

# **Fourier Series**

Chapter Intended Learning Outcomes:

- (i) Represent continuous-time periodic signals using Fourier series
- (ii) Understand the properties of Fourier series
- (iii) Understand the relationship between Fourier series and linear time-invariant system

## Periodic Signal Representation in Frequency Domain

Fourier series can be considered as the frequency domain representation of a continuous-time periodic signal.

Recall (2.6) that  $x(t)$  is said to be periodic if there exists  $T_p > 0$  such that

$$x(t) = x(t + T_p), \quad t \in (-\infty, \infty) \quad (4.1)$$

The smallest  $T_p$  for which (4.1) holds is called the fundamental period.

Using (2.26), the fundamental frequency is related to  $T_p$  as:

$$\Omega_0 = \frac{2\pi}{T_p} \quad (4.2)$$

According to Fourier series,  $x(t)$  is represented as:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}, \quad t \in (-\infty, \infty) \quad (4.3)$$

where

$$a_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-jk\Omega_0 t} dt, \quad k = \dots - 1, 0, 1, 2, \dots \quad (4.4)$$

are called **Fourier series coefficients**. Note that the integration can be done for any period, e.g.,  $(0, T_p)$ ,  $(-T_p, 0)$ .

That is, a **periodic** signal can be expressed as a sum of **harmonically related complex sinusoids** with frequencies  $\dots - \Omega_0, 0, \Omega_0, 2\Omega_0, 3\Omega_0, \dots$ , where the fundamental frequency  $\Omega_0$  is called the **first harmonic**,  $2\Omega_0$  is called the **second harmonic**, and so on.

This means that  $x(t)$  only contains frequencies  $\dots - \Omega_0, 0, \Omega_0, 2\Omega_0, \dots$  with 0 being the DC component.

Note that the sinusoids are **complex-valued** with both **positive** and **negative** frequencies.

Note also that  $a_k$  is generally **complex** and we can also use magnitude and phase for its representation:

$$|a_k| = \sqrt{(\Re\{a_k\})^2 + (\Im\{a_k\})^2} \quad (4.5)$$

and

$$\angle(a_k) = \tan^{-1} \left( \frac{\Im\{a_k\}}{\Re\{a_k\}} \right) \quad (4.6)$$

From (4.3),  $\{a_k\}$  can be used to represent  $x(t)$ .

### Example 4.1

Find the Fourier series coefficients for  $x(t) = \cos(10\pi t) + \cos(20\pi t)$ .

It is clear that the fundamental frequency of  $x(t)$  is  $\Omega_0 = 10\pi$ . According to (4.2), the fundamental period is thus equal to  $T_p = 2\pi/\Omega_0 = 1/5$ , which is validated as follows:

$$\begin{aligned}x\left(t + \frac{1}{5}\right) &= \cos\left(10\pi\left(t + \frac{1}{5}\right)\right) + \cos\left(20\pi\left(t + \frac{1}{5}\right)\right) \\ &= \cos(10\pi t + 2\pi) + \cos(20\pi t + 4\pi) \\ &= \cos(10\pi t) + \cos(20\pi t)\end{aligned}$$

With the use of Euler formula in (2.29):

$$\cos(u) = \frac{e^{ju} + e^{-ju}}{2}$$

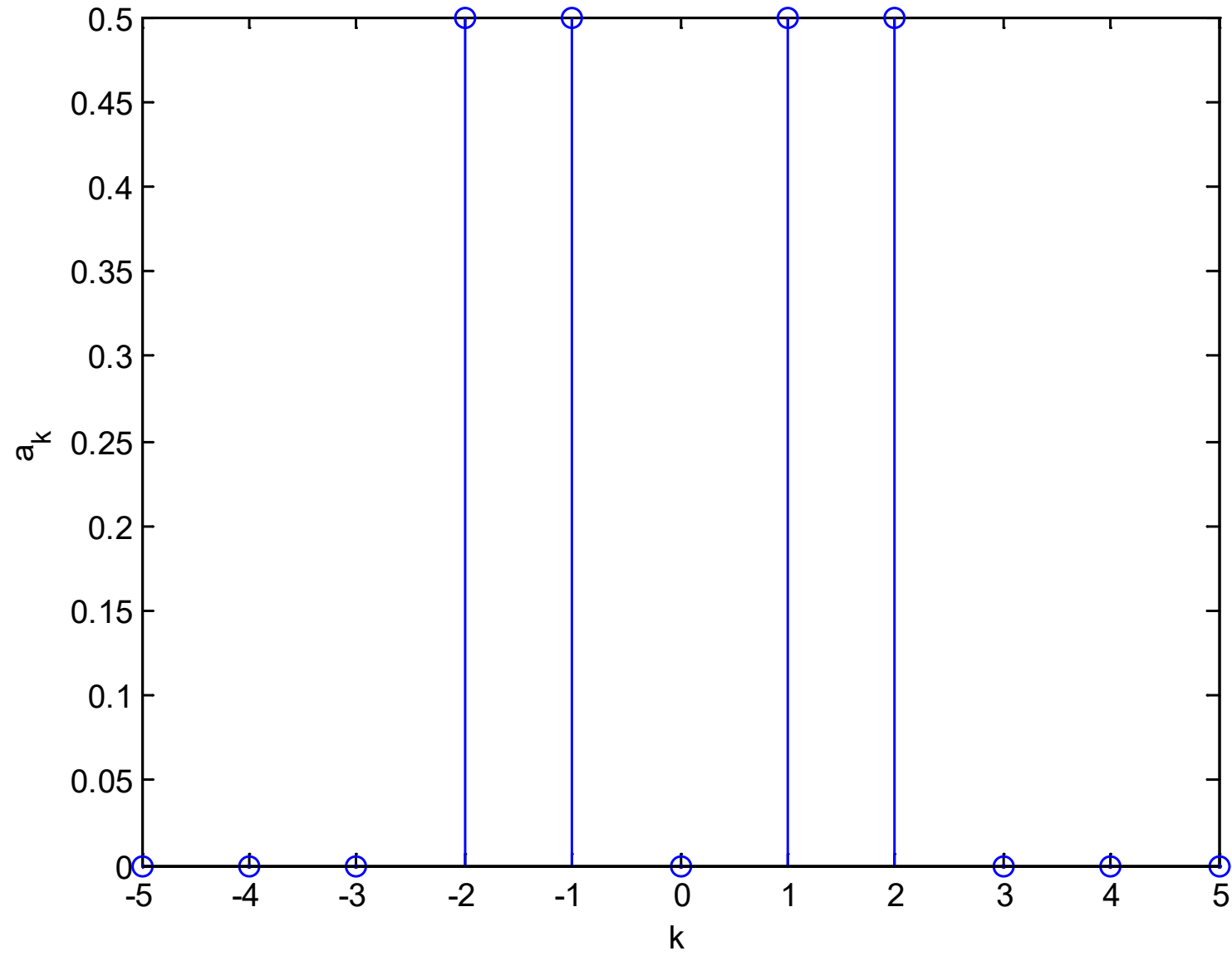
We can express  $x(t)$  as:

