

Fourier Transform

Chapter Intended Learning Outcomes:

- (i) Represent continuous-time aperiodic signals using Fourier transform
- (ii) Understand the properties of Fourier transform
- (iii) Understand the relationship between Fourier transform and linear time-invariant system

Aperiodic Signal Representation in Frequency Domain

For a periodic continuous-time signal, it can be represented in frequency domain using Fourier series.

But in general, signals are **aperiodic**. To address this, we use Fourier transform:

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt \quad (5.1)$$

where $X(j\Omega)$ is a function of frequency Ω , also known as **spectrum**, and we can study the signal frequency components from it.

Generally speaking, Fourier transform exists if $|X(j\Omega)| < \infty$:

$$\left| \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt \right| \leq \int_{-\infty}^{\infty} |x(t)e^{-j\Omega t}| dt = \int_{-\infty}^{\infty} |x(t)| dt < \infty \quad (5.2)$$

That is, $x(t)$ is **absolutely integrable**.

Unlike Fourier series, $X(j\Omega)$ is continuous in frequency, i.e., defined on a continuous range of Ω .

The inverse transform is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega \quad (5.3)$$

As in (4.7), we may write:

$$x(t) \leftrightarrow X(j\Omega) \quad (5.4)$$

Both $x(t)$ and $X(j\Omega)$ represent the **same** signal: $x(t)$ is the time domain representation while $X(j\Omega)$ is the frequency domain representation.

