, one has $\sum_{r \in \mathcal{N}_i} a_{ir}[\tilde{x}^+(t) - \tilde{x}_r^k(t)] > 0, \sum_{s \in \mathcal{N}_j} a_{js}[\tilde{x}^-(t) - \tilde{x}_s^l(t)] < 0$, and for

$$\partial V(\tilde{x}^+(t), \tilde{x}^-(t)) = \{(1, -1)\},$$

one has

$$\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}\left[(f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha) - (f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha)\right].$$

Since $| f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) | \leq \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, it follows from Lemma 2 that

$$\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0.$$

- *Case (iii)*: $\tilde{x}^+(t) < 0$ and $\tilde{x}^-(t) < 0$.

Since $\tilde{x}^-(t) < 0$ and Assumption 1 holds, one has $\sum_{s\in\mathcal{N}_j} a_{js}[\tilde{x}^-(t) - \tilde{x}_s^l(t)] < 0$, and for

$$\partial V(\tilde{x}^+(t), \tilde{x}^-(t)) = \{(0, -1)\},$$

one has

$$\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}\left[-(f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha)\right].$$

Since $\mid f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) \mid \leq \parallel f_j(t, x_j(t)) - f_0(t, x_0(t)) \parallel, \forall t \in \boldsymbol{R}^+, \forall j = 1, 2, \cdots, N, \forall l = 1, 2, \cdots, n$, it follows from Lemma 2 that

$$\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0.$$

- *Case (iv)*: $\tilde{x}^+(t) > 0$ and $\tilde{x}^-(t) = 0$.

Since $\tilde{x}^+(t) > 0, \tilde{x}^-(t) = 0$ and Assumption 1 holds, one has $\sum_{r\in\mathcal{N}_i} a_{ir}[\tilde{x}^+(t) - \tilde{x}_r^k(t)] > 0, \sum_{s\in\mathcal{N}_j} a_{js}[\tilde{x}^-(t) - \tilde{x}_s^l(t)] \leq 0$. So, if $v \in \mathcal{F}(\tilde{x}^+(t), \tilde{x}^-(t))$, then $v^T = (v_1, v_2)$ with $v_1 \in \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha]$ and $v_2 \in \mathcal{K}[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha] \cup \mathcal{K}[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t))]$. For

$$\partial V(\tilde{x}^+(t), \tilde{x}^-(t)) = \{1\} \times [-1, 0],$$

if $\zeta \in \partial V(\tilde{x}^+(t), \tilde{x}^-(t))$, then $\zeta^T = (1, y)$ with $y \in [-1, 0]$. Therefore,

$$\zeta^T v = v_1 + y v_2.$$

If there exists an element a satisfying that $\zeta^T v = a$ for all $y \in [-1, 0]$, then $v_2 = 0$. So, if $v_2 \neq 0$, one has $\tilde{\mathcal{L}}_{\mathcal{F}} V = \emptyset$; if $v_2 = 0$, one has $\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha]$, and then it follows from Lemma 2 that $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$. Thus, $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$ or $\tilde{\mathcal{L}}_{\mathcal{F}} V = \emptyset$ in this case.

- *Case (v)*: $\tilde{x}^+(t) = 0$ and $\tilde{x}^-(t) < 0$.

Since $\tilde{x}^+(t) = 0, \tilde{x}^-(t) < 0$ and Assumption 1 holds, one has $\sum_{r\in\mathcal{N}_i} a_{ir}[\tilde{x}^+(t) - \tilde{x}_r^k(t)] \geq 0, \sum_{s\in\mathcal{N}_j} a_{js}[\tilde{x}^-(t) - \tilde{x}_s^l(t)] < 0$. So, if $v \in \mathcal{F}(\tilde{x}^+(t), \tilde{x}^-(t))$, then $v^T = (v_1, v_2)$ with $v_1 \in \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha] \cup \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t))]$ and $v_2 \in \mathcal{K}[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha]$. For

$$\partial V(\tilde{x}^+(t), \tilde{x}^-(t)) = [0, 1] \times \{-1\},$$

if $\zeta \in \partial V(\tilde{x}^+(t), \tilde{x}^-(t))$, then $\zeta^T = (y, -1)$ with $y \in [0, 1]$. Therefore,

$$\zeta^T v = y v_1 - v_2.$$

If there exists an element a satisfying that $\zeta^T v = a$ for all $y \in [0, 1]$, then $v_1 = 0$. So, if $v_1 \neq 0$, one has $\tilde{\mathcal{L}}_{\mathcal{F}} V = \emptyset$; if $v_1 = 0$, one has $\tilde{\mathcal{L}}_{\mathcal{F}} V = -\mathcal{K}[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha]$, and then it follows from Lemma 2 that $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$. Thus, $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$ or $\tilde{\mathcal{L}}_{\mathcal{F}} V = \emptyset$ in this case.

Combining the above five cases, it can be concluded that $\max \tilde{\mathcal{L}}_F V < 0$ for all $(\tilde{x}^+(t), \tilde{x}^-(t)) \in \mathcal{D} \setminus \{(0, 0)\}$.

The proof is now completed.

6 Proof of Theorem 1

Proof The nonsmooth function V, which was given by (6) in the manuscript, is chosen as the Lyapunov function. If Assumptions 1 - 4 hold, then Lemma 6 holds. By using Lemma 1, it follows from Lemmas 3 - 6 that $(\tilde{x}^+(t), \tilde{x}^-(t)) = (0, 0)$ is a globally stable equilibrium point for system (2).

Next, the maximal converging time is considered.

- *Case (i)*: $\tilde{x}^+(t) > 0$ and $\tilde{x}^-(t) \geq 0$.

In this case, $V = \tilde{x}^+(t)$ and $\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}\left[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha\right]$. By the proof of Lemma 2, one has $\| f_i(t, x_i(t)) - f_0(t, x_0(t)) \| \leq P(t), P(t) \leq P(0), \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$. Since $| f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) | \leq \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, one has

$$\begin{aligned} \max \tilde{\mathcal{L}}_{\mathcal{F}} V &\leq -(\alpha - P(t)) \\ &\leq -(\alpha - P(0)). \end{aligned}$$

Therefore, the converging time satisfies

$$T_1 \leq \frac{1}{\alpha - P(0)} \tilde{x}^+(0).$$

- *Case (ii)*: $\tilde{x}^+(t) > 0$ and $\tilde{x}^-(t) < 0$.

In this case, $V = \tilde{x}^+(t) - \tilde{x}^-(t)$ and $\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}\Big[(f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha) - (f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha)\Big]$. By the proof of Lemma 2, one has $\| f_i(t, x_i(t)) - f_0(t, x_0(t)) \| \leq P(t), P(t) \leq$

$P(0), \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$. Since $\mid f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) \mid \leq \parallel f_i(t, x_i(t)) - f_0(t, x_0(t)) \parallel$ $, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, one has

$$\begin{aligned} \max \tilde{\mathcal{L}}_{\mathcal{F}} V &\leq -2(\alpha - P(t)) \\ &\leq -2(\alpha - P(0)). \end{aligned}$$

Therefore, the converging time satisfies

$$\begin{aligned} T_2 &\leq \frac{1}{2(\alpha - P(0))} (\tilde{x}^+(0) - \tilde{x}^-(0)) \\ &\leq \frac{1}{\alpha - P(0)} \max\{\tilde{x}^+(0), -\tilde{x}^-(0)\}. \end{aligned}$$

- *Case (iii)*: $\tilde{x}^+(t) \leq 0$ and $\tilde{x}^-(t) < 0$.

In this case, $V = -\tilde{x}^-(t)$. Since $\tilde{x}^-(t) < 0$ and Assumption 1 holds, one has $\sum_{s \in \mathcal{N}_j} a_{js} [\tilde{x}^-(t) - \tilde{x}_s^l(t)] < 0$, then $\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}\Big[- (f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha) \Big]$. By the proof of Lemma 2, one has $\parallel f_j(t, x_j(t)) - f_0(t, x_0(t)) \parallel \leq P(t), P(t) \leq P(0), \forall t \in \boldsymbol{R}^+, \forall j = 1, 2, \cdots, N$. Since $\mid f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) \mid \leq \parallel f_j(t, x_j(t)) - f_0(t, x_0(t)) \parallel, \forall t \in \boldsymbol{R}^+, \forall j = 1, 2, \cdots, N, \forall l = 1, 2, \cdots, n$, one has

$$\begin{aligned} \max \tilde{\mathcal{L}}_{\mathcal{F}} V &\leq -(\alpha - P(t)) \\ &\leq -(\alpha - P(0)). \end{aligned}$$

Therefore, the converging time satisfies

$$T_3 \leq -\frac{1}{\alpha - P(0)} \tilde{x}^-(0).$$

Combining the above three cases, the maximal converging time is obtained as

$$T = \frac{1}{\alpha - P(0)} \max_{\substack{i=1,2,\cdots,N \\ k=1,2,\cdots,n}} \{\mid x_i^k(0) - x_0^k(0) - x_i^{*k} \mid\}.$$

The proof is now completed.