$$\begin{aligned}x(t) &= \cos(10\pi t) + \cos(20\pi t) \\&= \frac{e^{j10\pi t} + e^{-j10\pi t}}{2} + \frac{e^{j20\pi t} + e^{-j20\pi t}}{2} \\&= \frac{1}{2}e^{-j20\pi t} + \frac{1}{2}e^{-j10\pi t} + \frac{1}{2}e^{j10\pi t} + \frac{1}{2}e^{j20\pi t}\end{aligned}$$

which only contains four frequencies. Comparing with (4.3):

$$a_k = \begin{cases} 0.5, & k = -2 \\ 0.5, & k = -1 \\ 0.5, & k = 1 \\ 0.5, & k = 2 \\ 0, & \text{otherwise} \end{cases}$$

**Can we use (4.4)? Why?**



## Example 4.2

Find the Fourier series coefficients for  $x(t) = 1 + \sin(\Omega_0 t) + 2 \cos(\Omega_0 t) + \cos(3\Omega_0 t + \pi/4)$ .

With the use of Euler formulas in (2.29)-(2.30),  $x(t)$  can be written as:

$$\begin{aligned} x(t) &= 1 + \left(1 + \frac{1}{2j}\right) e^{j\Omega_0 t} + \left(1 - \frac{1}{2j}\right) e^{-j\Omega_0 t} + \frac{1}{2} e^{j\pi/4} e^{3j\Omega_0 t} + \frac{1}{2} e^{-j\pi/4} e^{-3j\Omega_0 t} \\ &= \frac{\sqrt{2}}{4} (1 - j) e^{-3j\Omega_0 t} + \left(1 + j\frac{1}{2}\right) e^{-j\Omega_0 t} + 1 + \left(1 - j\frac{1}{2}\right) e^{j\Omega_0 t} \\ &\quad + \frac{\sqrt{2}}{4} (1 + j) e^{3j\Omega_0 t} \end{aligned}$$

Again, comparing with (4.3) yields:

$$a_k = \begin{cases} \frac{\sqrt{2}}{4}(1 - j), & k = -3 \\ 1 + \frac{j}{2}, & k = -1 \\ 1, & k = 0 \\ 1 - \frac{j}{2}, & k = 1 \\ \frac{\sqrt{2}}{4}(1 + j), & k = 3 \\ 0, & \text{otherwise} \end{cases}$$

To plot  $\{a_k\}$ , we may compute  $|a_k|$  and  $\angle(a_k)$  for all  $k$ , e.g.,