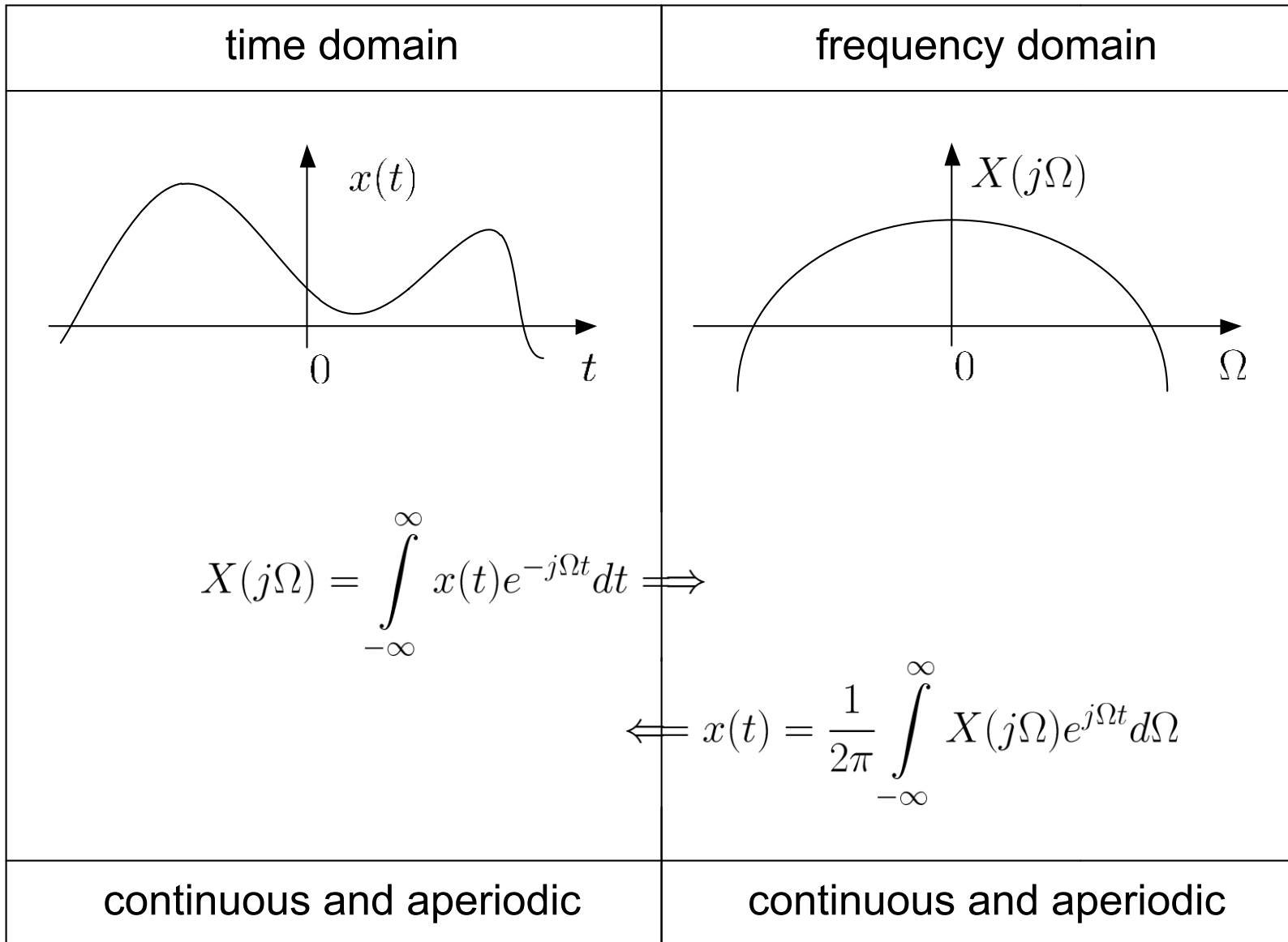


Fig.5.1: Illustration of Fourier transform

Derivation of Fourier Transform

Fourier transform can be derived from Fourier series as follows.

We start with an **aperiodic** $x(t)$ and then construct its **periodic** version $\tilde{x}(t)$ with period T .

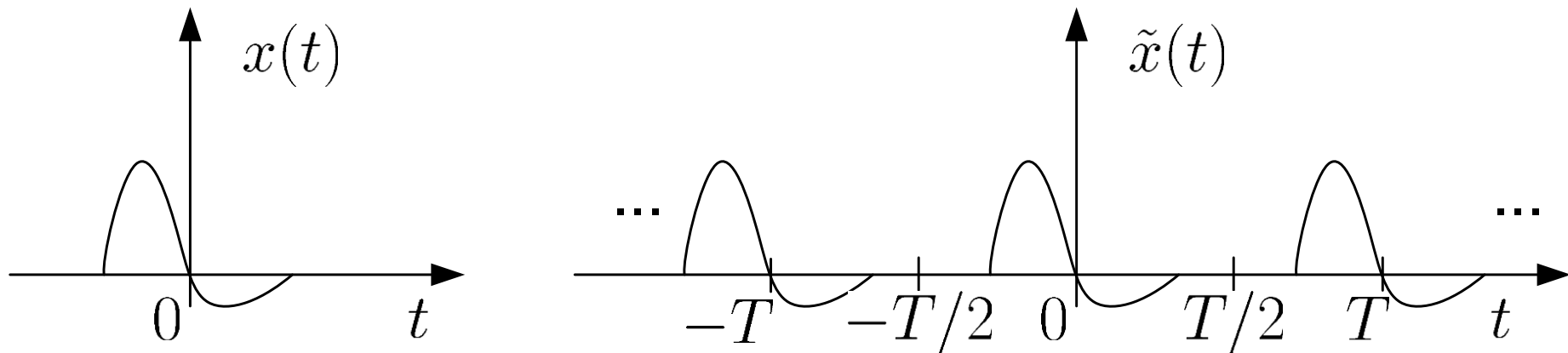


Fig.5.2: Constructing $\tilde{x}(t)$ from $x(t)$

According to (4.4), the Fourier series coefficients of $\tilde{x}(t)$ are:

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\Omega_0 t} dt \quad (5.5)$$

where $\Omega_0 = 2\pi/T$.