7 Supplementary Lemma i

Supplementary Lemma i If Assumptions 1, 5 and 6 hold, then $\alpha > \parallel f_i(t, x_i(t)) - f_0(t, x_0(t)) \parallel$ $, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$.

Proof Based on Assumption 5, for any $i = 1, 2, \cdots, N$ and each $t \in \boldsymbol{R}^+$, one has

$$\begin{aligned}
&\| f_i(t, x_i(t)) - f_0(t, x_0(t)) \| \\
=&\| f_0(t, x_i(t)) - f_0(t, x_0(t)) \| \\
\le& L_J^L (\| x_i(t) - x_0(t) \|) \\
\le& L_J^L (\| x_i(t) - x_0(t) - x_i^* \| + \| x_i^* \|) \\
\le& L_J^L \left(\sqrt{n} \max\{| \tilde{x}^+(t) |, | \tilde{x}^-(t) |\} + \max_{i=1,2,\cdots,N} \{\| x_i^* \|\} \right).
\end{aligned} \tag{5}$$

Let

$$Q(t) = L_J^L \left(\sqrt{n} \max\{| \tilde{x}^+(t) |, | \tilde{x}^-(t) |\} + \max_{i=1,2,\cdots,N} \| x_i^* \| \right). \tag{6}$$

If $\alpha > Q(t), \forall t \in \boldsymbol{R}^+$, then $\alpha > \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$.

Now, it can be proved that if $\alpha > Q(0)$ then $\alpha > Q(t)$, $\forall t \in \boldsymbol{R}^+$. Because $\alpha > Q(0)$ and $Q(t)$ are continuously changing, suppose that $t_1 \in \boldsymbol{R}^+$ is the first time at which $\alpha = Q(t)$. Since α, L_J^L and $\max_{i=1,2,\cdots,N}\{\| x_i^* \|\}$ are constants, one has $\max\{| \tilde{x}^+(t_1) |, | \tilde{x}^-(t_1) |\} > \max\{| \tilde{x}^+(0) |, | \tilde{x}^-(0) |\}$. So, there must exist a $t_2 \in [0, t_1)$ such that the derivative of $\max\{| \tilde{x}^+(t) |, | \tilde{x}^-(t) |\}$ is greater than zero.

Now, consider the following three cases.

- *Case (i)*: $\{\tilde{x}^+(t) > 0, \tilde{x}^-(t) \ge 0\}$.

In this case, $\max\{| \tilde{x}^+(t) |, | \tilde{x}^-(t) |\} = \tilde{x}^+(t)$, and the derivative of $\max\{| \tilde{x}^+(t) |, | \tilde{x}^-(t) |\}$ is $\dot{\tilde{x}}^+(t)$. Since Assumption 1 holds and $\tilde{x}^+(t) > 0$, one has $\sum_{r \in \mathcal{N}_i} a_{ir}[\tilde{x}^+(t) - \tilde{x}_r^k(t)] > 0$. Thus,

$$\dot{\tilde{x}}^+(t) \in \mathcal{K}\left[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha \right].$$

If the derivative of $\max\{| \tilde{x}^+(t) |, | \tilde{x}^-(t) |\}$ is greater than zero at $t_2 \in [0, t_1)$, one has $\dot{\tilde{x}}^+(t_2) > 0$. Then, there must exist $i \in \{1, 2, \cdots, N\}$ and $k \in \{1, 2, \cdots, n\}$ such that $f_i^k(t_2, x_i(t_2)) - f_0^k(t_2, x_0(t_2)) > 0$ and the positive constant $\alpha < | f_i^k(t_2, x_i(t_2)) - f_0^k(t_2, x_0(t_2)) |$. Since $| f_i^k(t_2, x_i(t_2)) - f_0^k(t_2, x_0(t_2)) | \le \| f_i(t_2, x_i(t_2)) - f_0(t_2, x_0(t_2)) \|$, one has $\alpha < \| f_i(t_2, x_i(t_2)) - f_0(t_2, x_0(t_2)) \|$. It follows that $\alpha < Q(t_2)$ based on (5). Because $\alpha > Q(0)$ and $Q(t)$ are continuously changing, there must be a $t_3 \in [0, t_2)$ such that $\alpha = Q(t_3)$. It contradicts the assumption that $t_1 \in \boldsymbol{R}^+$ is the first time at which $\alpha = Q(t)$.

- *Case (ii)*: $\{\tilde{x}^+(t) \le 0, \tilde{x}^-(t) < 0\}$.

In this case, $\max\{\mid \tilde{x}^+(t) \mid, \mid \tilde{x}^-(t) \mid\} = -\tilde{x}^-(t)$, and the derivative of $\max\{\mid \tilde{x}^+(t) \mid, \mid \tilde{x}^-(t) \mid\}$ is $-\dot{\tilde{x}}^-(t)$. Since Assumption 1 holds and $\tilde{x}^-(t) < 0$, one has $\sum_{s\in\mathcal{N}_j} a_{js}[\tilde{x}^-(t) - \tilde{x}_s^l(t)] < 0$. Thus,

$$\dot{\tilde{x}}^-(t) \in \mathcal{K}\left[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha\right].$$

If the derivative of $\max\{\mid \tilde{x}^+(t) \mid, \mid \tilde{x}^-(t) \mid\}$ is greater than zero at $t_2 \in [0, t_1)$, one has $\dot{\tilde{x}}^-(t_2) < 0$. Then, there must exist $j \in \{1, 2, \cdots, N\}$ and $l \in \{1, 2, \cdots, n\}$ such that $f_j^l(t_2, x_j(t_2)) - f_0^l(t_2, x_0(t_2)) < 0$ and the positive constant $\alpha < \mid f_j^l(t_2, x_j(t_2)) - f_0^l(t_2, x_0(t_2)) \mid$. Since $\mid f_j^l(t_2, x_j(t_2)) - f_0^l(t_2, x_0(t_2)) \mid \leq \parallel f_j(t_2, x_j(t_2)) - f_0(t_2, x_0(t_2)) \parallel$, one has $\alpha < \parallel f_j(t_2, x_j(t_2)) - f_0(t_2, x_0(t_2)) \parallel$. It follows that $\alpha < Q(t_2)$ based on (5). Because $\alpha > Q(0)$ and $Q(t)$ are continuously changing, there must be a $t_3 \in [0, t_2)$ such that $\alpha = Q(t_3)$. It contradicts the assumption that $t_1 \in \boldsymbol{R}^+$ is the first time at which $\alpha = Q(t)$.

- *Case (iii)*: $\{\tilde{x}^+(t) > 0, \tilde{x}^-(t) < 0\}$.

(i) If $\{\tilde{x}^+(t) \geq -\tilde{x}^-(t)\}$, then $\max\{\mid \tilde{x}^+(t) \mid, \mid \tilde{x}^-(t) \mid\} = \tilde{x}^+(t)$. So, the proof is the same as that in Case (i).

(ii) If $\{\tilde{x}^+(t) < -\tilde{x}^-(t)\}$, then $\max\{\mid \tilde{x}^+(t) \mid, \mid \tilde{x}^-(t) \mid\} = -\tilde{x}^-(t)$. So, the proof is the same as that in Case (ii).

Combining the above three cases, it can be concluded that the derivative of $\max\{\mid \tilde{x}^+(t) \mid, \mid \tilde{x}^-(t) \mid\}$ will not be greater than zero. Hence, if $\alpha > Q(0)$, i.e., Assumption 6 holds, then $\alpha > Q(t)$, $\forall t \in \boldsymbol{R}^+$. It follows that $\alpha > \parallel f_i(t, x_i(t)) - f_0(t, x_0(t)) \parallel, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$, based on (5).

The proof is now completed.

8 Supplementary Lemma ii

Supplementary Lemma ii Let $\mathcal{F}$ denote the set-valued map. If Assumptions 1, 5 and 6 hold, then the set-valued Lie derivative $\tilde{\mathcal{L}}_{\mathcal{F}}V$ of V with respect to $\mathcal{F}$ satisfies that $\max \tilde{\mathcal{L}}_{\mathcal{F}}V < 0$ for all $(\tilde{x}^+(t), \tilde{x}^-(t)) \in \mathcal{D} \setminus \{(0, 0)\}$.

Proof If Assumptions 1, 5 and 6 hold, then Supplementary Lemma i holds, i.e., $\alpha > \parallel f_i(t, x_i(t)) - f_0(t, x_0(t)) \parallel$,
$\forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$. Five cases are discussed as follows:

- *Case (i)*: $\tilde{x}^+(t) > 0$ and $\tilde{x}^-(t) > 0$.