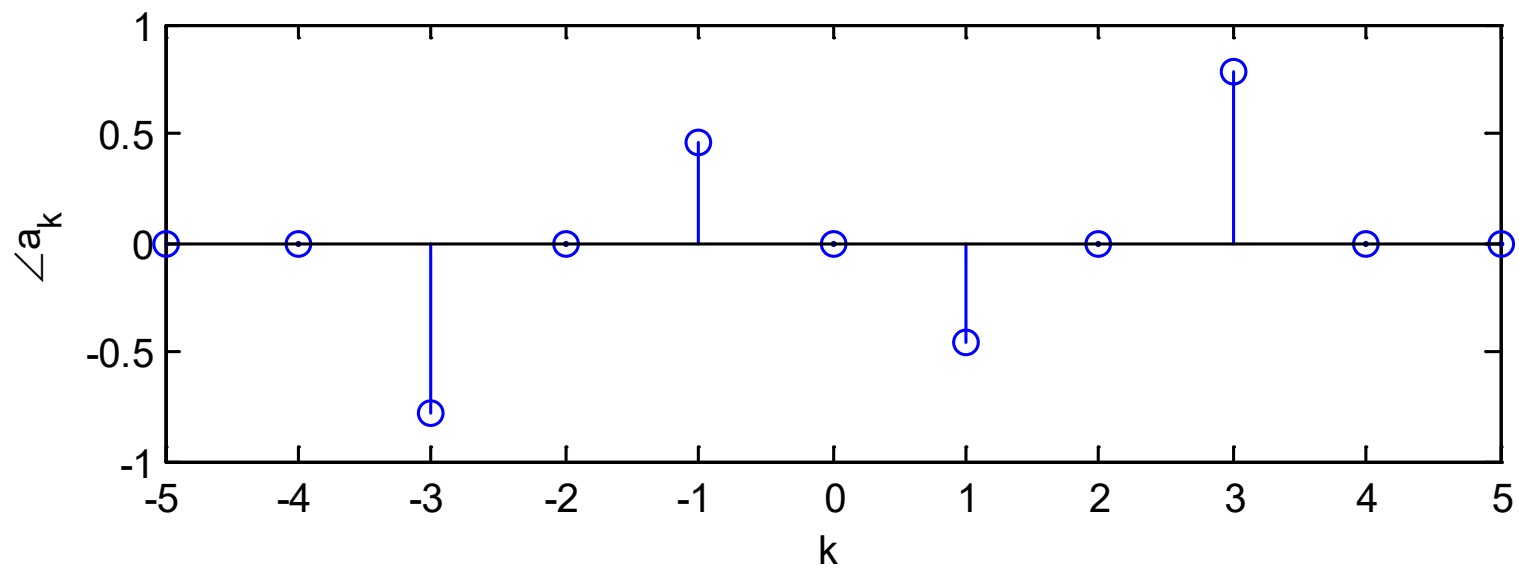
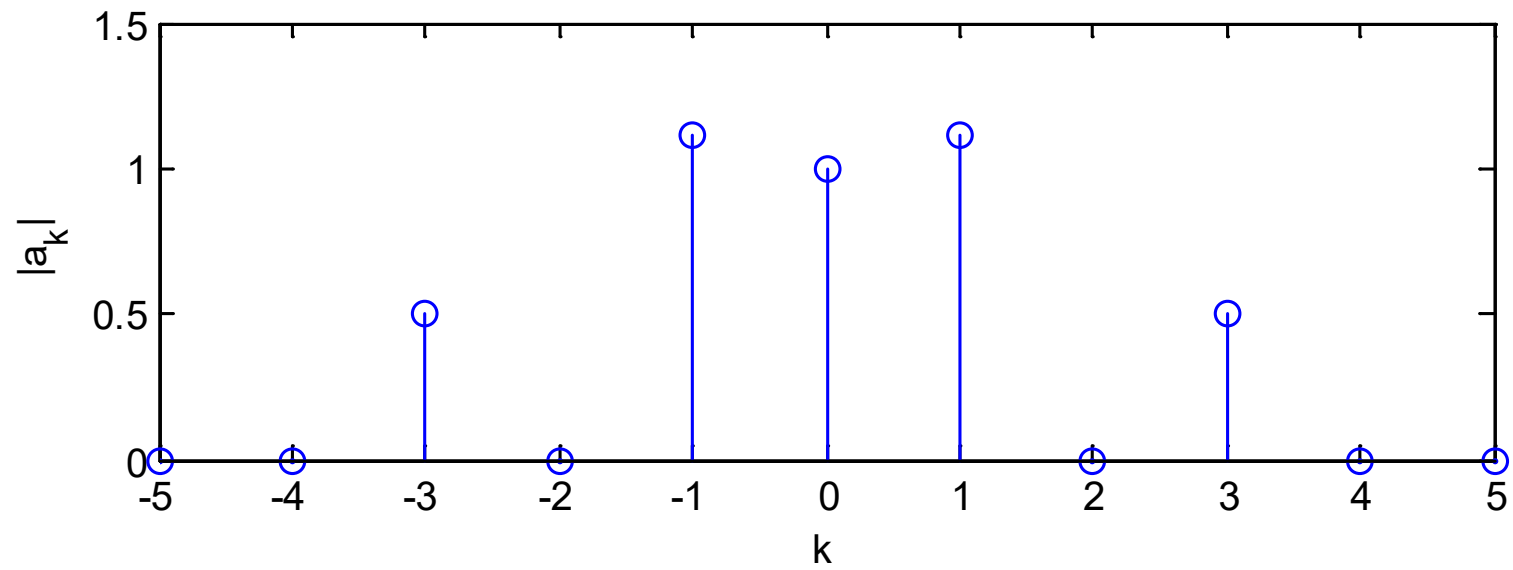
$$|a_{-3}| = \sqrt{\left(\frac{\sqrt{2}}{4}\right)^2 + \left(-\frac{\sqrt{2}}{4}\right)^2} = \frac{1}{2}$$

and

$$\angle(a_{-3}) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

We can also use MATLAB commands `abs` and `angle` to compute the magnitude and phase, respectively. After constructing a vector  $\mathbf{x}$  containing  $\{a_k\}$ , we can plot  $|a_k|$  and  $\angle(a_k)$  using:

```
subplot(2,1,1)
stem(n,abs(x))
xlabel('k')
ylabel('|a_k|')
subplot(2,1,2)
stem(n,angle(x))
xlabel('k')
ylabel('\angle{a_k}')
```

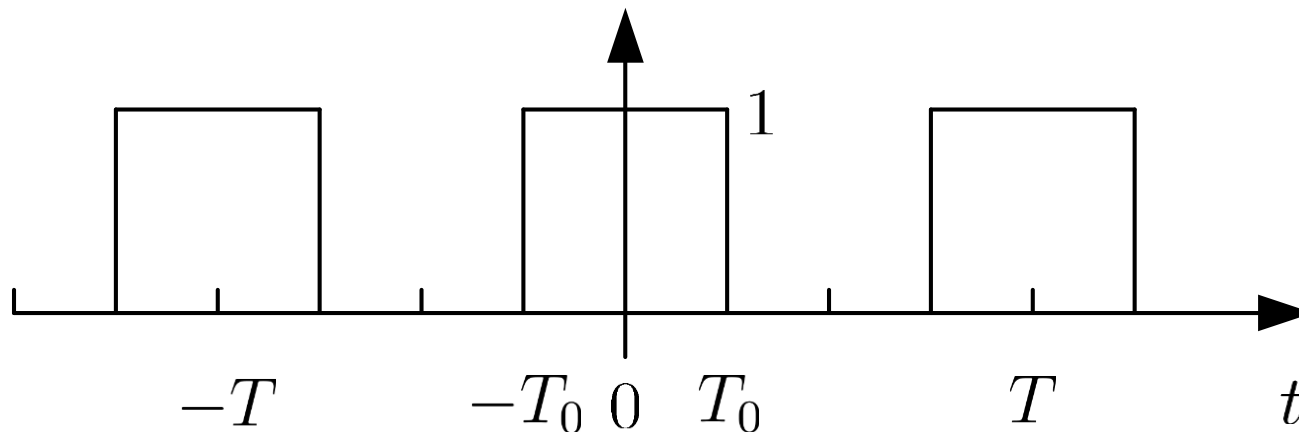


### Example 4.3

Find the Fourier series coefficients for  $x(t)$ , which is a periodic continuous-time signal of fundamental period  $T$  and is a pulse with a width of  $2T_0$  in each period. Over the specific period from  $-T/2$  to  $T/2$ ,  $x(t)$  is:

$$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & \text{otherwise} \end{cases}$$

with  $T > 2T_0$ .