Noting that $x(t) = \tilde{x}(t)$ for $|t| < T/2$ and $x(t) = 0$ for $|t| > T/2$, (5.5) can be expressed as:

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\Omega_0 t} dt = \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-jk\Omega_0 t} dt \quad (5.6)$$

As $X(j\Omega)$ is function of Ω , substituting $\Omega = k\Omega_0$ in (5.1) gives

$$X(jk\Omega_0) = \int_{-\infty}^{\infty} x(t) e^{-jk\Omega_0 t} dt \quad (5.7)$$

We can express a_k as:

$$a_k = \frac{1}{T} X(jk\Omega_0) \quad (5.8)$$

According to (4.3) and using (5.8), we get the Fourier series expansion for $\tilde{x}(t)$:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\Omega_0) e^{jk\Omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0) e^{jk\Omega_0 t} \quad (5.9)$$

As $T \rightarrow \infty$ or $\Omega_0 \rightarrow 0$, $\tilde{x}(t) \rightarrow x(t)$.

Considering $\Omega_0 X(jk\Omega_0) e^{jk\Omega_0 t}$ as the area of a rectangle whose height is $X(jk\Omega_0) e^{jk\Omega_0 t}$ and width corresponds to the interval of $[k\Omega_0, (k+1)\Omega_0]$, we obtain:

$$\begin{aligned}
 x(t) &= \lim_{\Omega_0 \rightarrow 0} \tilde{x}(t) = \lim_{\Omega_0 \rightarrow 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0) e^{jk\Omega_0 t} \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega
 \end{aligned}
 \tag{5.10}$$

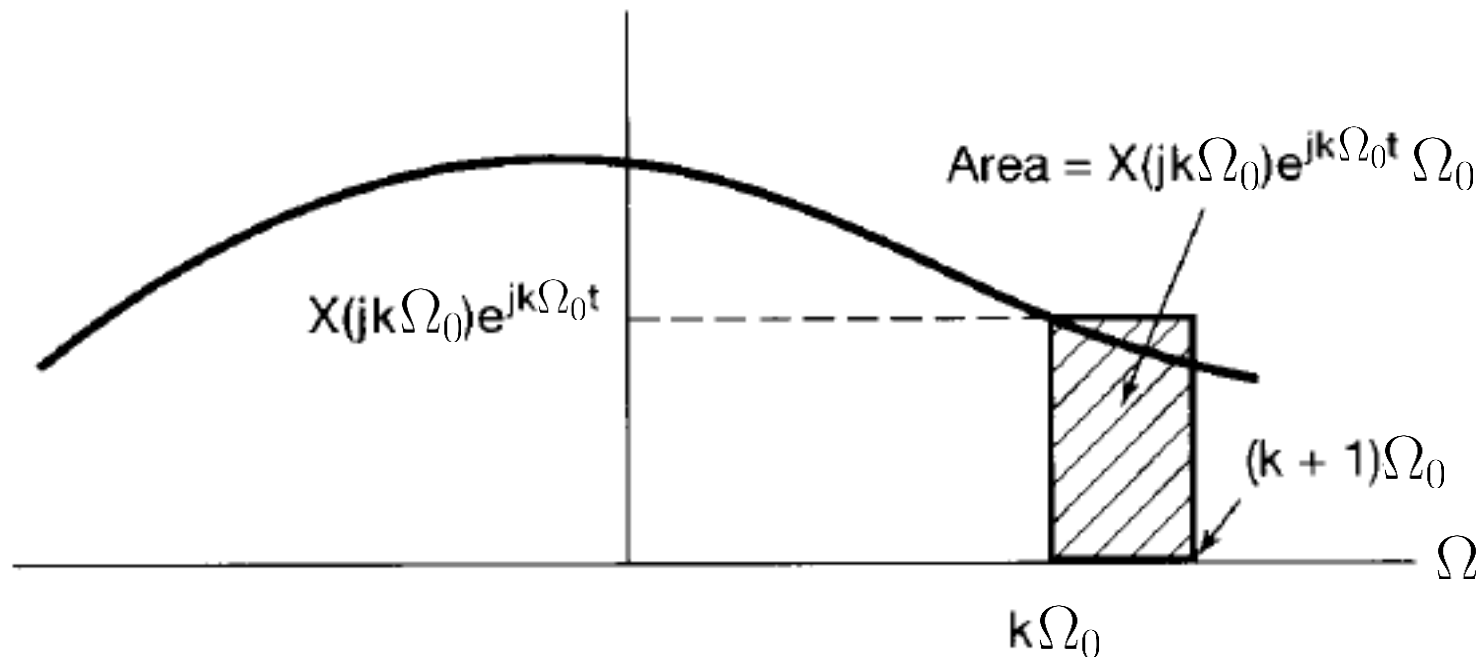


Fig. 5.3: Fourier transform from Fourier series

Example 5.1

Find the Fourier transform of $x(t)$ which is a rectangular pulse of the form:

$$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & \text{otherwise} \end{cases}$$

Note that the signal is of finite length and corresponds to one period of the periodic function in Example 4.3. Applying (5.1) on $x(t)$ yields:

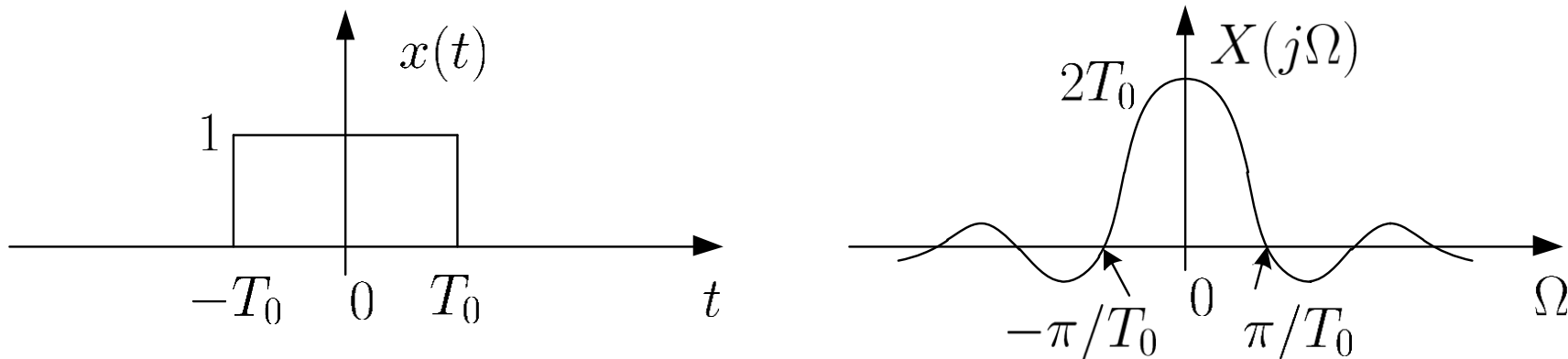
$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt = \int_{-T_0}^{T_0} e^{-j\Omega t} dt = \frac{2 \sin(\Omega T_0)}{\Omega}$$

Define the sinc function:

$$\text{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$$

It is seen that $X(j\Omega)$ is a scaled sinc function:

$$X(j\Omega) = \frac{2 \sin(\Omega T_0)}{\Omega} = 2T_0 \text{sinc}\left(\frac{\Omega T_0}{\pi}\right)$$



We can see that $X(j\Omega)$ is continuous in frequency. When the pulse width decreases, it covers more frequencies and vice versa.