Since $\tilde{x}^+(t) > 0$ and Assumption 1 holds, one has $\sum_{r\in\mathcal{N}_i} a_{ir}[\tilde{x}^+(t) - \tilde{x}_r^k(t)] > 0$, and for

$$\partial V(\tilde{x}^+(t), \tilde{x}^-(t)) = \{(1, 0)\},$$

one has

$$\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha].$$

Since $| f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) | \leq \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, it follows from Supplementary Lemma i that

$$\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0.$$

- *Case (ii)*: $\tilde{x}^+(t) > 0$ and $\tilde{x}^-(t) < 0$.

Since $\tilde{x}^+(t) > 0, \tilde{x}^-(t) < 0$ and Assumption 1 holds, one has $\sum_{r\in\mathcal{N}_i} a_{ir}[\tilde{x}^+(t) - \tilde{x}_r^k(t)] > 0, \sum_{s\in\mathcal{N}_j} a_{js}[\tilde{x}^-(t) - \tilde{x}_s^l(t)] < 0$, and for

$$\partial V(\tilde{x}^+(t), \tilde{x}^-(t)) = \{(1, -1)\},$$

one has

$$\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}\left[(f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha) - (f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha)\right].$$

Since $| f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) | \leq \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, it follows from Supplementary Lemma i that

$$\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0.$$

- *Case (iii)*: $\tilde{x}^+(t) < 0$ and $\tilde{x}^-(t) < 0$.

Since $\tilde{x}^-(t) < 0$ and Assumption 1 holds, one has $\sum_{s\in\mathcal{N}_j} a_{js}[\tilde{x}^-(t) - \tilde{x}_s^l(t)] < 0$, and for

$$\partial V(\tilde{x}^+(t), \tilde{x}^-(t)) = \{(0, -1)\},$$

one has

$$\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}\left[-(f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha)\right].$$

Since $| f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) | \leq \| f_j(t, x_j(t)) - f_0(t, x_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall j = 1, 2, \cdots, N, \forall l = 1, 2, \cdots, n$, it follows from Supplementary Lemma i that

$$\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0.$$

- *Case* (*iv*): $\tilde{x}^+(t) > 0$ and $\tilde{x}^-(t) = 0$.

Since $\tilde{x}^+(t) > 0, \tilde{x}^-(t) = 0$ and Assumption 1 holds, one has $\sum_{r\in\mathcal{N}_i} a_{ir}[\tilde{x}^+(t) - \tilde{x}_r^k(t)] > 0, \sum_{s\in\mathcal{N}_j} a_{js}[\tilde{x}^-(t) - \tilde{x}_s^l(t)] \leq 0$. So, if $v \in \mathcal{F}(\tilde{x}^+(t), \tilde{x}^-(t))$, then $v^T = (v_1, v_2)$ with $v_1 \in \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha]$ and $v_2 \in \mathcal{K}[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha] \cup \mathcal{K}[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t))]$. For

$$\partial V(\tilde{x}^+(t), \tilde{x}^-(t)) = \{1\} \times [-1, 0],$$

if $\zeta \in \partial V(\tilde{x}^+(t), \tilde{x}^-(t))$, then $\zeta^T = (1, y)$ with $y \in [-1, 0]$. Therefore,

$$\zeta^T v = v_1 + y v_2.$$

If there exists an element a satisfying that $\zeta^T v = a$ for all $y \in [-1, 0]$, then $v_2 = 0$. So, if $v_2 \neq 0$, one has $\tilde{\mathcal{L}}_{\mathcal{F}} V = \emptyset$; if $v_2 = 0$, one has $\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha]$, and then it follows from Supplementary Lemma i that $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$. Thus, $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$ or $\tilde{\mathcal{L}}_{\mathcal{F}} V = \emptyset$ in this case.

- *Case* (*v*): $\tilde{x}^+(t) = 0$ and $\tilde{x}^-(t) < 0$.

Since $\tilde{x}^+(t) = 0, \tilde{x}^-(t) < 0$ and Assumption 1 holds, one has $\sum_{r\in\mathcal{N}_i} a_{ir}[\tilde{x}^+(t) - \tilde{x}_r^k(t)] \geq 0, \sum_{s\in\mathcal{N}_j} a_{js}[\tilde{x}^-(t) - \tilde{x}_s^l(t)] < 0$. So, if $v \in \mathcal{F}(\tilde{x}^+(t), \tilde{x}^-(t))$, then $v^T = (v_1, v_2)$ with $v_1 \in \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha] \cup \mathcal{K}[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t))]$ and $v_2 \in \mathcal{K}[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha]$. For

$$\partial V(\tilde{x}^+(t), \tilde{x}^-(t)) = [0, 1] \times \{-1\},$$

if $\zeta \in \partial V(\tilde{x}^+(t), \tilde{x}^-(t))$, then $\zeta^T = (y, -1)$ with $y \in [0, 1]$. Therefore,

$$\zeta^T v = y v_1 - v_2.$$

If there exists an element a satisfying that $\zeta^T v = a$ for all $y \in [0, 1]$, then $v_1 = 0$. So, if $v_1 \neq 0$, one has $\tilde{\mathcal{L}}_{\mathcal{F}} V = \emptyset$; if $v_1 = 0$, one has $\tilde{\mathcal{L}}_{\mathcal{F}} V = -\mathcal{K}[f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha]$, and then it follows from Supplementary Lemma i that $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$. Thus, $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$ or $\tilde{\mathcal{L}}_{\mathcal{F}} V = \emptyset$ in this case.

Combining the above five cases, it can be concluded that $\max \tilde{\mathcal{L}}_F V < 0$ for all $(\tilde{x}^+(t), \tilde{x}^-(t)) \in \mathcal{D} \setminus \{(0, 0)\}$.

The proof is now completed.

9 Proof of Corollary 1

Proof The nonsmooth function V, which was given by (6) in the manuscript, is chosen as the Lyapunov function. If Assumptions 1, 5 and 6 hold, then Supplementary Lemma ii holds. By using Lemma 1, it follows from Lemmas 3 - 5 and Supplementary Lemma ii that $(\tilde{x}^+(t), \tilde{x}^-(t)) = (0, 0)$ is a globally stable equilibrium point for system (2).

Next, the maximal converging time is considered.

- *Case (i)*: $\tilde{x}^+(t) > 0$ and $\tilde{x}^-(t) \geq 0$.

In this case, $V = \tilde{x}^+(t)$ and $\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}\Big[f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha\Big]$. By the proof of Supplementary Lemma i, one has $\| f_i(t, x_i(t)) - f_0(t, x_0(t)) \| \leq Q(t), Q(t) \leq Q(0), \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$. Since $| f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) | \leq \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, one has

$$\begin{aligned}\max \tilde{\mathcal{L}}_{\mathcal{F}} V &\leq -(\alpha - Q(t)) \\ &\leq -(\alpha - Q(0)).\end{aligned}$$

Therefore, the converging time satisfies

$$T_1 \leq \frac{1}{\alpha - Q(0)} \tilde{x}^+(0).$$

- *Case (ii)*: $\tilde{x}^+(t) > 0$ and $\tilde{x}^-(t) < 0$.

In this case, $V = \tilde{x}^+(t) - \tilde{x}^-(t)$ and $\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}\Big[(f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) - \alpha) - (f_j^l(t, x_j(t)) - f_0^l(t, x_0(t)) + \alpha)\Big]$. By the proof of Supplementary Lemma i, one has $\| f_i(t, x_i(t)) - f_0(t, x_0(t)) \| \leq Q(t), Q(t) \leq Q(0), \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$. Since $| f_i^k(t, x_i(t)) - f_0^k(t, x_0(t)) | \leq \| f_i(t, x_i(t)) - f_0(t, x_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, one has

$$\begin{aligned}\max \tilde{\mathcal{L}}_{\mathcal{F}} V &\leq -2(\alpha - Q(t)) \\ &\leq -2(\alpha - Q(0)).\end{aligned}$$

Therefore, the converging time satisfies

$$\begin{aligned}T_2 &\leq \frac{1}{2(\alpha - Q(0))}(\tilde{x}^+(0) - \tilde{x}^-(0)) \\ &\leq \frac{1}{\alpha - Q(0)} \max\{\tilde{x}^+(0), -\tilde{x}^-(0)\}.\end{aligned}$$

- *Case (iii)*: $\tilde{x}^+(t) \leq 0$ and $\tilde{x}^-(t) < 0$.