Noting that the fundamental frequency is  $\Omega_0 = 2\pi/T$  and using (4.4), we get:

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\Omega_0 t} dt = \frac{1}{T} \int_{-T_0}^{T_0} e^{-jk\Omega_0 t} dt$$

For  $k = 0$ :

$$a_0 = \frac{1}{T} \int_{-T_0}^{T_0} 1 dt = \frac{2T_0}{T}$$

For  $k \neq 0$ :

$$a_k = \frac{1}{T} \int_{-T_0}^{T_0} e^{-jk\Omega_0 t} dt = -\frac{1}{jk\Omega_0 T} e^{-jk\Omega_0 t} \Big|_{-T_0}^{T_0} = \frac{\sin(k\Omega_0 T_0)}{k\pi} = \frac{\sin(2\pi k T_0 / T)}{k\pi}$$

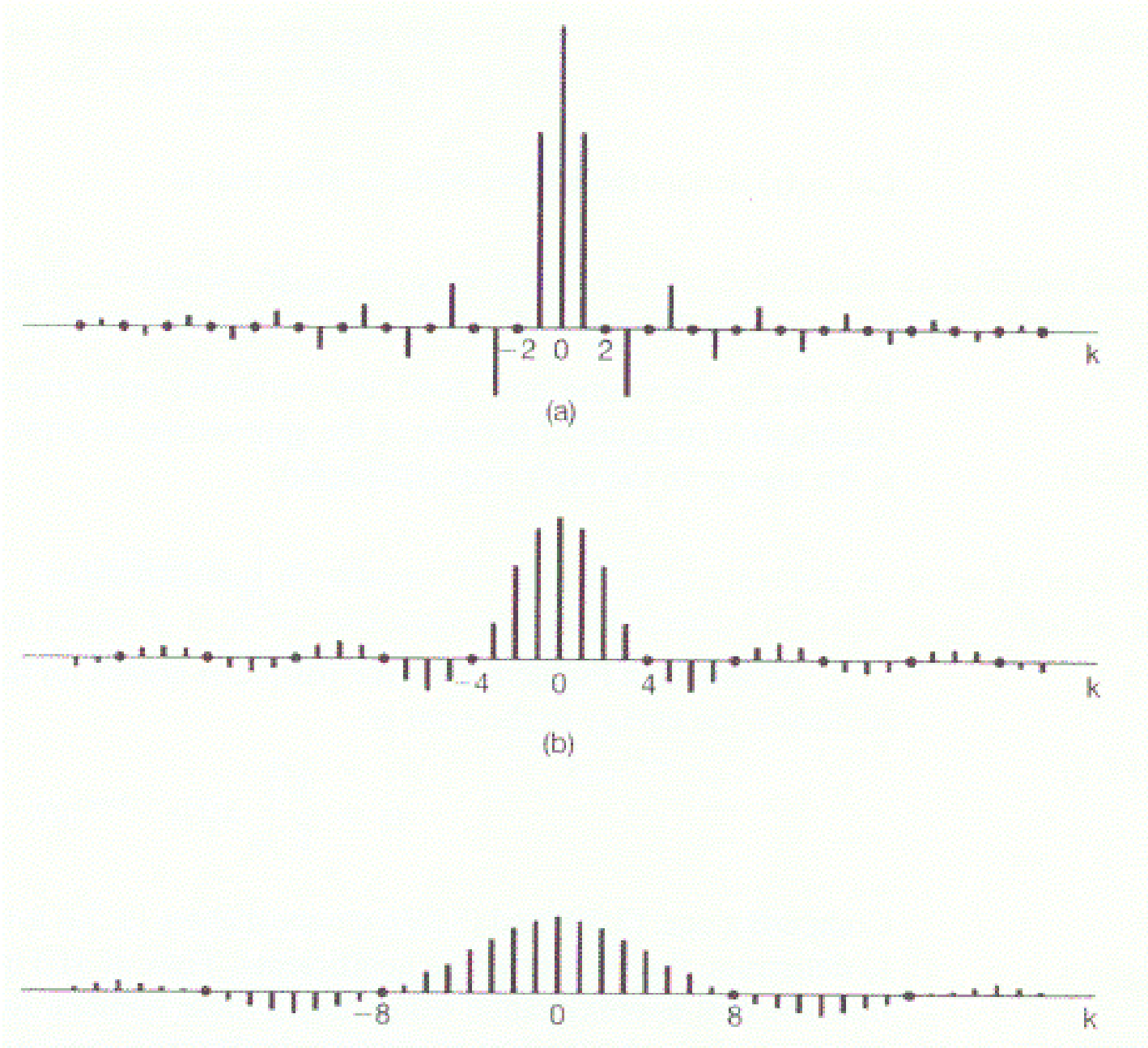
The reason of separating the cases of  $k = 0$  and  $k \neq 0$  is to facilitate the computation of  $a_0$ , whose value is not straightforwardly obtained from the general expression which involves "0/0".

Nevertheless, using L'Hôpital's rule:

$$\lim_{k \rightarrow 0} \frac{\sin(2\pi k T_0/T)}{k\pi} = \lim_{k \rightarrow 0} \frac{\frac{d \sin(2\pi k T_0/T)}{dk}}{\frac{dk\pi}{dk}} = \lim_{k \rightarrow 0} \frac{2\pi T_0/T \cos((2\pi k T_0/T))}{\pi} = \frac{2T_0}{T}$$

An investigation on the values of  $\{a_k\}$  with respect to relative pulse width  $T_0/T$  is performed as follows.

We see that when  $T_0/T$  decreases,  $\{a_k\}$  seem to be stretched.



### Example 4.4

Find the Fourier series coefficients for the following continuous-time periodic signal  $x(t)$ :

$$x(t) = \begin{cases} 1.5, & 0 < t < 1 \\ -1.5, & 1 < t < 2 \end{cases}$$

where the fundamental period is  $T_p = 2$  and fundamental frequency is  $\Omega_0 = \pi$ .