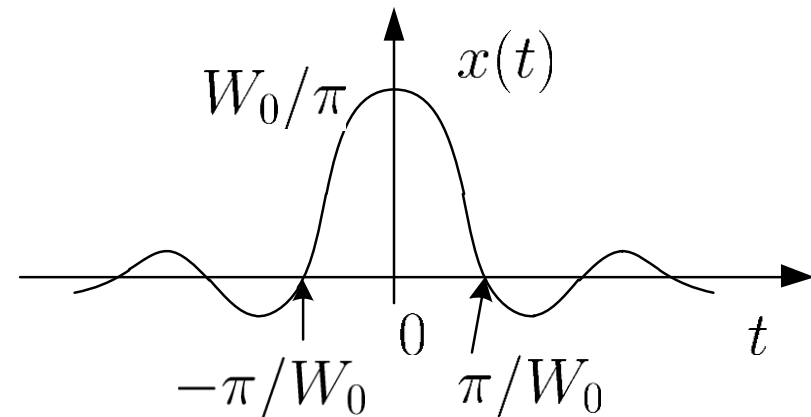
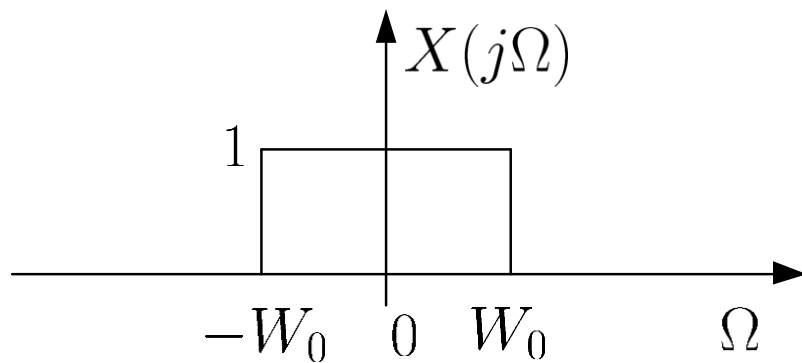
Example 5.2

Find the inverse Fourier transform of $X(j\Omega)$ which is a rectangular pulse of the form:

$$X(j\Omega) = \begin{cases} 1, & -W_0 < \Omega < W_0 \\ 0, & \text{otherwise} \end{cases}$$

Using (5.3), we get:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega = \frac{1}{2\pi} \int_{-W_0}^{W_0} e^{j\Omega t} d\Omega = \frac{\sin(W_0 t)}{\pi t} = \frac{W_0}{\pi} \operatorname{sinc}\left(\frac{W_0 t}{\pi}\right)$$



Example 5.3

Find the Fourier transform of $x(t) = e^{-at}u(t)$ with $a > 0$.

Employing the property of $u(t)$ in (2.22) and (5.1), we get:

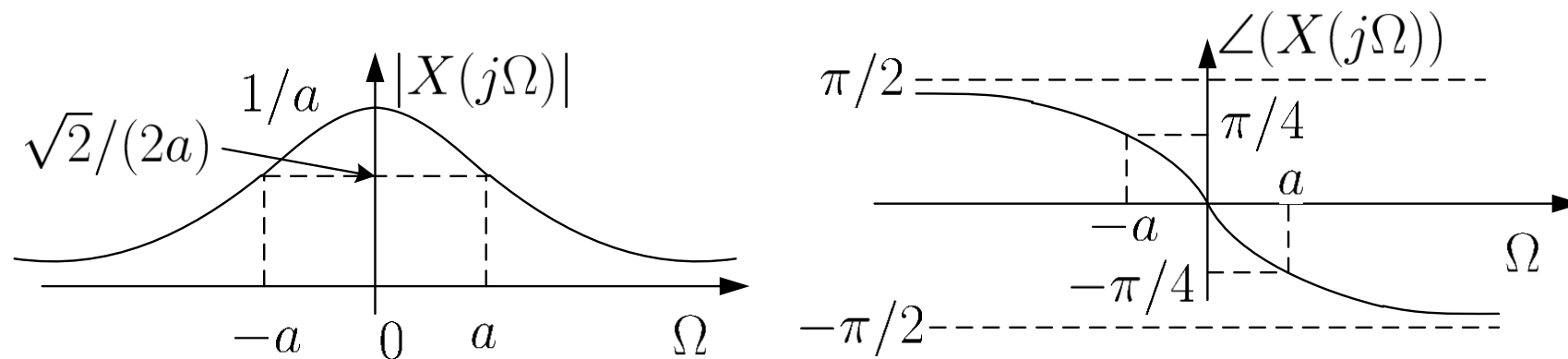
$$X(j\Omega) = \int_0^{\infty} e^{-at} e^{-j\Omega t} dt = -\frac{1}{a + j\Omega} e^{-(a+j\Omega)t} \Big|_0^{\infty} = \frac{1}{a + j\Omega} = \frac{a - j\Omega}{a^2 + \Omega^2}$$

Note that when $t \rightarrow \infty$, $e^{-at} \rightarrow 0$.

$$|X(j\Omega)| = \frac{1}{\sqrt{a^2 + \Omega^2}}$$

and

$$\angle(X(j\Omega)) = -\tan^{-1} \left(\frac{\Omega}{a} \right)$$



Example 5.4

Find the Fourier transform of the impulse $x(t) = \delta(t)$.