In this case, $V = -\tilde{x}^-(t)$ and $\tilde{\mathcal{L}}_{\mathcal{F}}V = \mathcal{K}\Big[-(f_j^l(t,x_j(t)) - f_0^l(t,x_0(t)) + \alpha)\Big]$. By the proof of Supplementary Lemma i, one has $\| f_j(t,x_j(t)) - f_0(t,x_0(t)) \| \le Q(t), Q(t) \le Q(0), \forall t \in \boldsymbol{R}^+, \forall j = 1,2,\cdots,N$. Since $| f_j^l(t,x_j(t)) - f_0^l(t,x_0(t)) | \le \| f_j(t,x_j(t)) - f_0(t,x_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall j = 1,2,\cdots,N, \forall l = 1,2,\cdots,n$, one has

$$\begin{aligned}\max \tilde{\mathcal{L}}_{\mathcal{F}}V &\le -(\alpha - Q(t)) \\ &\le -(\alpha - Q(0)).\end{aligned}$$

Therefore, the converging time satisfies

$$T_3 \le -\frac{1}{\alpha - Q(0)}\tilde{x}^-(0).$$

Combining the above three cases, the maximal converging time is obtained as

$$T = \frac{1}{\alpha - Q(0)} \max_{\substack{i=1,2,\cdots,N \\ k=1,2,\cdots,n}} \{| x_i^k(0) - x_0^k(0) - x_i^{*k} |\}.$$

The proof is now completed.

10 Proof of Lemma 7

Proof Define the formation position errors $\tilde{r}_i(t) = r_i(t) - r_0(t) - r_i^*$ and the velocity errors $\tilde{v}_i(t) = v_i(t) - v_0(t), i = 1,2,\cdots,N$, with $\tilde{r}_0(t) = 0$ and $\tilde{v}_0(t) = 0$. Sliding mode is designed as $S_i(t) = \tilde{r}_i(t) + \tilde{v}_i(t)$. The Filippov solution of $S_i(t)$ is defined as the absolutely continuous solution of the differential inclusion

$$\dot{S}_i(t) \in \mathcal{K}\left[F_i(t,r_i(t),v_i(t)) - F_0(t,r_0(t),v_0(t)) - \alpha\,\mathrm{sgn}\left\{\sum_{j\in\mathcal{N}_i} a_{ij}[S_i(t) - S_j(t)]\right\}\right], \quad \forall i = 1,2,\cdots,N.$$

Based on Assumption 1, one follower must receive information from other followers or the leader, in other words, it is connected with other followers or the leader. Define $S^+(t)$ as the maximal error component which is connected with non-maximal error components of the followers or connected with the component of the leader. Similarly, define $S^-(t)$ as the minimal error component which is connected with non-minimal error components of the followers or connected with the component of the leader. Suppose that, at any time t, $S^+(t)$ is the kth error component of agent

i and $S^-(t)$ is the lth error component of agent j, where $i,j \in \{1,2,\cdots,N\}$, $k,l \in \{1,2,\cdots,n\}$. The Filippov solutions of $S^+(t)$ and $S^-(t)$ can be described by

$$\begin{aligned}\dot{S}^+(t) \in& \mathcal{K}\left[F_i^k(t,r_i(t),v_i(t)) - F_0^k(t,r_0(t),v_0(t)) - \alpha\,\mathrm{sgn}\left\{\sum_{r\in\mathcal{N}_i} a_{ir}[S^+(t)-S_r^k(t)]\right\}\right],\\ \dot{S}^-(t) \in& \mathcal{K}\left[F_j^l(t,r_j(t),v_j(t)) - F_0^l(t,r_0(t),v_0(t)) - \alpha\,\mathrm{sgn}\left\{\sum_{s\in\mathcal{N}_j} a_{js}[S^-(t)-S_s^l(t)]\right\}\right].\end{aligned} \tag{7}$$

Based on Assumptions 7 and 8, for any $i=1,2,\cdots,N$ and each $t\in\boldsymbol{R}^+$, one has

$$\begin{aligned}
&\parallel F_i(t,r_i(t),v_i(t)) - F_0(t,r_0(t),v_0(t)) \parallel\\
=&\parallel f_i(t,r_i(t),v_i(t)) + v_i(t) - f_0(t,r_0(t),v_0(t)) - v_0(t) \parallel\\
=&\parallel f_i(t,r_i(t),v_i(t)) - f_i(t,r_0(t),v_0(t)) + f_i(t,r_0(t),v_0(t)) - f_0(t,r_0(t),v_0(t)) + v_i(t) - v_0(t) \parallel\\
\le&\parallel f_i(t,r_i(t),v_i(t)) - f_i(t,r_0(t),v_0(t)) \parallel + \parallel f_i(t,r_0(t),v_0(t)) \parallel\\
&+\parallel f_0(t,r_0(t),v_0(t)) \parallel + \parallel v_i(t) - v_0(t) \parallel\\
\le&\parallel f_i(t,r_i(t),v_i(t)) - f_i(t,r_0(t),v_0(t)) \parallel + \parallel f_i(t,r_0(t),v_0(t)) - f_i(t,r_i^E,v_i^E) \parallel\\
&+\parallel f_0(t,r_0(t),v_0(t)) - f_0(t,r_0^E,v_0^E) \parallel + \parallel v_i(t) - v_0(t) \parallel\\
\le& L_J^F(\parallel r_i(t)-r_0(t) \parallel + \parallel v_i(t)-v_0(t) \parallel) + L_J^F\left(\parallel r_0(t)-r_i^E \parallel + \parallel v_0(t)-v_i^E \parallel\right)\\
&+L_J^L\left(\parallel r_0(t)-r_0^E \parallel + \parallel v_0(t)-v_0^E \parallel\right) + (\parallel v_i(t)-v_0(t) \parallel)\\
\le& L_J^F(\parallel r_i(t)-r_0(t)-r_i^* \parallel + \parallel r_i^* \parallel + \parallel v_i(t)-v_0(t) \parallel) + L_J^F\left(\parallel r_0(t)-r_i^E \parallel + \parallel v_0(t)-v_i^E \parallel\right)\\
&+L_J^L\left(\parallel r_0(t)-r_0^E \parallel + \parallel v_0(t)-v_0^E \parallel\right) + (\parallel v_i(t)-v_0(t) \parallel)\\
\le& L_J^F \parallel \tilde{r}_i(t) \parallel + (L_J^F+1) \parallel \tilde{v}_i(t) \parallel + L_J^F(\parallel r_i^* \parallel + \parallel r_i^E \parallel + \parallel v_i^E \parallel + \beta_r + \beta_v)\\
&+L_J^L(\parallel r_0^E \parallel + \parallel v_0^E \parallel + \beta_r + \beta_v).
\end{aligned}$$

Let

$$G = L_J^F\Big(\max_{i=1,2,\cdots,N}\{\parallel r_i^* \parallel + \parallel r_i^E \parallel + \parallel v_i^E \parallel\} + \beta_r + \beta_v\Big) + L_J^L(\parallel r_0^E \parallel + \parallel v_0^E \parallel + \beta_r + \beta_v).$$

Clearly, G is a constant. Since $\tilde{v}_i(t) = S_i(t) - \tilde{r}_i(t)$, one has $\parallel \tilde{v}_i(t) \parallel \le \parallel S_i(t) \parallel + \parallel \tilde{r}_i(t) \parallel$. Thus,

$$
\begin{aligned}
&\| F_i(t,r_i(t),v_i(t)) - F_0(t,r_0(t),v_0(t)) \| \\
\leq & L_J^F \| \tilde{r}_i(t) \| + \left(L_J^F+1\right) \| (S_i(t)-\tilde{r}_i(t)) \| + G \\
\leq & \left(2L_J^F+1\right) \| \tilde{r}_i(t) \| + \left(L_J^F+1\right) \| S_i(t) \| + G \\
\leq & \left(2L_J^F+1\right)\sqrt{n} \max_{\substack{i=1,2,\cdots,N \\ k=1,2,\cdots,n}} \{ | \tilde{r}_i^k(t) |, | S_i^k(t) | \} + \left(L_J^F+1\right)\sqrt{n} \max_{\substack{i=1,2,\cdots,N \\ k=1,2,\cdots,n}} \{ | S_i^k(t) | \} + G
\end{aligned} \tag{8}
$$

Let

$$
M(t) = (2L_J^F+1)\sqrt{n} \max_{\substack{i=1,2,\cdots,N \\ k=1,2,\cdots,n}} \{ | \tilde{r}_i^k(t) |, | S_i^k(t) | \} + (L_J^F+1)\sqrt{n} \max_{\substack{i=1,2,\cdots,N \\ k=1,2,\cdots,n}} \{ | S_i^k(t) | \} + G.
$$

If $\alpha > M(t), \forall t \in \boldsymbol{R}^+$, then $\alpha > \| F_i(t,r_i(t),v_i(t)) - F_0(t,r_0(t),v_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall i = 1,2,\cdots,N$.