Using (4.4) with the period from  $t = -1$  to  $t = 1$ :

$$\begin{aligned} a_k &= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\Omega_0 t} dt \\ &= \frac{1}{2} \int_{-1}^0 (-1.5) e^{-jk\pi t} dt + \frac{1}{2} \int_0^1 1.5 e^{-jk\pi t} dt \end{aligned}$$

For  $k = 0$ :

$$a_k = \frac{1}{2} \int_{-1}^0 (-1.5) dt + \frac{1}{2} \int_0^1 1.5 dt = \frac{1}{2} (-1.5 + 1.5) = 0$$

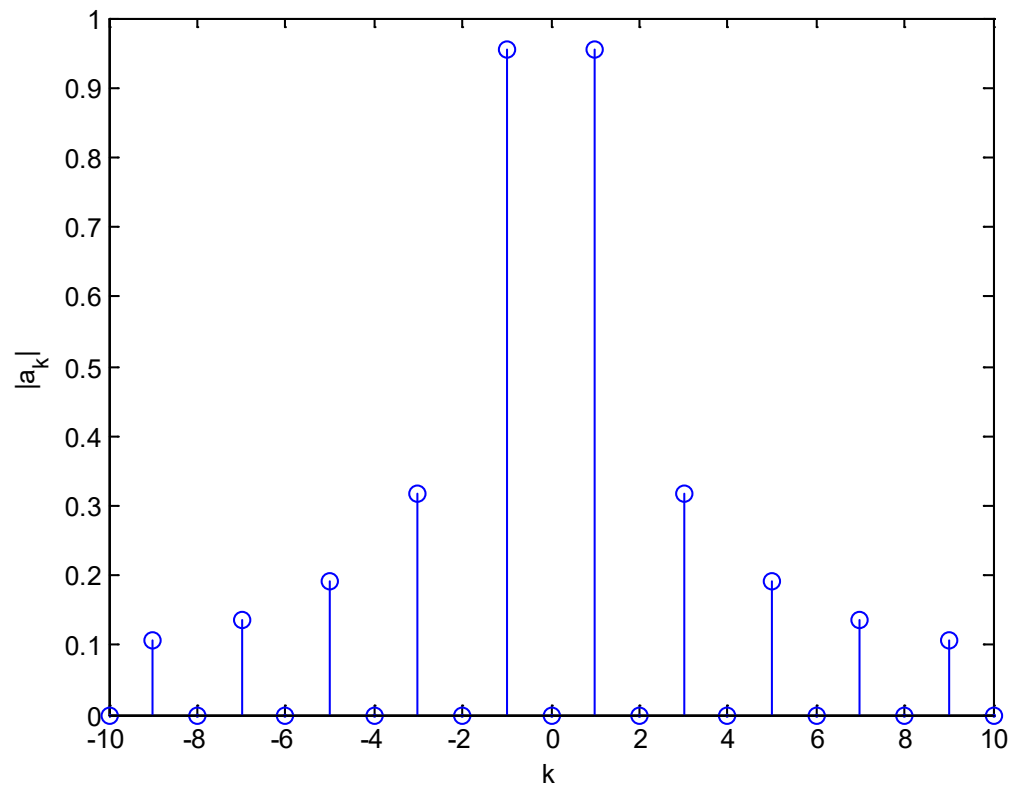
For  $k \neq 0$ :

$$\begin{aligned} a_k &= \frac{1}{2} \int_{-1}^0 (-1.5) e^{-jk\pi t} dt + \frac{1}{2} \int_0^1 1.5 e^{-jk\pi t} dt \\ &= \frac{3}{4} \left[ \int_{-1}^0 -e^{-jk\pi t} dt + \int_0^1 e^{-jk\pi t} dt \right] \\ &= \frac{3}{4} \left[ -\frac{1}{-jk\pi} e^{-jk\pi t} \Big|_{-1}^0 + \frac{1}{-jk\pi} e^{-jk\pi t} \Big|_0^1 \right] \\ &= \frac{3}{4jk\pi} [1 - e^{jk\pi} - e^{-jk\pi} + 1] \\ &= \frac{3}{2jk\pi} [1 - \cos(k\pi)] \end{aligned}$$

MATLAB can be used to validate the answer. First we have:

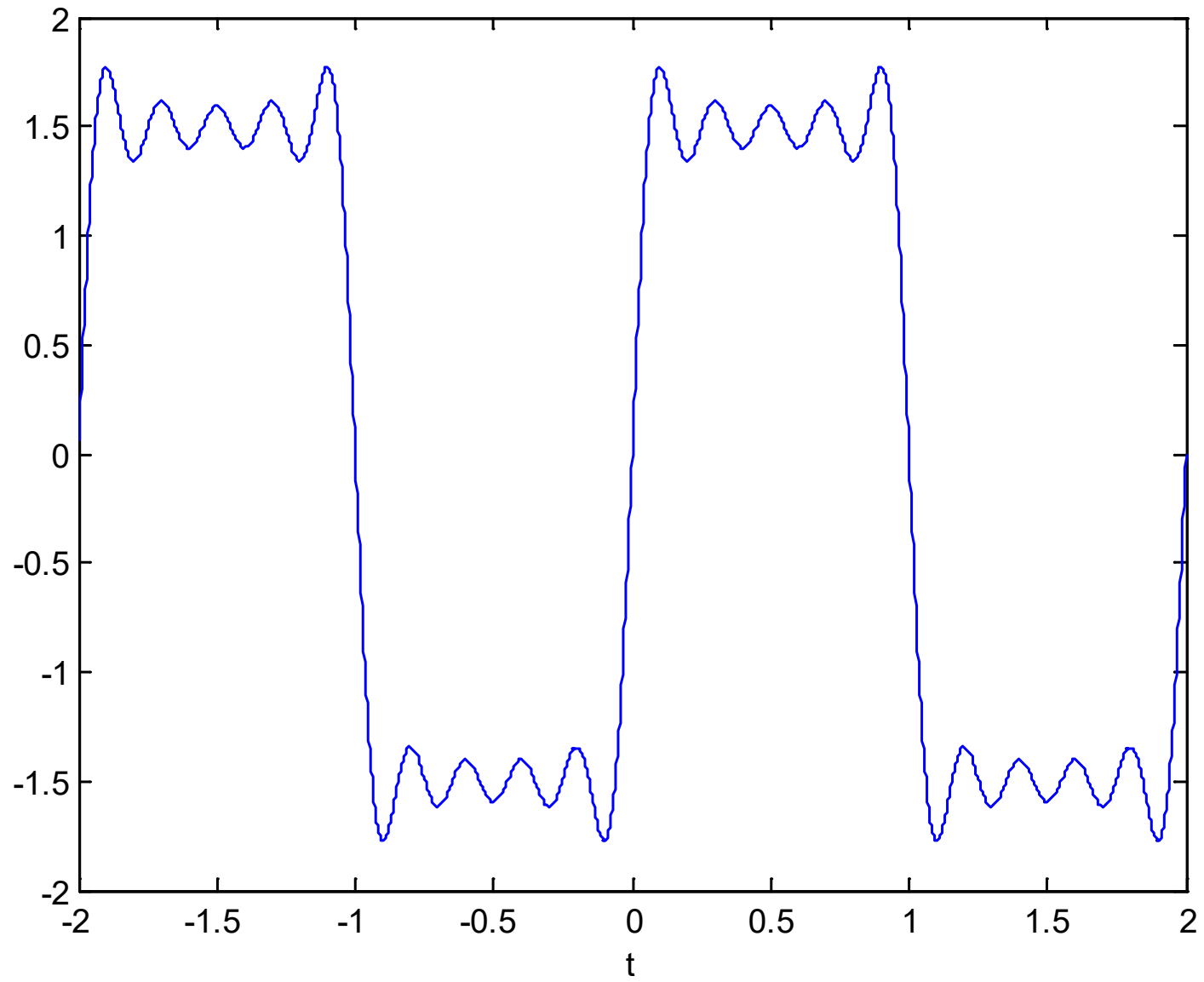
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \approx \sum_{k=-K}^K a_k e^{jk\Omega_0 t}$$