Using (2.19) and (2.20) with $x(t) = e^{-j\Omega t}$ and $t_0 = 0$, we get:

$$X(j\Omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega \cdot 0} dt = e^{-j\Omega \cdot 0} \int_{-\infty}^{\infty} \delta(t) dt = e^{-j\Omega \cdot 0} = 1$$

Spectrum of $\delta(t)$ has **unit amplitude** at **all frequencies**. This aligns with Example 5.1 when $T_0 \rightarrow 0$. On the other hand, at $T_0 \rightarrow \infty$, $x(t)$ will be a constant and only contains frequency 0.

Periodic Signal Representation using Fourier Transform

Fourier transform can be used to represent continuous-time periodic signals with the use of $\delta(t)$.

Instead of time domain, we consider an impulse in the **frequency domain**:

$$X(j\Omega) = 2\pi\delta(\Omega - \Omega_0) \quad (5.11)$$

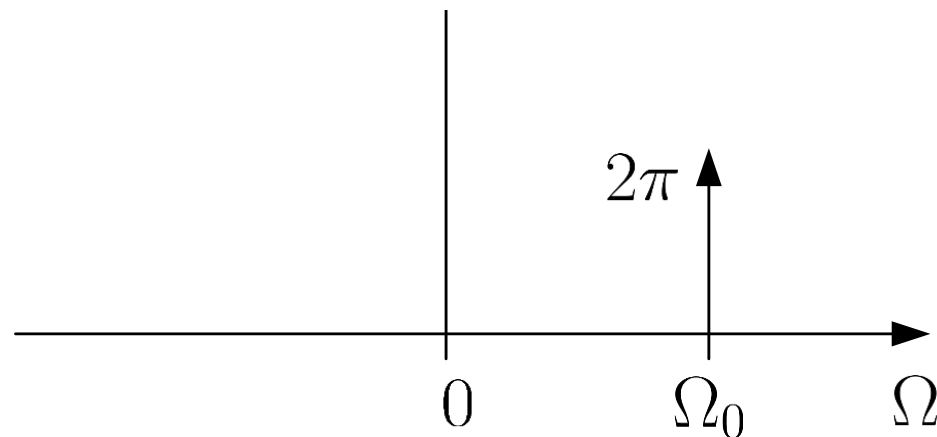


Fig.5.4: Impulse in frequency domain

Taking the inverse Fourier transform of $X(j\Omega)$ and employing the result in Example 5.4, $x(t)$ is computed as:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0)e^{j\Omega t} d\Omega = e^{j\Omega_0 t} \quad (5.12)$$

As a result, the Fourier transform pair is:

$$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega - \Omega_0) \quad (5.13)$$

From (4.3) and (5.13), the Fourier transform pair for a continuous-time periodic signal is:

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\Omega - k\Omega_0) \quad (5.14)$$