Now, it can be proved that if $\alpha > M(0)$ then $\alpha > M(t)$, $\forall t \in \boldsymbol{R}^+$. Because $\alpha > M(0)$ and $M(t)$ are continuously changing, suppose that $t_1 \in \boldsymbol{R}^+$ is the first time at which $\alpha = M(t)$. Thus, $M(t_1) > M(0)$.

Now, consider the following two cases.

- *Case (i)*: The signs of $\tilde{r}_i^k(t)$ and $\tilde{v}_i^k(t)$ are the same. In this case, $| \tilde{r}_i^k(t) |$ will increase. Since $| \tilde{r}_i^k(t) | + | \tilde{v}_i^k(t) | = | \tilde{r}_i^k(t) + \tilde{v}_i^k(t) | = | S_i^k(t) |$, it follows that $| \tilde{r}_i^k(t) | \leq | S_i^k(t) |$.

- *Case (ii)*: The signs of $\tilde{r}_i^k(t)$ and $\tilde{v}_i^k(t)$ are opposite. In this case, one has $| \tilde{r}_i^k(t) | + | \tilde{v}_i^k(t) | = | \tilde{r}_i^k(t) - \tilde{v}_i^k(t) | \geq | S_i^k(t) |$. Both $| \tilde{r}_i^k(t) | \leq | S_i^k(t) |$ and $| \tilde{r}_i^k(t) | \geq | S_i^k(t) |$ are possible. For $\tilde{v}_i^k(t) = \dot{\tilde{r}}_i^k(t)$ and their signs are opposite, $| \tilde{r}_i^k(t) |$ must decrease.

Combining the above two cases, it can be concluded that $| \tilde{r}_i^k(t) |$ must be decreasing when $| \tilde{r}_i^k(t) | \geq | S_i^k(t) |$. Since α, L_J^F and G are constants, if $M(t_1) > M(0)$, then $\max_{\substack{i=1,2,\cdots,N \\ k=1,2,\cdots,n}} \{ | S_i^k(t_1) | \}$ must be larger than $\max_{\substack{i=1,2,\cdots,N \\ k=1,2,\cdots,n}} \{ | S_i^k(0) | \}$. So, there must exist a $t_2 \in [0, t_1)$ such that the derivative of $\max\{ | S^+(t) |, | S^-(t) | \}$ is greater than zero.

Now, consider the following three cases.

- *Case (i)*: $\{S^+(t) > 0, S^-(t) \geq 0\}$.

In this case, $\max\{ | S^+(t) |, | S^-(t) | \} = S^+(t)$, and the derivative of $\max\{ | S^+(t) |, | S^-(t) | \}$ is $\dot{S}^+(t)$. Since Assumption 1 holds and $S^+(t) > 0$, one has $\sum_{r \in \mathcal{N}_i} a_{ir}[S^+(t) - S_r^k(t)] > 0$. Thus,

$$
\dot{S}^+(t) \in \mathcal{K}\left[F_i^k(t,r_i(t),v_i(t)) - F_0^k(t,r_0(t),v_0(t)) - \alpha \right].
$$

If the derivative of $\max\{ | S^+(t) |, | S^-(t) | \}$ is greater than zero at $t_2 \in [0, t_1)$, one has $\dot{S}^+(t_2) > 0$. Then, there must exist $i \in \{1,2,\cdots,N\}$ and $k \in \{1,2,\cdots,n\}$ such that $F_i^k(t_2, r_i(t_2), v_i(t_2)) -$

$F_0^k(t_2, r_0(t_2), v_0(t_2)) > 0$ and the positive constant $\alpha <\mid F_i^k(t_2, r_i(t_2), v_i(t_2)) - F_0^k(t_2, r_0(t_2), v_0(t_2)) \mid$. Since $\mid F_i^k(t_2, r_i(t_2), v_i(t_2)) - F_0^k(t_2, r_0(t_2), v_0(t_2)) \mid \leq \parallel F_i(t_2, r_i(t_2), v_i(t_2)) - F_0(t_2, r_0(t_2), v_0(t_2)) \parallel$, one has $\alpha < \parallel F_i(t_2, r_i(t_2), v_i(t_2)) - F_0(t_2, r_0(t_2), v_0(t_2)) \parallel$. It follows that $\alpha < M(t_2)$ based on (8). Because $\alpha > M(0)$ and $M(t)$ are continuously changing, there must be a $t_3 \in [0, t_2)$ such that $\alpha = M(t_3)$. It contradicts the assumption that $t_1 \in \boldsymbol{R}^+$ is the first time at which $\alpha = M(t)$.

- *Case (ii)*: $\{S^+(t) \leq 0, S^-(t) < 0\}$.

In this case, $\max\{\mid S^+(t) \mid, \mid S^-(t) \mid\} = -S^-(t)$, and the derivative of $\max\{\mid S^+(t) \mid, \mid S^-(t) \mid \}$ is $-\dot{S}^-(t)$. Since Assumption 1 holds and $S^-(t) < 0$, one has $\sum_{s \in \mathcal{N}_j} a_{js}[S^-(t) - S_s^l(t)] < 0$. Thus,

$$\dot{S}^-(t) \in \mathcal{K}\left[F_j^l(t, r_j(t), v_j(t)) - F_0^l(t, r_0(t), v_0(t)) + \alpha\right].$$

If the derivative of $\max\{\mid S^+(t) \mid, \mid S^-(t) \mid\}$ is greater than zero at $t_2 \in [0, t_1)$, one has $\dot{S}^-(t_2) < 0$. Then, there must exist $j \in \{1, 2, \cdots, N\}$ and $l \in \{1, 2, \cdots, n\}$ such that $F_j^l(t_2, r_j(t_2), v_j(t_2)) - F_0^l(t_2, r_0(t_2), v_0(t_2)) < 0$ and the positive constant $\alpha <\mid F_j^l(t_2, r_j(t_2), v_j(t_2)) - F_0^l(t_2, r_0(t_2), v_0(t_2)) \mid$. Since $\mid F_j^l(t_2, r_j(t_2), v_j(t_2)) - F_0^l(t_2, r_0(t_2), v_0(t_2)) \mid \leq \parallel F_j(t_2, r_j(t_2), v_j(t_2)) - F_0(t_2, r_0(t_2), v_0(t_2)) \parallel$, one has $\alpha < \parallel F_j(t_2, r_j(t_2), v_j(t_2)) - F_0(t_2, r_0(t_2), v_0(t_2)) \parallel$. It follows that $\alpha < M(t_2)$ based on (8). Because $\alpha > M(0)$ and $M(t)$ are continuously changing, there must be a $t_3 \in [0, t_2)$ such that $\alpha = M(t_3)$. It contradicts the assumption that $t_1 \in \boldsymbol{R}^+$ is the first time at which $\alpha = M(t)$.

- *Case (iii)*: $\{S^+(t) > 0, S^-(t) < 0\}$.

(*a*) If $\{S^+(t) \geq -S^-(t)\}$, then $\max\{\mid S^+(t) \mid, \mid S^-(t) \mid\} = S^+(t)$. So, the proof is the same as that in Case (i).

(*b*) If $\{S^+(t) < -S^-(t)\}$, then $\max\{\mid S^+(t) \mid, \mid S^-(t) \mid\} = -S^-(t)$. So, the proof is the same as that in Case (ii).

Combining the above three cases, it can be concluded that the derivative of $\max\{\mid S^+(t) \mid, \mid S^-(t) \mid\}$ will not be greater than zero. Hence, if $\alpha > M(0)$, i.e., Assumption 9 holds, then $\alpha > M(t)$, $\forall t \in \boldsymbol{R}^+$. It follows that $\alpha > \parallel F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \parallel, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$, based on (8).

The proof is now completed.

11 Proof of Lemma 8

Proof If Assumptions 1 and 7 - 9 hold, then Lemma 7 holds, i.e., $\alpha > \parallel F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \parallel, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$. Based on (6) in the manuscript, the nonsmooth function $V(S^+(t), S^-(t)) : \boldsymbol{R}^2 \rightarrow \boldsymbol{R}$ is

$$V(S^+(t), S^-(t)) = \begin{cases} S^+(t) & S^+(t) \geq 0, S^-(t) \geq 0 \\ S^+(t) - S^-(t) & S^+(t) > 0, S^-(t) < 0 \\ -S^-(t) & S^+(t) \leq 0, S^-(t) < 0. \end{cases} \tag{9}$$

Five cases are discussed as follows:

- *Case* (*i*): $S^+(t) > 0$ and $S^-(t) > 0$.

Since Assumption 1 holds and $S^+(t) > 0$, one has $\sum_{r \in \mathcal{N}_i} a_{ir}[S^+(t) - S_r^k(t)] > 0$, and for

$$\partial V(S^+(t), S^-(t)) = \{(1, 0)\},$$

one has

$$\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}\left[F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) - \alpha\right].$$

Since $\mid F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) \mid \leq \parallel F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \parallel, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, it follows from Lemma 7 that

$$\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0.$$

- *Case* (*ii*): $S^+(t) > 0$ and $S^-(t) < 0$.