for sufficiently large  $K$  because  $|a_k|$  is decreasing with  $k$

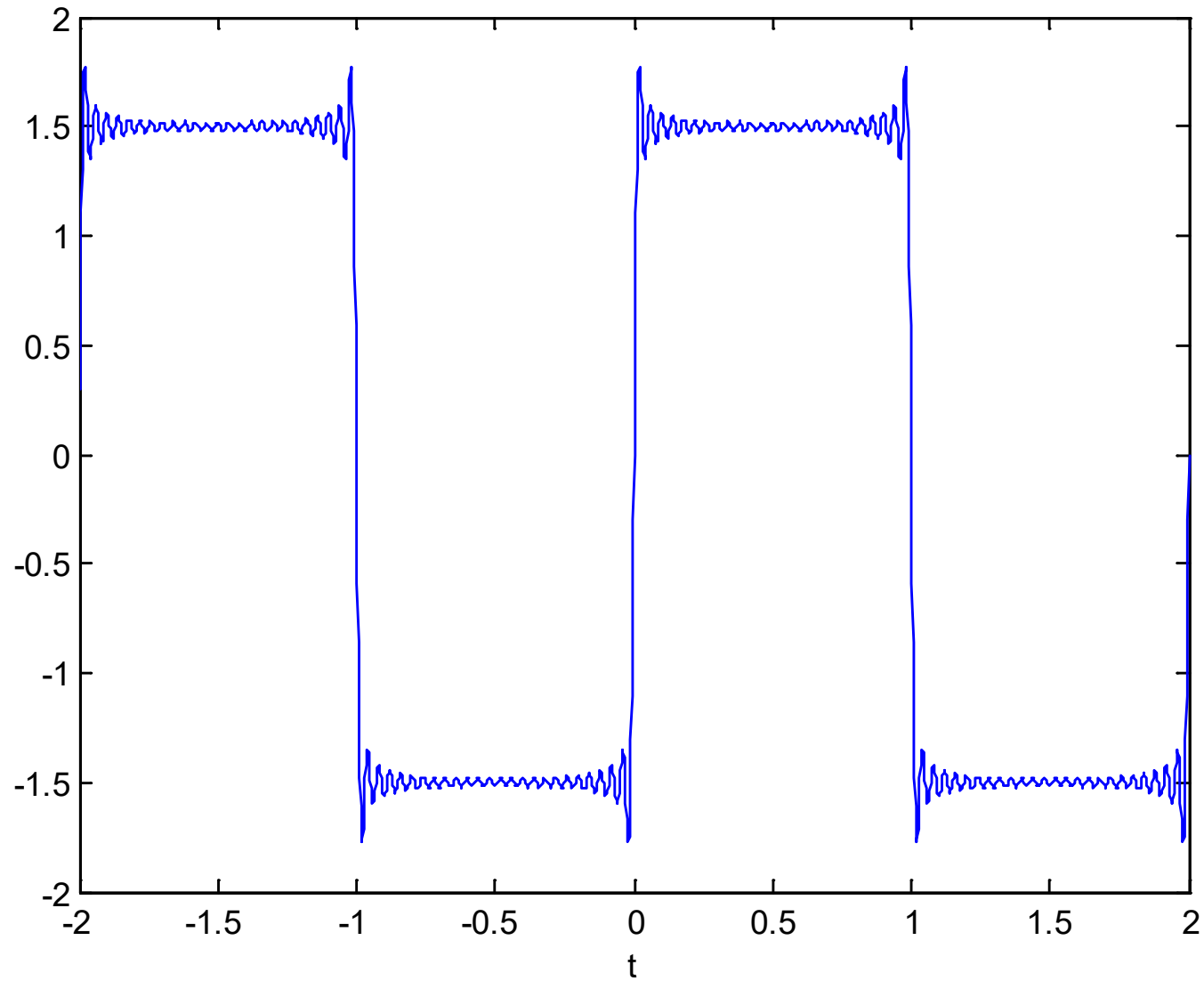


Setting  $K = 10$ , we may use the following code:

```
K=10;
a_p = 3./(j.*2.*[1:K].*pi).*(1-cos([1:K].*pi)); % +ve a_k
a_n = 3./(j.*2.*[-K:-1].*pi).*(1-cos([-K:-1].*pi)); %-ve a_k
a = [a_n 0 a_p]; %construct vector of a_k
for n=1:2000
    t=(n-1000)/500; %time interval of (-2,2);
        %small sampling interval of 1/500 to approximate x(t);
    e = (exp(j.*[-K:K].*pi.*t)).'; %construct exponential vector
    x(n) = a*e;
end
x=real(x); %remove imaginary parts due to precision error
n=1:2000;
t=(n-1000)./500;
plot(t,x)
xlabel('t')
```



For  $K = 50$ :



In summary, if  $x(t)$  is periodic, it can be represented as a **sum** of complex harmonics with amplitudes  $\{a_k\}$ .