Example 5.5

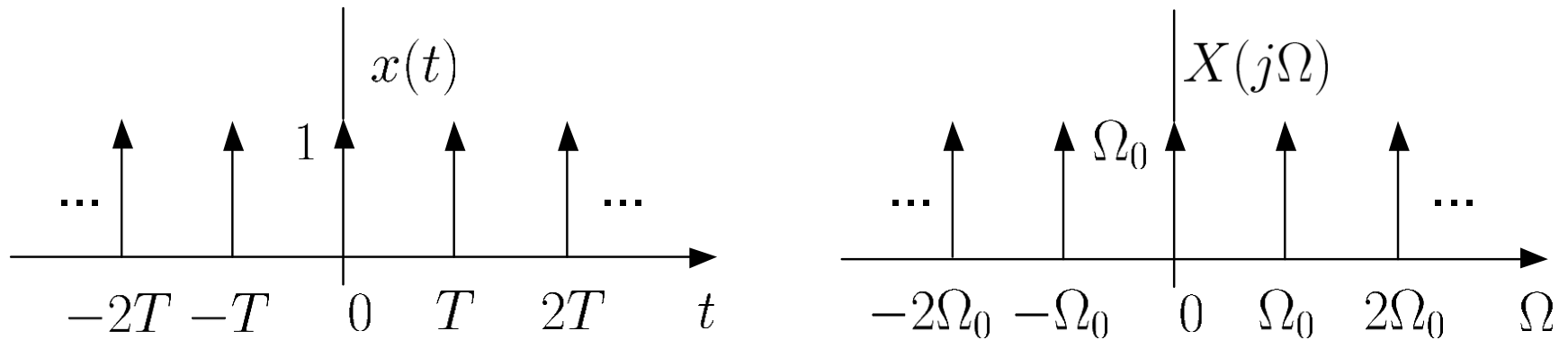
Find the Fourier transform of $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$ which is called an impulse train.

Clearly, $x(t)$ is a periodic signal with a period of T . Using (4.4) and Example 5.4, the Fourier series coefficients are:

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\Omega_0 t} dt = \frac{1}{T}$$

with $\Omega_0 = 2\pi/T$. According to (5.14), the Fourier transform is:

$$X(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{T}\right) = \Omega_0 \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_0)$$



Properties of Fourier Transform

Linearity

Let $x(t) \leftrightarrow X(j\Omega)$ and $y(t) \leftrightarrow Y(j\Omega)$ be two Fourier transform pairs. We have:

$$ax(t) + by(t) \leftrightarrow aX(j\Omega) + bY(j\Omega) \quad (5.15)$$

Time Shifting

A shift of t_0 in $x(t)$ causes a multiplication of $e^{-j\Omega t_0}$ in $X(j\Omega)$:

$$x(t) \leftrightarrow X(j\Omega) \Rightarrow x(t - t_0) \leftrightarrow e^{-j\Omega t_0} X(j\Omega) \quad (5.16)$$

Time Reversal

$$x(t) \leftrightarrow X(j\Omega) \Rightarrow x(-t) \leftrightarrow X(-j\Omega) \quad (5.17)$$

Time Scaling

For a time-scaled version of $x(t)$, $x(\alpha t)$ where $\alpha \neq 0$ is a real number, we have:

$$x(t) \leftrightarrow X(j\Omega) \Rightarrow x(\alpha t) \leftrightarrow \frac{1}{|\alpha|} X\left(\frac{j\Omega}{\alpha}\right) \quad (5.18)$$

Multiplication

Let $x(t) \leftrightarrow X(j\Omega)$ and $y(t) \leftrightarrow Y(j\Omega)$ be two Fourier transform pairs. We have:

$$x(t) \cdot y(t) \leftrightarrow \frac{1}{2\pi} X(j\Omega) \otimes Y(j\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\tau) Y(j(\Omega - \tau)) d\tau \quad (5.19)$$

Conjugation

$$x(t) \leftrightarrow X(j\Omega) \Rightarrow x^*(t) \leftrightarrow X^*(-j\Omega) \quad (5.20)$$

Multiplication by an Exponential Signal

Multiplying $x(t)$ by $e^{j\Omega_0 t}$ in time domain corresponds to a shift of Ω_0 in the frequency domain:

$$x(t) \leftrightarrow X(j\Omega) \Rightarrow e^{j\Omega_0 t} x(t) \leftrightarrow X(j(\Omega - \Omega_0)) \quad (5.21)$$

Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega \quad (5.22)$$

The **energy** of $x(t)$ can be computed in either time or frequency domain.