Since $S^+(t) > 0, S^-(t) < 0$ and Assumption 1 holds, one has $\sum_{r \in \mathcal{N}_i} a_{ir}[S^+(t) - S_r^k(t)] > 0, \sum_{s \in \mathcal{N}_j} a_{js}[S^-(t) - S_s^l(t)] < 0$, and for

$$\partial V(S^+(t), S^-(t)) = \{(1, -1)\},$$

one has

$$\begin{aligned} \tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}\Big[&(F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) - \alpha) \\ &- (F_j^l(t, r_j(t), v_j(t)) - F_0^l(t, r_0(t), v_0(t)) + \alpha)\Big]. \end{aligned}$$

Since $\mid F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) \mid \leq \parallel F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \parallel, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, it follows from Lemma 7 that

$$\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0.$$

- *Case (iii)*: $S^+(t) < 0$ and $S^-(t) < 0$.

Since $S^-(t) < 0$ and Assumption 1 holds, one has $\sum_{s\in\mathcal{N}_j} a_{js}[S^-(t) - S_s^l(t)] < 0$, and for

$$\partial V(S^+(t), S^-(t)) = \{(0, -1)\},$$

one has

$$\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}\left[-(F_j^l(t, r_j(t), v_j(t)) - F_0^l(t, r_0(t), v_0(t)) + \alpha)\right].$$

Since $\mid F_j^l(t, r_j(t), v_j(t)) - F_0^l(t, r_0(t), v_0(t)) \mid \leq \| F_j(t, r_j(t), v_j(t)) - F_0(t, r_0(t), v_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall j = 1, 2, \cdots, N, \forall l = 1, 2, \cdots, n$, it follows from Lemma 7 that

$$\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0.$$

- *Case (iv)*: $S^+(t) > 0$ and $S^-(t) = 0$.

Since $S^+(t) > 0, S^-(t) = 0$ and Assumption 1 holds, one has $\sum_{r\in\mathcal{N}_i} a_{ir}[S^+(t) - S_r^k(t)] > 0, \sum_{s\in\mathcal{N}_j} a_{js}[S^-(t) - S_s^l(t)] \leq 0$. So, if $v \in \mathcal{F}(S^+(t), S^-(t))$, then $v^T = (v_1, v_2)$ with $v_1 \in \mathcal{K}[F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) - \alpha]$ and $v_2 \in \mathcal{K}[F_j^l(t, r_j(t), v_j(t)) - F_0^l(t, r_0(t), v_0(t)) + \alpha] \cup \mathcal{K}[F_j^l(t, r_j(t), v_j(t)) - F_0^l(t, r_0(t), v_0(t))]$. For

$$\partial V(S^+(t), S^-(t)) = \{1\} \times [-1, 0],$$

if $\zeta \in \partial V(S^+(t), S^-(t))$, then $\zeta^T = (1, y)$ with $y \in [-1, 0]$. Therefore,

$$\zeta^T v = v_1 + y v_2.$$

If there exists an element a satisfying that $\zeta^T v = a$ for all $y \in [-1, 0]$, then $v_2 = 0$. So, if $v_2 \neq 0$, one has $\tilde{\mathcal{L}}_{\mathcal{F}} V = \emptyset$; if $v_2 = 0$, one has $\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}[F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) - \alpha]$, and then it follows from Lemma 7 that $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$. Thus, $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$ or $\tilde{\mathcal{L}}_{\mathcal{F}} V = \emptyset$ in this case.

- *Case (v)*: $S^+(t) = 0$ and $S^-(t) < 0$.

Since $S^+(t) = 0, S^-(t) < 0$ and Assumption 1 holds, one has $\sum_{r\in\mathcal{N}_i} a_{ir}[S^+(t) - S_r^k(t)] \geq 0, \sum_{s\in\mathcal{N}_j} a_{js}[S^-(t) - S_s^l(t)] < 0$. So, if $v \in \mathcal{F}(S^+(t), S^-(t))$, then $v^T = (v_1, v_2)$ with $v_1 \in \mathcal{K}[F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) - \alpha] \cup \mathcal{K}[F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t))]$ and $v_2 \in \mathcal{K}[F_j^l(t, r_j(t), v_j(t)) - F_0^l(t, r_0(t), v_0(t)) + \alpha]$. For

$$\partial V(S^+(t), S^-(t)) = [0, 1] \times \{-1\},$$

if $\zeta \in \partial V(S^+(t), S^-(t))$, then $\zeta^T = (y, -1)$ with $y \in [0, 1]$. Therefore,

$$\zeta^T v = y v_1 - v_2.$$

If there exists an element a satisfying that $\zeta^T v = a$ for all $y \in [0,1]$, then $v_1 = 0$. So, if $v_1 \neq 0$, one has $\tilde{\mathcal{L}}_{\mathcal{F}} V = \emptyset$; if $v_1 = 0$, one has $\tilde{\mathcal{L}}_{\mathcal{F}} V = -\mathcal{K}[F_j^l(t, r_j(t), v_j(t)) - F_0^l(t, r_0(t), v_0(t)) + \alpha]$, and then it follows from Lemma 7 that $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$. Thus, $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$ or $\tilde{\mathcal{L}}_{\mathcal{F}} V = \emptyset$ in this case.

Combining the above five cases, it can be concluded that $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$ for all $(S^+(t), S^-(t)) \in \mathcal{D} \setminus \{(0,0)\}$.

The proof is now completed.

12 Proof of Theorem 2

Proof The nonsmooth function $V(S^+(t), S^-(t))$ in (9) is chosen as the Lyapunov function. If Assumptions 1 and 7 - 9 hold, then Lemma 8 holds. By Lemma 1, it follows from Lemmas 3 - 5 and 8 that $(S^+(t), S^-(t)) = (0, 0)$ is a globally stable equilibrium point for system (7).

Solving

$$S_i^k(t) = \tilde{r}_i^k(t) + \dot{\tilde{r}}_i^k(t) = 0,$$

one has

$$\tilde{r}_i^k(t) = ce^{-t}, \dot{\tilde{r}}_i^k(t) = -ce^{-t},$$

where c is a constant determined by the initial conditions. Therefore, the errors $\tilde{r}_i(t)$ and $\tilde{v}_i(t)$ converge to zero exponentially; that is, the second-order multi-agent system achieves the desired formation asymptotically.

The proof is now completed.

13 Supplementary Lemma iii

Supplementary Lemma iii If Assumptions 1, 10 and 11 hold, then $\alpha > \| F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$, where $F_i(t, r_i(t), v_i(t)) = v_i(t) + f_i(t, r_i(t), v_i(t))$

and $F_0(t, r_0(t), v_0(t)) = v_0(t) + f_0(t, r_0(t), v_0(t))$.

Proof Based on Assumption 10, for any $i = 1, 2, \cdots, N$ and each $t \in \boldsymbol{R}^+$, one has

$$\begin{aligned}
&\| F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \| \\
=&\| f_i(t, r_i(t), v_i(t)) + v_i(t) - f_0(t, r_0(t), v_0(t)) - v_0(t) \| \\
=&\| f_0(t, r_i(t), v_i(t)) - f_0(t, r_0(t), v_0(t)) + v_i(t) - v_0(t) \| \\
\leq&\| f_0(t, r_i(t), v_i(t)) - f_0(t, r_0(t), v_0(t)) \| + \| v_i(t) - v_0(t) \| \\
\leq& L_J^L(\| r_i(t) - r_0(t) \| + \| v_i(t) - v_0(t) \|) + \| v_i(t) - v_0(t) \| \\
\leq& L_J^L(\| r_i(t) - r_0(t) - r_i^* \| + \| r_i^* \| + \| v_i(t) - v_0(t) \|) + \| v_i(t) - v_0(t) \| \\
\leq& L_J^L(\| \tilde{r}_i(t) \| + \| r_i^* \| + \| \tilde{v}_i(t) \|) + \| \tilde{v}_i(t) \|
\end{aligned}$$

Since $\tilde{v}_i(t) = S_i(t) - \tilde{r}_i(t)$, one has $\| \tilde{v}_i(t) \| \leq \| S_i(t) \| + \| \tilde{r}_i(t) \|$. Thus,

$$\begin{aligned}
&\| F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \| \\
\leq&(2L_J^L + 1) \| \tilde{r}_i(t) \| + (L_J^L + 1) \| S_i(t) \| + L_J^L \| r_i^* \| \\
\leq&(2L_J^L + 1)\sqrt{n} \max_{\substack{i=1,2,\cdots,N \\ k=1,2,\cdots,n}} \{| \tilde{r}_i^k(t) |, | S_i^k(t) |\} + (L_J^L + 1)\sqrt{n} \max_{\substack{i=1,2,\cdots,N \\ k=1,2,\cdots,n}} \{| S_i^k(t) |\} \\
&+ L_J^L \max_{i=1,2,\cdots,N} \{\| r_i^* \|\}.
\end{aligned} \tag{10}$$

Let

$$\begin{aligned}
W(t) =&(2L_J^L + 1)\sqrt{n} \max_{\substack{i=1,2,\cdots,N \\ k=1,2,\cdots,n}} \{| \tilde{r}_i^k(t) |, | S_i^k(t) |\} \\
&+ (L_J^L + 1)\sqrt{n} \max_{\substack{i=1,2,\cdots,N \\ k=1,2,\cdots,n}} \{| S_i^k(t) |\} + L_J^L \max_{i=1,2,\cdots,N} \{\| r_i^* \|\}.
\end{aligned}$$

If $\alpha > W(t), \forall t \in \boldsymbol{R}^+$, then $\alpha > \| F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \|, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$.