That is,  $\{a_k\}$  correspond to the frequency domain representation of  $x(t)$  and we may write:

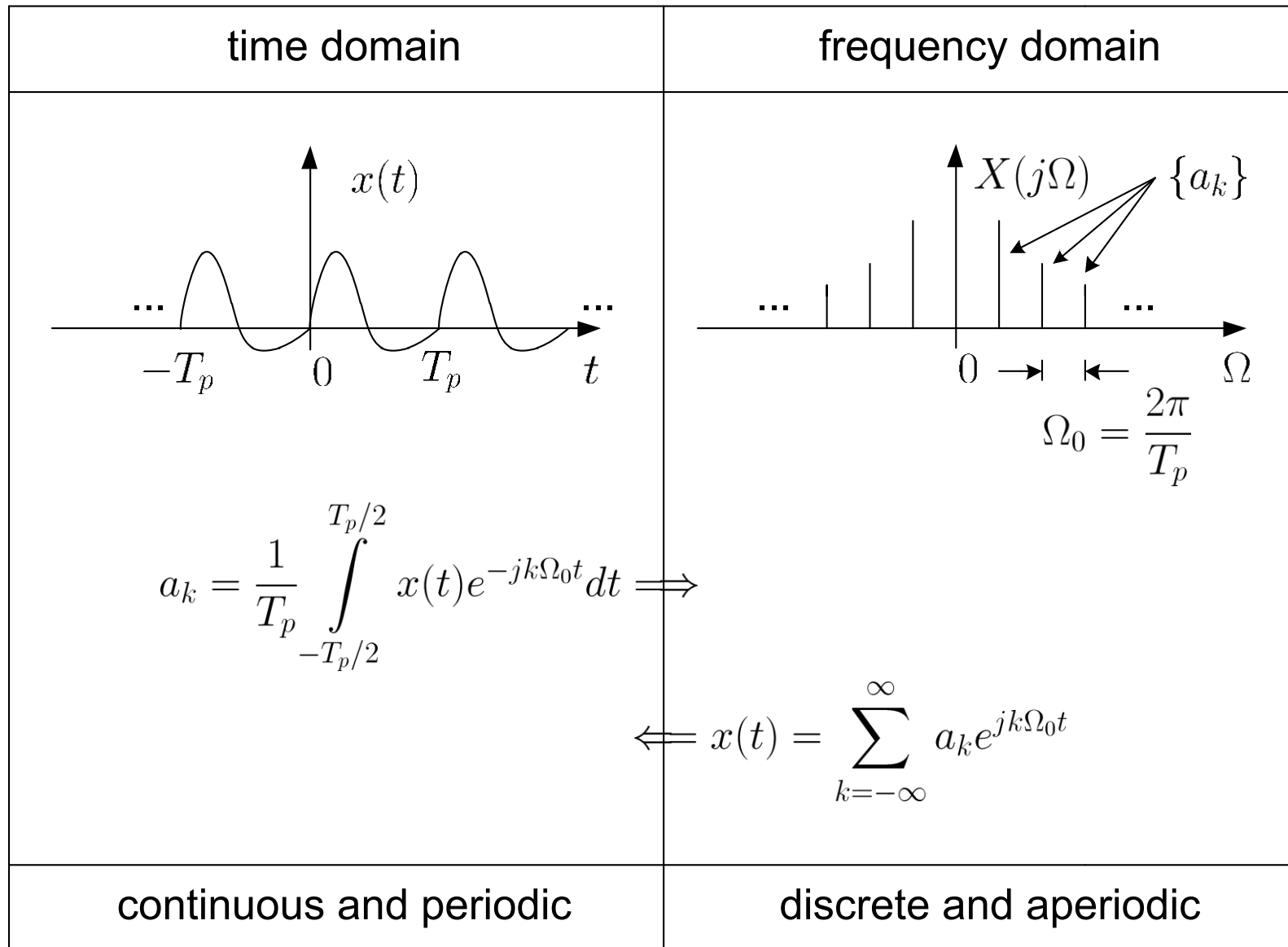
$$x(t) \leftrightarrow X(j\Omega) \quad \text{or} \quad x(t) \leftrightarrow a_k \quad (4.7)$$

where  $X(j\Omega)$  is characterized by  $\{a_k\}$ .

$x(t)$  and  $X(j\Omega)$  represent the **same** signal: we observe the former in time domain while the latter in frequency domain.

Generally speaking, Fourier series exist if  $|a_k| < \infty$ , i.e.,

$$\left| \int_{-T_p/2}^{T_p/2} x(t) e^{-jk\Omega_0 t} dt \right| \leq \int_{-T_p/2}^{T_p/2} |x(t) e^{-jk\Omega_0 t}| dt = \int_{-T_p/2}^{T_p/2} |x(t)| dt < \infty \quad (4.8)$$



**Fig.4.1: Illustration of Fourier series**

# Properties of Fourier Series

## Linearity

Let  $x(t) \leftrightarrow a_k$  and  $y(t) \leftrightarrow b_k$  be two Fourier series pairs with the same period of  $T_p$ . We have:

$$Ax(t) + By(t) \leftrightarrow Aa_k + Bb_k \quad (4.9)$$

This can be proved as follows. As  $x(t)$  and  $y(t)$  have the same fundamental period of  $T_p$  or fundamental frequency  $\Omega_0$ , we can write:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}, \quad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\Omega_0 t}$$