Convolution

Let $x(t) \leftrightarrow X(j\Omega)$ and $y(t) \leftrightarrow Y(j\Omega)$ be two Fourier transform pairs. We have:

$$x(t) \otimes y(t) \leftrightarrow X(j\Omega)Y(j\Omega) \quad (5.23)$$

The proof is given as follows.

$$\begin{aligned}
& \int_{-\infty}^{\infty} x(t) \otimes y(t) e^{-j\Omega t} dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) y(t - \tau) e^{-j\Omega t} d\tau dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) y(u) e^{-j\Omega\tau} e^{-j\Omega u} d\tau du, \quad u = t - \tau \\
&= \left[\int_{-\infty}^{\infty} x(\tau) e^{-j\Omega\tau} d\tau \right] \cdot \left[\int_{-\infty}^{\infty} y(u) e^{-j\Omega u} du \right] \\
&= X(j\Omega) \cdot Y(j\Omega)
\end{aligned} \tag{5.24}$$

Differentiation

Differentiating $x(t)$ with respect to t corresponds to multiplying $X(j\Omega)$ by $j\Omega$ in the frequency domain:

$$\frac{dx(t)}{dt} \leftrightarrow j\Omega X(j\Omega) \Rightarrow \frac{d^k x(t)}{dt^k} \leftrightarrow (j\Omega)^k X(j\Omega) \tag{5.25}$$

Integration

On the other hand, if we perform integration on $x(t)$, then the frequency domain representation becomes:

$$\int_{-\infty}^t x(\tau) d\tau \leftrightarrow \frac{1}{j\Omega} X(j\Omega) + \pi X(0) \delta(\Omega) \quad (5.26)$$

Fourier Transform and Linear Time-Invariant System

Recall in a linear time-invariant (LTI) system, the input-output relationship is characterized by convolution in (3.17):

$$\begin{aligned} y(t) &= x(t) \otimes h(t) \\ &= \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau \end{aligned} \quad (5.27)$$

Using (5.23), we can consider (5.27) in frequency domain:

$$y(t) = x(t) \otimes h(t) \leftrightarrow Y(j\Omega) = X(j\Omega)H(j\Omega) \quad (5.28)$$

This suggests apart from computing the output using time-domain approach via convolution, we can convert the input and impulse response to frequency domain, then $y(t)$ is computed from inverse Fourier transform of $X(j\Omega)H(j\Omega)$.

In fact, $H(j\Omega)$ represents the LTI system in the frequency domain, is called the **system frequency response**.

Recall (3.25) that the input and output of a LTI system satisfy the **differential equation**:

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k} \quad (5.29)$$

Taking the Fourier transform and using the linearity and differentiation properties, we get:

$$Y(j\Omega) \left[\sum_{k=0}^N a_k (j\Omega)^k \right] = X(j\Omega) \left[\sum_{k=0}^M b_k (j\Omega)^k \right] \quad (5.30)$$

The system frequency response can also be computed as:

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = \frac{\sum_{k=0}^M b_k (j\Omega)^k}{\sum_{k=0}^N a_k (j\Omega)^k} \quad (5.31)$$

Example 5.6

Determine the system frequency response for a LTI system described by the following differential equation:

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

Applying (5.31), we easily obtain:

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = \frac{1}{j\Omega + a}$$

Example 5.7

Find the differential equation corresponding to a continuous-time LTI system whose system frequency response is:

$$H(j\Omega) = \frac{j\Omega + 3}{(j\Omega + 1)(j\Omega + 2)}$$

Let $H(j\Omega) = Y(j\Omega)/X(j\Omega)$. Performing cross-multiplication and using (5.25), we obtain:

$$\begin{aligned}(j\Omega + 1)(j\Omega + 2)Y(j\Omega) &= (j\Omega + 3)X(j\Omega) \\ \Rightarrow ((j\Omega)^2 + 3j\Omega + 2)Y(j\Omega) &= (j\Omega + 3)X(j\Omega) \\ \Rightarrow \frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) &= \frac{dx(t)}{dt} + 3x(t)\end{aligned}$$

Examples 5.6 and 5.7 imply that the differential equation and system frequency response can be interchangeable.