Now, it can be proved that if $\alpha > W(0)$ then $\alpha > W(t)$, $\forall t \in \boldsymbol{R}^+$. Because $\alpha > W(0)$ and $W(t)$ are continuously changing, suppose that $t_1 \in \boldsymbol{R}^+$ is the first time at which $\alpha = W(t)$. Thus, $W(t_1) > W(0)$.

Now, consider the following two cases.

- *Case* (i): The signs of $\tilde{r}_i^k(t)$ and $\tilde{v}_i^k(t)$ are the same. In this case, $| \tilde{r}_i^k(t) |$ will increase. Since $| \tilde{r}_i^k(t) | + | \tilde{v}_i^k(t) | = | \tilde{r}_i^k(t) + \tilde{v}_i^k(t) | = | S_i^k(t) |$, it follows that $| \tilde{r}_i^k(t) | \leq | S_i^k(t) |$.

• *Case* (*ii*): The signs of $\tilde{r}_i^k(t)$ and $\tilde{v}_i^k(t)$ are opposite. In this case, one has $| \tilde{r}_i^k(t) | + | \tilde{v}_i^k(t) | = | \tilde{r}_i^k(t) - \tilde{v}_i^k(t) | \geq | S_i^k(t) |$. Both $| \tilde{r}_i^k(t) | \leq | S_i^k(t) |$ and $| \tilde{r}_i^k(t) | \geq | S_i^k(t) |$ are possible. For $\tilde{v}_i^k(t) = \dot{\tilde{r}}_i^k(t)$ and their signs are opposite, $| \tilde{r}_i^k(t) |$ must decrease.

Combining the above two cases, it can be concluded that $| \tilde{r}_i^k(t) |$ must be decreasing when $| \tilde{r}_i^k(t) | \geq | S_i^k(t) |$. Since α, L_J^L and $\max_{i=1,2,\cdots,N}\{\| r_i^* \|\}$ are constants, if $W(t_1) > W(0)$, then $\max_{\substack{i=1,2,\cdots,N\\k=1,2,\cdots,n}}\{| S_i^k(t_1) |\}$ must be larger than $\max_{\substack{i=1,2,\cdots,N\\k=1,2,\cdots,n}}\{| S_i^k(0) |\}$. So, there must exist a $t_2 \in [0, t_1)$ such that the derivative of $\max\{| S^+(t) |, | S^-(t) |\}$ is greater than zero.

Now, consider the following three cases.

• *Case* (*i*): $\{S^+(t) > 0, S^-(t) \geq 0\}$.

In this case, $\max\{| S^+(t) |, | S^-(t) |\} = S^+(t)$, and the derivative of $\max\{| S^+(t) |, | S^-(t) |\}$ is $\dot{S}^+(t)$. Since Assumption 1 holds and $S^+(t) > 0$, one has $\sum_{r\in\mathcal{N}_i} a_{ir}[S^+(t) - S_r^k(t)] > 0$. Thus,

$$\dot{S}^+(t) \in \mathcal{K}\left[F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) - \alpha\right].$$

If the derivative of $\max\{| S^+(t) |, | S^-(t) |\}$ is greater than zero at $t_2 \in [0, t_1)$, one has $\dot{S}^+(t_2) > 0$. Then, there must exist $i \in \{1, 2, \cdots, N\}$ and $k \in \{1, 2, \cdots, n\}$ such that $F_i^k(t_2, r_i(t_2), v_i(t_2)) - F_0^k(t_2, r_0(t_2), v_0(t_2)) > 0$ and the positive constant $\alpha < | F_i^k(t_2, r_i(t_2), v_i(t_2)) - F_0^k(t_2, r_0(t_2), v_0(t_2)) |$. Since $| F_i^k(t_2, r_i(t_2), v_i(t_2)) - F_0^k(t_2, r_0(t_2), v_0(t_2)) | \leq \| F_i(t_2, r_i(t_2), v_i(t_2)) - F_0(t_2, r_0(t_2), v_0(t_2)) \|$, one has $\alpha < \| F_i(t_2, r_i(t_2), v_i(t_2)) - F_0(t_2, r_0(t_2), v_0(t_2)) \|$. It follows that $\alpha < W(t_2)$ based on (10). Because $\alpha > W(0)$ and $W(t)$ are continuously changing, there must be a $t_3 \in [0, t_2)$ such that $\alpha = W(t_3)$. It contradicts the assumption that $t_1 \in \boldsymbol{R}^+$ is the first time at which $\alpha = W(t)$.

• *Case* (*ii*): $\{S^+(t) \leq 0, S^-(t) < 0\}$.

In this case, $\max\{| S^+(t) |, | S^-(t) |\} = -S^-(t)$, and the derivative of $\max\{| S^+(t) |, | S^-(t) |\}$ is $-\dot{S}^-(t)$. Since Assumption 1 holds and $S^-(t) < 0$, one has $\sum_{s\in\mathcal{N}_j} a_{js}[S^-(t) - S_s^l(t)] < 0$. Thus,

$$\dot{S}^-(t) \in \mathcal{K}\left[F_j^l(t, r_j(t), v_j(t)) - F_0^l(t, r_0(t), v_0(t)) + \alpha\right].$$

If the derivative of $\max\{| S^+(t) |, | S^-(t) |\}$ is greater than zero at $t_2 \in [0, t_1)$, one has $\dot{S}^-(t_2) < 0$. Then, there must exist $i \in \{1, 2, \cdots, N\}$ and $k \in \{1, 2, \cdots, n\}$ such that $F_j^l(t_2, r_j(t_2), v_j(t_2)) - F_0^l(t_2, r_0(t_2), v_0(t_2)) < 0$ and the positive constant $\alpha < | F_j^l(t_2, r_j(t_2), v_j(t_2)) - F_0^l(t_2, r_0(t_2), v_0(t_2)) |$. Since $| F_j^l(t_2, r_j(t_2), v_j(t_2)) - F_0^l(t_2, r_0(t_2), v_0(t_2)) | \leq \| F_j(t_2, r_j(t_2), v_j(t_2)) - F_0(t_2, r_0(t_2), v_0(t_2)) \|$, one has $\alpha < \| F_j(t_2, r_j(t_2), v_j(t_2)) - F_0(t_2, r_0(t_2), v_0(t_2)) \|$. It follows that $\alpha < W(t_2)$ based on

(10). Because $\alpha > W(0)$ and $W(t)$ are continuously changing, there must be a $t_3 \in [0, t_2)$ such that $\alpha = W(t_3)$. It contradicts the assumption that $t_1 \in \boldsymbol{R}^+$ is the first time at which $\alpha = W(t)$.

• *Case* (iii): $\{S^+(t) > 0, S^-(t) < 0\}$.

(a) If $\{S^+(t) \geq -S^-(t)\}$, then $\max\{\mid S^+(t) \mid, \mid S^-(t) \mid\} = S^+(t)$. So, the proof is the same as that in Case (i).

(b) If $\{S^+(t) < -S^-(t)\}$, then $\max\{\mid S^+(t) \mid, \mid S^-(t) \mid\} = -S^-(t)$. So, the proof is the same as that in Case (ii).

Combining the above three cases, it can be concluded that the derivative of $\max\{\mid S^+(t) \mid,$ $\mid S^-(t) \mid\}$ will not be greater than zero. Hence, if $\alpha > W(0)$, i.e., Assumption 11 holds, then $\alpha > W(t)$, $\forall t \in \boldsymbol{R}^+$. It follows that $\alpha > \parallel F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \parallel, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$, based on (10).

The proof is now completed.