Multiplying  $x(t)$  and  $y(t)$  by  $A$  and  $B$ , respectively, yields:

$$Ax(t) = A \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}, \quad By(t) = B \sum_{k=-\infty}^{\infty} b_k e^{jk\Omega_0 t}$$

Summing  $Ax(t)$  and  $By(t)$ , we get:

$$Ax(t) + By(t) = \sum_{k=-\infty}^{\infty} (Aa_k + Bb_k) e^{jk\Omega_0 t} \leftrightarrow Aa_k + Bb_k$$

## Time Shifting

A shift of  $t_0$  in  $x(t)$  causes a multiplication of  $e^{-jk\Omega_0 t_0}$  in  $a_k$ :

$$x(t) \leftrightarrow a_k \Rightarrow x(t - t_0) \leftrightarrow e^{-jk\Omega_0 t_0} a_k = e^{-jk(2\pi)/T_p t_0} a_k \quad (4.10)$$

## Time Reversal

$$x(t) \leftrightarrow a_k \Rightarrow x(-t) \leftrightarrow a_{-k} \quad (4.11)$$

(4.10) and (4.11) are proved as follows.

Recall (4.3):

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

Substituting  $t$  by  $t - t_0$ , we obtain:

$$x(t - t_0) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0(t-t_0)} = \sum_{k=-\infty}^{\infty} (e^{-jk\Omega_0 t_0} a_k) e^{jk\Omega_0 t} \leftrightarrow e^{-jk\Omega_0 t_0} a_k$$

Substituting  $t$  by  $-t$  yields:

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0(-t)} = \sum_{l=-\infty}^{\infty} a_{-l} e^{jl\Omega_0 t} = \sum_{k=-\infty}^{\infty} a_{-k} e^{jk\Omega_0 t} \leftrightarrow a_{-k}$$

## Time Scaling

For a time-scaled version of  $x(t)$ ,  $x(\alpha t)$  where  $\alpha > 0$  is a real number, we have:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \Rightarrow x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\Omega_0)t} \quad (4.12)$$

## Multiplication

Let  $x(t) \leftrightarrow a_k$  and  $y(t) \leftrightarrow b_k$  be two Fourier series pairs with the same period of  $T_p$ . We have:

$$x(t)y(t) \leftrightarrow \sum_{l=-\infty}^{\infty} a_l b_{k-l} = a_k \otimes b_k \quad (4.13)$$

(4.13) is proved as follows.

Applying (4.3) again, the product of  $x(t)$  and  $y(t)$  is:

$$\begin{aligned}x(t)y(t) &= \sum_{l=-\infty}^{\infty} a_l e^{jl\Omega_0 t} \sum_{n=-\infty}^{\infty} b_n e^{jn\Omega_0 t} \\&= \sum_{l=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_l b_n e^{j(l+n)\Omega_0 t} \\&= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_l b_{k-l} e^{jk\Omega_0 t}, \quad k = l + n \\&= \sum_{k=-\infty}^{\infty} \left( \sum_{l=-\infty}^{\infty} a_l b_{k-l} \right) e^{jk\Omega_0 t} \leftrightarrow \sum_{l=-\infty}^{\infty} a_l b_{k-l}\end{aligned}$$

## Conjugation

$$x(t) \leftrightarrow a_k \Rightarrow x^*(t) \leftrightarrow a_{-k}^* \quad (4.14)$$

## Parseval's Relation

The **power** of  $x(t)$  can be represented in two forms:

$$\frac{1}{T_p} \int_{-T_p/2}^{T_p/2} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 \quad (4.15)$$

That is, we can compute the power in either the time domain or frequency domain.

**What is the power of the signal in Example 4.1?**

## Example 4.5

Prove the Parseval's relation.

Using (4.3), we have:

$$\begin{aligned}\frac{1}{T_p} \int_{-T_p/2}^{T_p/2} |x(t)|^2 dt &= \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} \left( \sum_{m=-\infty}^{\infty} a_m e^{jm\Omega_0 t} \right) \left( \sum_{n=-\infty}^{\infty} a_n e^{jn\Omega_0 t} \right)^* dt \\ &= \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} \left( \sum_{m=-\infty}^{\infty} a_m e^{jm\Omega_0 t} \right) \left( \sum_{n=-\infty}^{\infty} a_n^* e^{-jn\Omega_0 t} \right) dt \\ &= \frac{1}{T_p} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_m a_n^* \int_{-T_p/2}^{T_p/2} e^{j(m-n)\Omega_0 t} dt \\ &= \frac{1}{T_p} \sum_{m=-\infty}^{\infty} |a_m|^2 \int_{-T_p/2}^{T_p/2} dt = \sum_{m=-\infty}^{\infty} |a_m|^2\end{aligned}$$

## Linear Time-Invariant System with Periodic Input

Recall for a linear time-invariant (LTI) system, the input-output relationship is characterized by convolution in (3.17):

$$\begin{aligned} y(t) &= x(t) \otimes h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau \end{aligned} \quad (4.16)$$

If the input to the system with impulse response  $h(t)$  is  $x(t) = e^{j\Omega_0 t}$ , then the output is:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{j\Omega_0(t-\tau)}d\tau \\ &= e^{j\Omega_0 t} \int_{-\infty}^{\infty} h(\tau)e^{-j\Omega_0\tau}d\tau \end{aligned} \quad (4.17)$$

Note that  $\int_{-\infty}^{\infty} h(\tau)e^{-j\Omega_0\tau}d\tau$  is independent of  $t$  but a function of  $\Omega_0$  and we may denote it as  $H(j\Omega_0)$ :

$$y(t) = e^{j\Omega_0 t} H(j\Omega_0) = H(j\Omega_0)x(t) \quad (4.18)$$

If we input a sinusoid through a LTI system, there is **no change in frequency** in the output but amplitude and phase are modified.

Generalizing the result to any periodic signal in (4.3) yields:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\Omega_0) e^{jk\Omega_0 t} \quad (4.19)$$

where only the Fourier series coefficients are modified.

Note that discrete Fourier series is used to represent discrete periodic signal in (2.7) but it will not be discussed.