14 Supplementary Lemma iv

Supplementary Lemma iv Let $\mathcal{F}$ denote the set-valued map. If Assumptions 1, 10 and 11 hold, then the set-valued Lie derivative $\tilde{\mathcal{L}}_{\mathcal{F}}V$ of V with respect to $\mathcal{F}$ satisfies that $\max \tilde{\mathcal{L}}_{\mathcal{F}}V < 0$ for all $(S^+(t), S^-(t)) \in \mathcal{D} \setminus \{(0, 0)\}$.

Proof If Assumptions 1, 10 and 11 hold, then Supplementary Lemma iii holds, i.e., $\alpha > \parallel F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \parallel, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N$. The nonsmooth function $V(S^+(t), S^-(t)) : \boldsymbol{R}^2 \to \boldsymbol{R}$ was given by (9).

Five cases are discussed as follows:

• *Case* (i): $S^+(t) > 0$ and $S^-(t) > 0$.

Since Assumption 1 holds and $S^+(t) > 0$, one has $\sum_{r \in \mathcal{N}_i} a_{ir}[S^+(t) - S_r^k(t)] > 0$, and for

$$\partial V(S^+(t), S^-(t)) = \{(1, 0)\},$$

one has

$$\tilde{\mathcal{L}}_{\mathcal{F}}V = \mathcal{K}\left[F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) - \alpha\right].$$

Since $\mid F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) \mid \leq \parallel F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \parallel, \forall t \in$

$\boldsymbol{R}^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, it follows from Supplementary Lemma iii that

$$\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0.$$

- *Case (ii)*: $S^+(t) > 0$ and $S^-(t) < 0$.

Since $S^+(t) > 0, S^-(t) < 0$ and Assumption 1 holds, one has $\sum_{r \in \mathcal{N}_i} a_{ir}[S^+(t) - S^k_r(t)] > 0, \sum_{s \in \mathcal{N}_j} a_{js}[S^-(t) - S^l_s(t)] < 0$, and for

$$\partial V(S^+(t), S^-(t)) = \{(1, -1)\},$$

one has

$$\begin{aligned}\tilde{\mathcal{L}}_F V = \mathcal{K}\Big[&(F^k_i(t, r_i(t), v_i(t)) - F^k_0(t, r_0(t), v_0(t)) - \alpha) \\ &- (F^l_j(t, r_j(t), v_j(t)) - F^l_0(t, r_0(t), v_0(t)) + \alpha)\Big].\end{aligned}$$

Since $\mid F^k_i(t, r_i(t), v_i(t)) - F^k_0(t, r_0(t), v_0(t)) \mid \leq \parallel F_i(t, r_i(t), v_i(t)) - F_0(t, r_0(t), v_0(t)) \parallel, \forall t \in \boldsymbol{R}^+, \forall i = 1, 2, \cdots, N, \forall k = 1, 2, \cdots, n$, it follows from Supplementary Lemma iii that

$$\max \tilde{\mathcal{L}}_F V < 0.$$

- *Case (iii)*: $S^+(t) < 0$ and $S^-(t) < 0$.

Since $S^-(t) < 0$ and Assumption 1 holds, one has $\sum_{s \in \mathcal{N}_j} a_{js}[S^-(t) - S^l_s(t)] < 0$, and for

$$\partial V(S^+(t), S^-(t)) = \{(0, -1)\},$$

one has

$$\tilde{\mathcal{L}}_F V = \mathcal{K}\left[-(F^l_j(t, r_j(t), v_j(t)) - F^l_0(t, r_0(t), v_0(t)) + \alpha)\right].$$

Since $\mid F^l_j(t, r_j(t), v_j(t)) - F^l_0(t, r_0(t), v_0(t)) \mid \leq \parallel F_j(t, r_j(t), v_j(t)) - F_0(t, r_0(t), v_0(t)) \parallel, \forall t \in \boldsymbol{R}^+, \forall j = 1, 2, \cdots, N, \forall l = 1, 2, \cdots, n$, it follows from Supplementary Lemma iii that

$$\max \tilde{\mathcal{L}}_F V < 0.$$

- *Case (iv)*: $S^+(t) > 0$ and $S^-(t) = 0$.

Since $S^+(t) > 0, S^-(t) = 0$ and Assumption 1 holds, one has $\sum_{r \in \mathcal{N}_i} a_{ir}[S^+(t) - S^k_r(t)] > 0, \sum_{s \in \mathcal{N}_j} a_{js}[S^-(t) - S^l_s(t)] \leq 0$. So, if $v \in \mathcal{F}(S^+(t), S^-(t))$, then $v^T = (v_1, v_2)$ with $v_1 \in$

$\mathcal{K}[F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) - \alpha]$ and $v_2 \in \mathcal{K}[F_j^l(t, r_j(t), v_j(t)) - F_0^l(t, r_0(t), v_0(t)) + \alpha] \cup \mathcal{K}[F_j^l(t, r_j(t), v_j(t)) - F_0^l(t, r_0(t), v_0(t))]$. For

$$\partial V(S^+(t), S^-(t)) = \{1\} \times [-1, 0],$$

if $\zeta \in \partial V(S^+(t), S^-(t))$, then $\zeta^T = (1, y)$ with $y \in [-1, 0]$. Therefore,

$$\zeta^T v = v_1 + y v_2.$$

If there exists an element a satisfying that $\zeta^T v = a$ for all $y \in [-1, 0]$, then $v_2 = 0$. So, if $v_2 \neq 0$, one has $\tilde{\mathcal{L}}_{\mathcal{F}} V = \emptyset$; if $v_2 = 0$, one has $\tilde{\mathcal{L}}_{\mathcal{F}} V = \mathcal{K}[F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) - \alpha]$, and then it follows from Supplementary Lemma iii that $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$. Thus, $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$ or $\tilde{\mathcal{L}}_{\mathcal{F}} V = \emptyset$ in this case.

- *Case (v)*: $S^+(t) = 0$ and $S^-(t) < 0$.

Since $S^+(t) = 0, S^-(t) < 0$ and Assumption 1 holds, one has $\sum_{r \in \mathcal{N}_i} a_{ir}[S^+(t) - S_r^k(t)] \geq 0, \sum_{s \in \mathcal{N}_j} a_{js}[S^-(t) - S_s^l(t)] < 0$. So, if $v \in \mathcal{F}(S^+(t), S^-(t))$, then $v^T = (v_1, v_2)$ with $v_1 \in \mathcal{K}[F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t)) - \alpha] \cup \mathcal{K}[F_i^k(t, r_i(t), v_i(t)) - F_0^k(t, r_0(t), v_0(t))]$ and $v_2 \in \mathcal{K}[F_j^l(t, r_j(t), v_j(t)) - F_0^l(t, r_0(t), v_0(t)) + \alpha]$. For

$$\partial V(S^+(t), S^-(t)) = [0, 1] \times \{-1\},$$

if $\zeta \in \partial V(S^+(t), S^-(t))$, then $\zeta^T = (y, -1)$ with $y \in [0, 1]$. Therefore,

$$\zeta^T v = y v_1 - v_2.$$

If there exists an element a satisfying that $\zeta^T v = a$ for all $y \in [0, 1]$, then $v_1 = 0$. So, if $v_1 \neq 0$, one has $\tilde{\mathcal{L}}_{\mathcal{F}} V = \emptyset$; if $v_1 = 0$, one has $\tilde{\mathcal{L}}_{\mathcal{F}} V = -\mathcal{K}[F_j^l(t, r_j(t), v_j(t)) - F_0^l(t, r_0(t), v_0(t)) + \alpha]$, and then it follows from Supplementary Lemma iii that $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$. Thus, $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$ or $\tilde{\mathcal{L}}_{\mathcal{F}} V = \emptyset$ in this case.

Combining the above five cases, it can be concluded that $\max \tilde{\mathcal{L}}_{\mathcal{F}} V < 0$ for all $(S^+(t), S^-(t)) \in \mathcal{D} \setminus \{(0, 0)\}$.

The proof is now completed.

15 Proof of Corollary 2

Proof The nonsmooth function $V(S^+(t), S^-(t))$ in (9) is chosen as the Lyapunov function. If Assumptions 1, 10 and 11 hold, then Supplementary Lemma iv holds. By Lemma 1, it follows from Lemmas 3 - 5 and Supplementary Lemma iv that $(S^{+}(t), S^{-}(t)) = (0, 0)$ is a globally stable equilibrium point for system (7).

Solving

$$S_i^k(t) = \tilde{r}_i^k(t) + \dot{\tilde{r}}_i^k(t) = 0,$$

one has

$$\tilde{r}_i^k(t) = ce^{-t}, \dot{\tilde{r}}_i^k(t) = -ce^{-t},$$

where c is a constant determined by the initial conditions. Therefore, the errors $\tilde{r}_i(t)$ and $\tilde{v}_i(t)$ converge to zero exponentially; that is, the second-order multi-agent system achieves the desired formation asymptotically.

The proof is now completed.